Overgroups of the Automorphism Group of the Rado Graph

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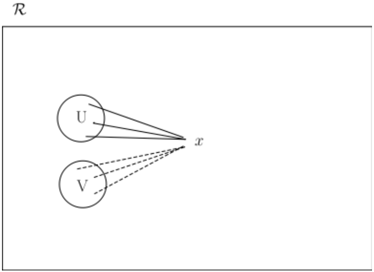
Rado Graph

The Rado graph $\mathcal{R}$ is the (unique) countable graph with the propery that:

For all finite disjoint $U, V \subseteq \mathcal{R}$, there is a vertex x connected to all vertices of U and none of V.



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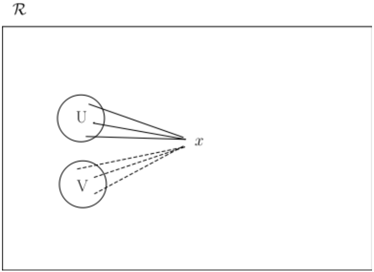

Definition

Let $W_{\mathcal{R}}(U, V)$ be the collection of all these witness x

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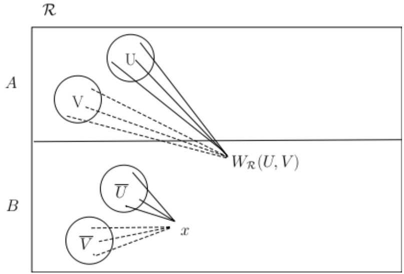
  Proof: If A is not Rado with bad pair U, V, then $W_{\mathcal{R}}(U, V) \subseteq B$.
  But $W_{\mathcal{R}}(U \cup \overline{U}, V \cup \overline{V}) = W_{\mathcal{R}}(U, V) \cap W_{\mathcal{R}}(\overline{U}, \overline{V})$.



$\square$

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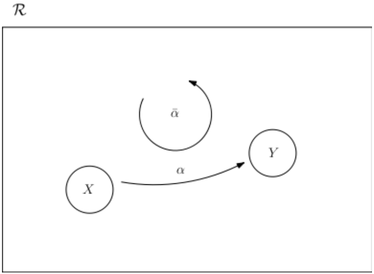
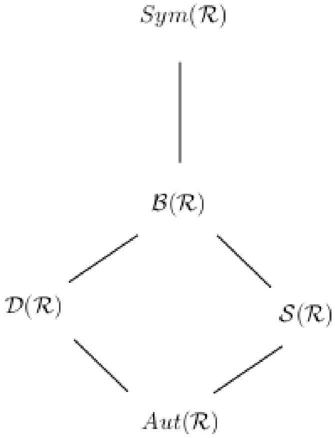


S. Thomas [91]

The closed subgroups of $Sym(\mathcal{R})$ containing $Aut(\mathcal{R})$ (the reducts) are:



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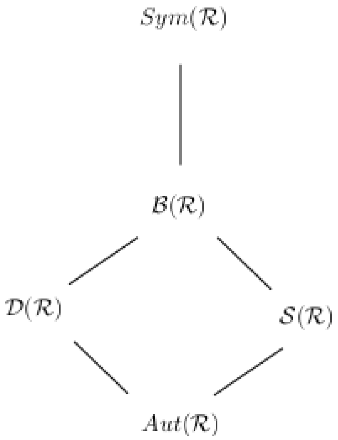

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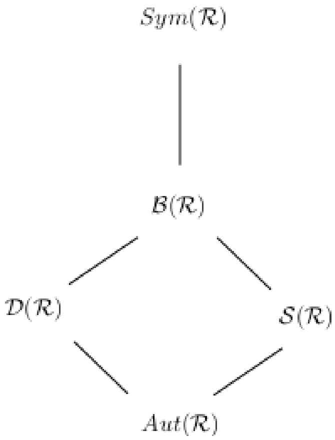

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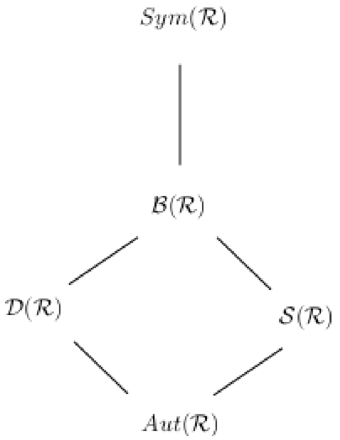

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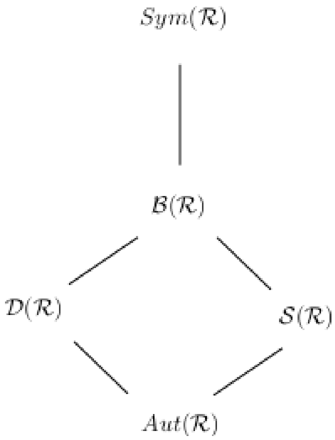

