Homogeneous coloured multipartite graphs

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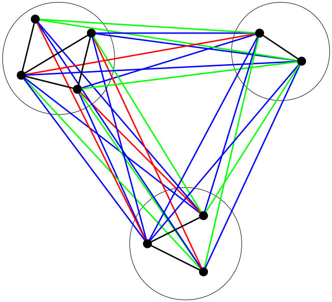
n-graphs

Definition

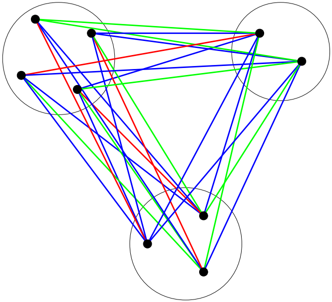
For n a positive integer, an *n-graph* is a graph on n pairwise disjoint sets of vertices $V_0, V_1, \ldots, V_{n-1}$ (called *parts*) each of which is an ordinary countable graph, with finitely many edge-types between pairs of parts (i.e. finite set of colours C_{ij} on $V_i \times V_j$).



Multipartite graphs

Definition

An *n-partite graph* is an n-graph for which each part is null (contains no edges within parts).



Homogeneous n-graphs

Problem (Cherlin)

Classify the countable homogeneous n-graphs.

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- DL, John Truss – coloured multipartite graphs
  [This talk, and a paper soon!]

Coloured multipartite graphs

From now on, let L be a finite relational language for coloured multipartite graphs. So L determines the n parts of the graph, and contains the finite sets of colours C_{ij} for edges between each pair of parts $V_i \times V_j$. We refer to graphs in this language as L-graphs.

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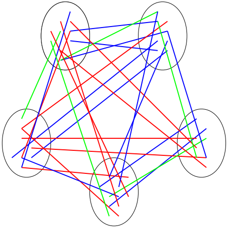
Lemma

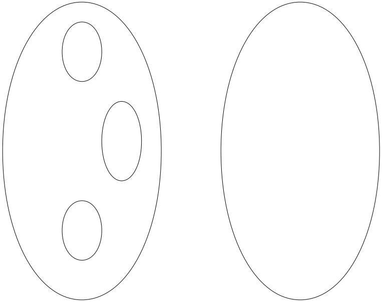
Any restriction of a homogeneous multipartite graph to a subset of its set of parts is homogeneous.



Bipartite graphs

Definition

C-edge-coloured bipartite graph G is *generic* if both parts are infinite and for any finite subset U of a part, and map $f : U \to C$, there is a vertex x in the other part such that for each $u \in U$ the edge xu is $f(u)$-coloured.



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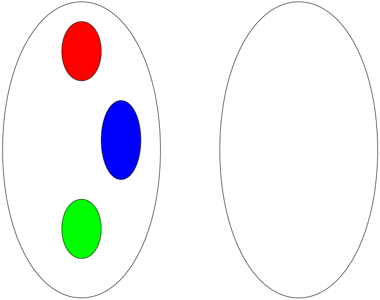

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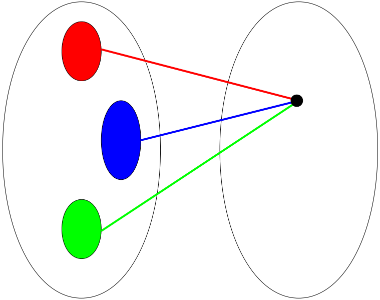

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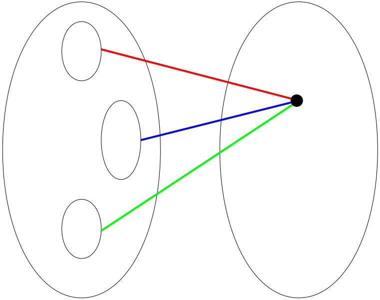

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A generic bipartite graph is homogeneous.