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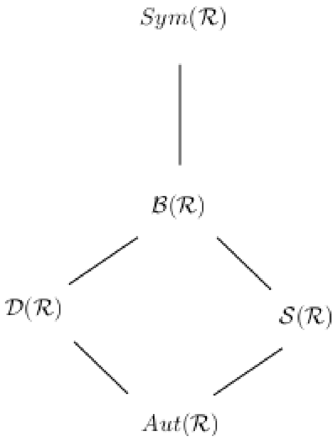

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Switching

For $X \subset \mathcal{R}$, consider the new graph $S(X)$ on the same vertex set as $\mathcal{R}$, but adjacencies between X and X^c are switched.
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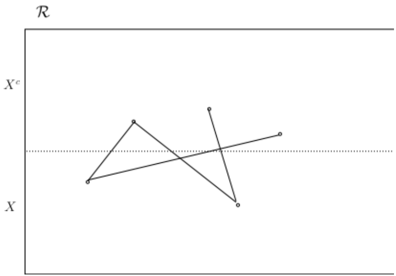

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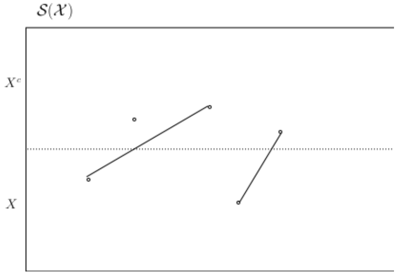

Transitivity of Reducts

- $Aut(\mathcal{R})$ is 1-transitive, not 2-transitive.
- $\mathcal{D}(\mathcal{R})$ is 2-transitive, not 3-transitive.
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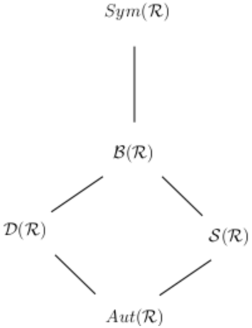
- $Sym(\mathcal{R})$ is highly transitive.



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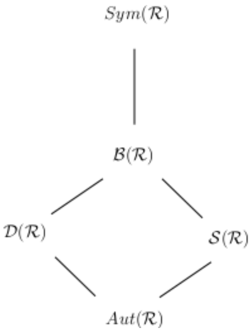

Cameron

Any overgroup of $Aut(\mathcal{R})$ not contained in $\mathcal{B}(\mathcal{R})$ is highly transitive.

Hypergraph of Copies and relatives

Let $\mathcal{H} = \mathcal{H}_{\mathcal{R}}$, the hypergraph of copies of $\mathcal{R}$:

- $Aut(\mathcal{H}_{\mathcal{R}}) =$
  $$\{\sigma \in Sym(\mathcal{R}) : \forall E \in \mathcal{H}\ E\sigma \text{ and } E\sigma^{-1} \in \mathcal{H}\}$$
- $\mathrm{FAut}(\mathcal{H}_{\mathcal{R}}) =$
  $$\{\sigma \in Sym(\mathcal{R}) :\ \exists F \text{ finite } \ \forall E \in \mathcal{H}\ (E \setminus F)\sigma \text{ and } (E \setminus F)\sigma^{-1} \in \mathcal{H}\}$$
- $Aut^*(\mathcal{H}_{\mathcal{R}}) =$
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CLPTW

$Aut(\mathcal{R}) < Aut(\mathcal{H}_{\mathcal{R}}) < \mathrm{FAut}(\mathcal{H}_{\mathcal{R}}) < Aut^*(\mathcal{H}_{\mathcal{R}}) < Sym(\mathcal{R})$.

Diversion (L-Pouzet-Sauer)

$Aut(\mathcal{H}_{\mathbb{Q}})$ 'is' $Aut(\mathbb{Q}, <)$

If $f : \mathbb{Q} \to \mathbb{Q}$ is a bijection preserving copies of the rationals (and conversely), then f is order preserving or reverse order preserving.

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$Aut(\mathcal{H}_{\mathcal{R}})$ is 'large'!

e.g. if X, Y are two thin subsets of $\mathcal{R}$, then any bijection $\alpha : X \to Y$ extends to an automorphism $\overline{\alpha}$ of $\mathcal{H}_{\mathcal{R}}$.

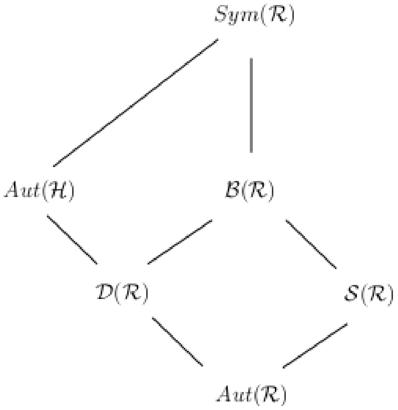
Reducts and $Aut(\mathcal{H})$

Observation

$\mathcal{D}(\mathcal{R}) \leq Aut(\mathcal{H}_{\mathcal{R}})$

Cameron

$\mathcal{S}(\mathcal{R}) \not\leq Aut(\mathcal{H}_{\mathcal{R}})$



CLPTW

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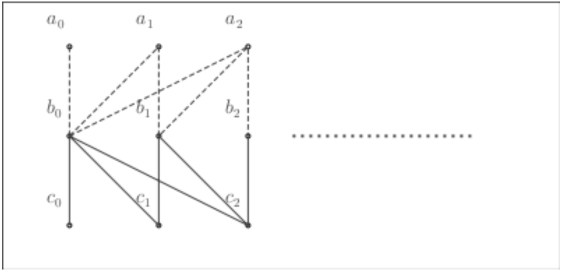

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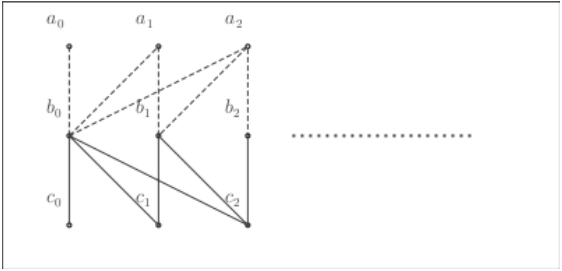

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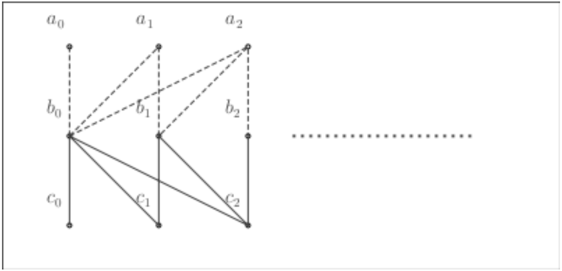

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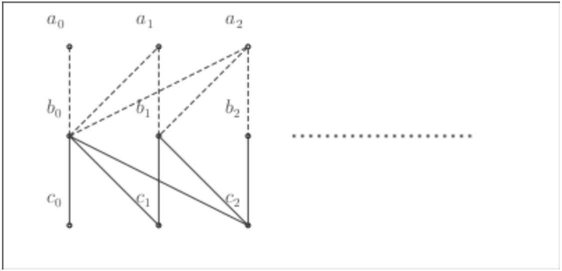

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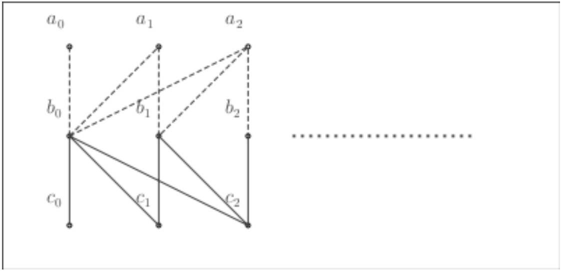

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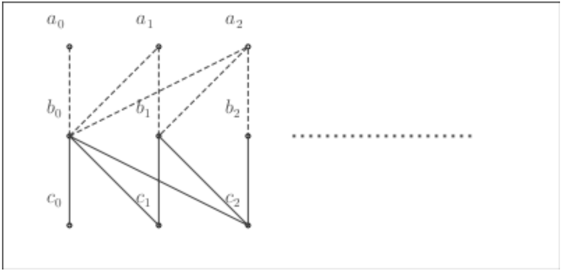

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- $S(C)$ is the Rado graph.
- In $S(C)$, b_n is isolated in E_n.
- For any finite set F, choose n large enough so that $E_n = E_n \setminus F$.

Then E_n is a copy in $\mathcal{R}$, but E_n is not a copy in $S(C)$.