Theorem (J,S,T)

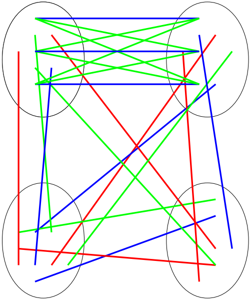
If G is a countable homogeneous C-edge-coloured bipartite graph, then one of the following holds:

1. $|C| = 1$ *and all edges are the same colour;*
2. $|C| = 2$ *and edges of one colour are a perfect matching, and edges of the other colour are its complement;*
3. $|C| \geq 2$ *and G is generic.*

Perfect matchings in multipartite graphs

Theorem

Let G be a multipartite graph such that parts V_i and V_j are related by a perfect matching. Then G is homogeneous if and only if $G - V_j$ is homogeneous and the map from $G - V_j$ to $G - V_i$ induced by the perfect matching is a colour-isomorphism.



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Reduce problem to the generic case — given a homogeneous multipartite graph first eliminate perfect matchings on bipartite restrictions, and then eliminate finite parts.

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Reduce problem to the generic case — given a homogeneous multipartite graph first eliminate perfect matchings on bipartite restrictions, and then eliminate finite parts.

Definition

A countable m-partite graph G is *m-generic* if each bipartite restriction is generic.

We aim to classify the homogeneous m-generic graphs.

m-generic graphs

So let L be a language for m-partite graphs, and let G be a homogeneous m-generic L-graph. Each bipartite restriction of G is generic, and so every possible finite bipartite L-graph embeds in G. However, not all L-graphs defined on at least three parts will necessarily embed in G.

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So let L be a language for m-partite graphs, and let G be a homogeneous m-generic L-graph. Each bipartite restriction of G is generic, and so every possible finite bipartite L-graph embeds in G. However, not all L-graphs defined on at least three parts will necessarily embed in G.

Example

Suppose we work in a language for 3-partite graphs, such that $C_{01} = C_{02} = C_{12} = \{\text{red, green, blue}\}$. There is a homogeneous 3-generic graph which does not embed a red triangle or a blue triangle, but does embed all other possible finite L-graphs.

Amalgamation

Definition

An *amalgamation class* $\mathcal{C}$ is a family of finite structures in a countable relational language, which is closed under isomorphism and taking substructures, and which has the *amalgamation property* (AP):
for $A, B_1, B_2 \in \mathcal{C}$, if there exist embeddings $f_1 : A \to B_1$ and $f_2 : A \to B_2$, then there exists $C \in \mathcal{C}$ and embeddings $g_1 : B_1 \to C$ and $g_2 : B_2 \to C$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

So, two structures in $\mathcal{C}$ with isomorphic substructures can be "glued together" so that the substructures are identified, to give a larger structure in $\mathcal{C}$.

Fraïssé's Theorem

Definition

The *age* of a countable relational structure M is the class of all finite structures which embed in M.

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Theorem (Fraïssé)

The age of any countable homogeneous structure is an amalgamation class. Conversely, if $\mathcal{C}$ is an amalgamation class, then there is a countable homogeneous structure M (unique up to isomorphism) with $Age(M) = \mathcal{C}$.

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Aim: Use Fraïssé's Theorem to classify m-generic graphs by classifying amalgamation classes.

Omitted graphs

Let L be a language for m-partite graphs, and let G be an L-graph.

Definitions

A finite L-graph A is *realized* in G if it embeds in G. Otherwise it is *omitted*.

A is *minimally omitted* if it is omitted and every proper induced subgraph is realized.

$O(G) :=$ family of finite L-graphs minimally omitted from G.

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Lemma

Two countable homogeneous L-graphs G_1, G_2 are isomorphic if and only if they minimally omit the same family of finite L-graphs.

(Note: can reduce the classification using "colour-isomorphism", but will not go into detail here.)

Forbidden families

Let $\mathcal{F}$ be a family of finite L-graphs.

Definition

$Forb(\mathcal{F}) :=$ family of all finite L-graphs which omit members of $\mathcal{F}$.

If G is a countable homogeneous L-graph with $O(G) = \mathcal{F}$, then $Age(G) = Forb(\mathcal{F})$.

Classifying countable homogeneous m-generic L-graphs

=

Classifying families $\mathcal{F}$ of finite L-graphs for which $Forb(\mathcal{F})$ is an amalgamation class

Non-monic realization

Theorem (Non-monic realization theorem)

If G is a homogeneous m-generic graph, then each member of $O(G)$ is monic.