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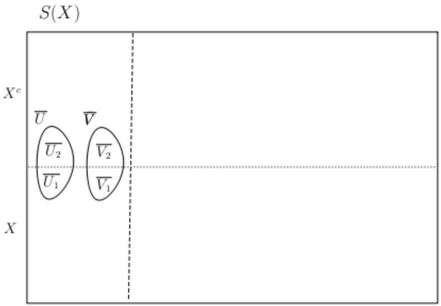

If $S(X) \notin \mathcal{H}$, then $W_\mathcal{R}(\overline{U_2} \cup \overline{V_1}, \overline{U_1} \cup \overline{V_2}) \subseteq X$

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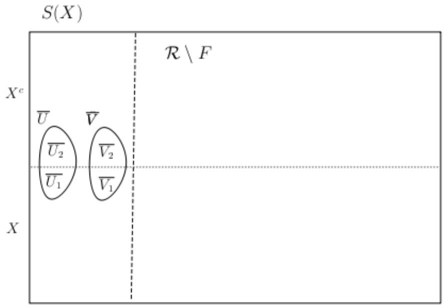

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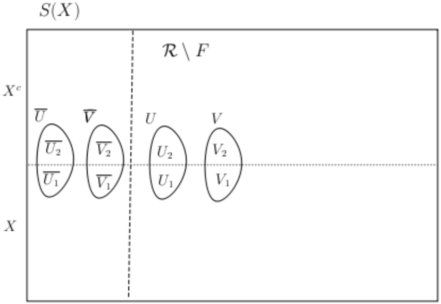

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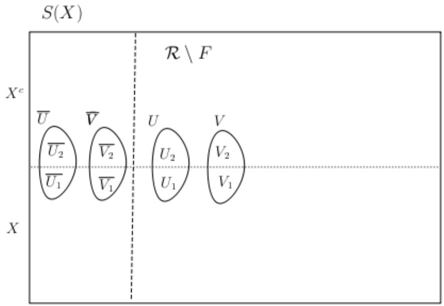

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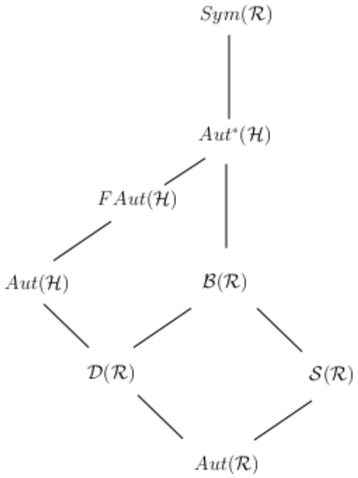
So $W_{S(X)}(U, V) \cap (\mathcal{R} \setminus F) \neq \emptyset$

CLPTW

$\mathcal{B}(\mathcal{R}) < Aut^*(\mathcal{H}_\mathcal{R})$



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CT

- $Aut(\mathcal{R}) < Aut_1(\mathcal{R}) < Aut_2(\mathcal{R}) < Aut_3(\mathcal{R})$
- $Aut_2(\mathcal{R}) \leq Aut(\mathcal{F}_\mathcal{R})$ but $Aut_3(\mathcal{R})$ and $Aut(\mathcal{F}_\mathcal{R})$ are incomparable.

Connections with $Aut(\mathcal{H})$

CLPTW

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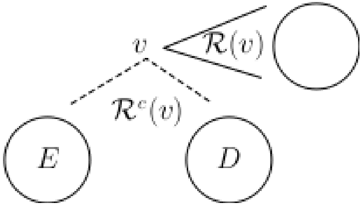

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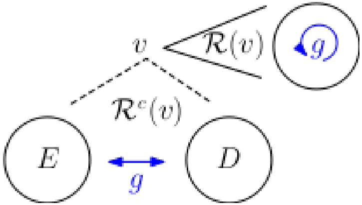
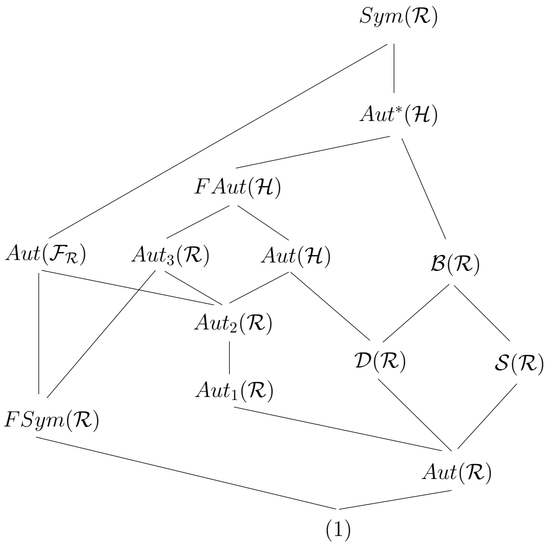


$$g \in Aut(\mathcal{F}_{\mathcal{R}}) \setminus Aut^*(\mathcal{H})$$

Overall picture



Question

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- *What insight does this bring us?*