Definitions

A multipartite graph is *monic* if it has at most one vertex in each part. A *triangle* is a 3-partite monic.

Non-monic realization

Theorem (Non-monic realization theorem)

If G is a homogeneous m-generic graph, then each member of $O(G)$ is monic.

Definitions

A multipartite graph is *monic* if it has at most one vertex in each part. A *triangle* is a 3-partite monic.

To prove this, the first (easy) step is to show:

Lemma

If G is a homogeneous m-generic graph and $A \in O(G)$, then each bipartite restriction of A is monochromatic.

Tripartite graphs

First consider 3-partite graphs.

Example

Recall the example: $|C_{01}| = |C_{02}| = |C_{12}| = 3$, $\mathcal{F} = \{$red triangle, blue triangle$\}$. $Forb(\mathcal{F})$ is an amalgamation class, so there is a homogeneous 3-generic graph G with $O(G) = \mathcal{F}$.

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Recall the example: $|C_{01}| = |C_{02}| = |C_{12}| = 3$, $\mathcal{F} = \{$red triangle, blue triangle$\}$. *Forb*$(\mathcal{F})$ is an amalgamation class, so there is a homogeneous 3-generic graph G with $O(G) = \mathcal{F}$.

Definitions

A multipartite graph A *covers* the colour $c \in C_{ij}$ on the restriction $V_i \times V_j$ if some E_{ij}-edge of A is c-coloured.
A set of multipartite graphs $\mathcal{A}$ *covers* C_{ij} if for each $c \in C_{ij}$ there is some $A \in \mathcal{A}$ which covers c on $V_i \times V_j$. Then we call $\mathcal{A}$ a C_{ij}*-cover set*.

3-generic graphs

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A 3-generic graph G is homogeneous if and only if $O(G)$ does not contain any cover sets.

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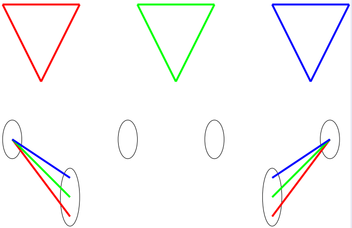
Proof.

($\Rightarrow$) Suppose we have a C_{01}-cover set $\mathcal{A}$. Since G is generic, we can realize two particular 2-partite graphs in G. Then by the homogeneity of G these can be realized as shown to form some amalgam. But each choice of colour for the new edge realizes some member of $\mathcal{A}$; a contadiction.



$\square$

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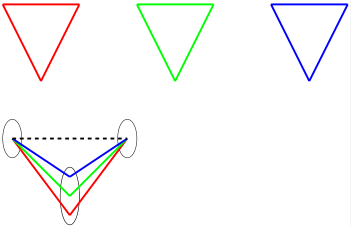

3-generic graphs

Theorem

A 3-generic graph G is homogeneous if and only if $O(G)$ does not contain any cover sets.

Proof.

($\Leftarrow$) Let $\mathcal{F}$ be a family of triangles which does not contain any cover sets. We verify that $Forb(\mathcal{F})$ is an amalgamation class. Only need two-point amalgamations: $B_1 = A \cup \{x\}$, $B_2 = A \cup \{y\}$. These amalgamations can always be done since there is always a free colour not covered by any member of $\mathcal{F}$, which we may assign to the edge xy. $\square$

3-generic graphs

Theorem

A 3-generic graph G is homogeneous if and only if $O(G)$ does not contain any cover sets.

Example

$|C_{01}| = |C_{02}| = |C_{12}| = 3$, $\mathcal{F}' := \{$triangles with no green edges$\}$.
If $\mathcal{F} \subseteq \mathcal{F}'$, then $Forb(\mathcal{F})$ is an amalgamation class.

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However we will see that if $m > 3$ and G is a homogeneous m-generic graph, then $O(G)$ may contain cover sets. But certain conditions must be satisfied ...

Omission sets

Definitions

A monic on parts $\bigcup_{i \in J} V_i$ is called a *J-monic*.

e.g. a triangle on V_0, V_1, V_2 is called a 012-*triangle*.

A C_{ij}^{kl}*-omission set* S_{ij} is a C_{ij}-cover set made up of ijk-triangles and ijl-triangles which agree on the colours of all edges other than the E_{ij}-edges (i.e. same colour $E_{ik}, E_{il}, E_{jk}, E_{jl}$-edges).
S_{ij} has *code* $(i, j, k, l; c_{ik}, c_{il}, c_{jk}, c_{jl})$.

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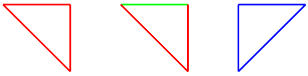
Example

Let $m = 4$, $|C_{ij}| = 3$ for each distinct $i, j \in \{0, 1, 2, 3\}$. Then the following is a C_{01}^{23}-omission set:



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S_{ij} has *code* $(i, j, k, l; c_{ik}, c_{il}, c_{jk}, c_{jl})$.

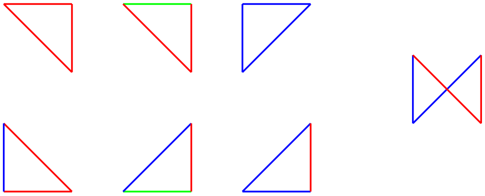
Note that no cover sets are defined on just 3 parts, so omission sets always have two types of triangles.

Corresponding omission sets

Definitions

If the C_{kl}^{ij}-omission set S_{kl} has the same code as S_{ij} (i.e. has the same colours on all edges other than the E_{kl}-edges), then we say that S_{ij} and S_{kl} are *corresponding* omission sets.

Example



Corresponding omission sets

Lemma

Let G be a homogeneous m-generic graph. If there is a C_{ij}^{kl}-omission set $\mathcal{S}$ in $O(G)$, then there is a corresponding C_{kl}^{ij}-omission set in $O(G)$.

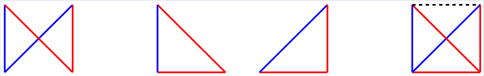
Corresponding omission sets

Lemma

Let G be a homogeneous m-generic graph. If there is a C_{ij}^{kl}-omission set $\mathcal{S}$ in $O(G)$, then there is a corresponding C_{kl}^{ij}-omission set in $O(G)$.

Proof.

If not, then for some colour $d \in C_{kl}$, the two triangles with d-coloured E_{kl}-edge which agree with the code of $\mathcal{S}$ are both realized in G.



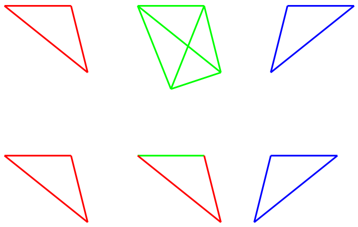
Then by the homogeneity of G, we can realize these sharing their d-coloured E_{kl}-edge. But then we have realized some member of $\mathcal{S}$; contradiction. □

'Based on' ordering

Definition

Let $\mathcal{A}$ be a C_{ij}-cover set. We say that $\mathcal{S}$ is a C_{ij}^{kl}-omission set *based on* $\mathcal{A}$ if there are $A, B \in \mathcal{A}$ with the colours on the E_{ik}, E_{jk}-edges of $\mathcal{S}$ agreeing with A, and the colours on the E_{il}, E_{jl}-edges of $\mathcal{S}$ agreeing with B.

Example



Non-complication

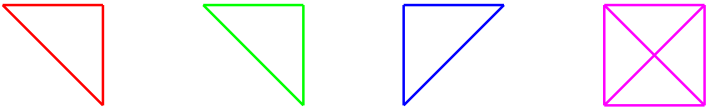
Omission sets are as complicated as things get.

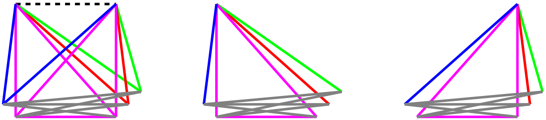
Theorem

Let G be a homogeneous m-generic graph. If there is a C_{ij}-cover set $\mathcal{A}$ in $O(G)$, then there is some C_{ij}^{kl}-omission set in $O(G)$ based on $\mathcal{A}$.

Non-complication

Proof outline for 4-partite case:

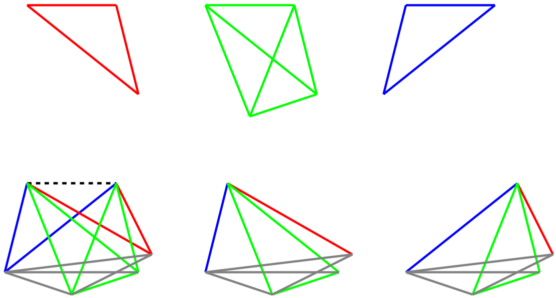




Assume for a contradiction that there are no omission sets based on the cover set. Then show that the above 3-partite graphs can be realised, giving the required contradiction.

Non-complication

Proof outline for m-partite case, for $m > 4$:



Assume for a contradiction that there are no omission sets based on the cover set. Now aim to show that the above $(m-1)$-partite graphs can be realised. This is possible, but much more complicated than in the 4-partite case!

m-generic graphs

Theorem

Let L be a language for coloured m-partite graphs, and let $\mathcal{F}$ be a family of monic L-graphs. Then there is a unique countable homogeneous m-generic L-graph G with $Age(G) = Forb(\mathcal{F})$ if and only if:

1. *If $\mathcal{A} \subset \mathcal{F}$ is a C_{ij}-cover set, then there is a C_{ij}^{kl}-omission set in $\mathcal{F}$ based on $\mathcal{A}$.*
2. *If there is a C_{ij}^{kl}-omission set in $\mathcal{F}$, then there is a corresponding C_{kl}^{ij}-omission set in $\mathcal{F}$.*

The theorem gives sufficient conditions for $\mathcal{F}$ to verify whether or not $Forb(\mathcal{F})$ is an amalgamation class.

Examples

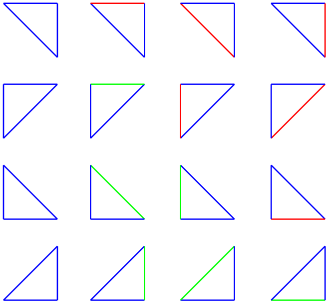
We can use the classification theorem to construct examples of families of monics $\mathcal{F}$ such that *Forb*$(\mathcal{F})$ is an amalgamation class.

The following examples will be 'maximal', i.e. there is no family $\mathcal{F}'$ with $\mathcal{F} \subset \mathcal{F}'$ such that *Forb*$(\mathcal{F}')$ is an amalgamation class.

For each of the examples, let L be a language for 4-partite graphs, and let $|C_{ij}| = 3$ for each distinct $i, j \in \{0, 1, 2, 3\}$.

Examples

Example ($\mathcal{F}_1$)

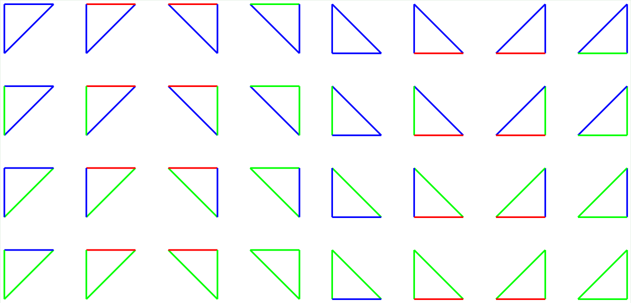


$Forb(\mathcal{F}_1)$ is an amalgamation class, giving homogeneous 4-generic L-graph G_1 with $O(G_1) = \mathcal{F}_1$.

$\mathcal{F}_1$ contains C_{ij}-cover sets for each distinct i, j.
$\mathcal{F}_1$ contains three pairs of corresponding omission sets.

Examples

Example ($\mathcal{F}_2$)



$Forb(\mathcal{F}_2)$ is an amalgamation class, giving homogeneous 4-generic L-graph G_2 with $O(G_2) = \mathcal{F}_2$.
$\mathcal{F}_2$ only contains C_{01}-cover sets and C_{23}-cover sets.
$\mathcal{F}_2$ contains 16 overlapping pairs of C_{01}^{23}-omission sets and corresponding C_{23}^{01}-omission sets.