

# Injectivity and retracts of Fraïssé limits

Wiesław Kubiś

Czech Academy of Sciences (CZECH REPUBLIC)

and

Jan Kochanowski University, Kielce (POLAND)

<http://www.math.cas.cz/~kubis/>

2nd Workshop on Homogeneous Structures

Praha, 27 July 2012

# Outline

- 1 The setup
  - Small and big objects
  - Mixed pushouts
  - Fraïssé limits
  - Injectivity
- 2 Main result
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- 4 Selected applications
  - Metric spaces
  - Banach spaces
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- 6 About the proof
  - Main Lemma
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We fix a pair of categories  $\mathcal{G} \subseteq \mathcal{B}$ .

The objects of  $\mathcal{G}$  will be called **small**, while the objects of  $\mathcal{B}$  will be called **big**.

We denote by  $\mathcal{G}^\varepsilon$  and  $\mathcal{B}^\varepsilon$  the same categories with restricted arrows, called **embeddings**.

We require that:

- 1 Every big object is the co-limit of a sequence of embeddings of small objects.
- 2 Every arrow between big objects is the co-limit of a sequence of arrows of small objects.

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## Definition

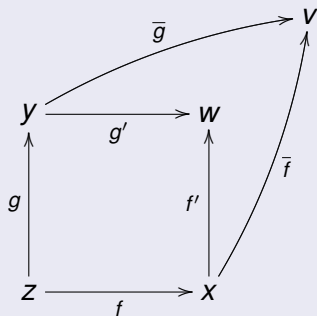
We say that  $\langle \mathfrak{G}^\varepsilon, \mathfrak{G} \rangle$  has the **mixed pushout property** if given a  $\mathfrak{G}^\varepsilon$ -arrow  $e: c \rightarrow a$ , and a  $\mathfrak{G}$ -arrow  $f: c \rightarrow b$ , there exist a  $\mathfrak{G}^\varepsilon$ -arrow  $e': b \rightarrow w$  and a  $\mathfrak{G}$ -arrow  $f': a \rightarrow w$  for which the diagram

$$\begin{array}{ccc}
 b & \xrightarrow{e'} & w \\
 \uparrow f & & \uparrow f' \\
 c & \xrightarrow{e} & a
 \end{array}$$

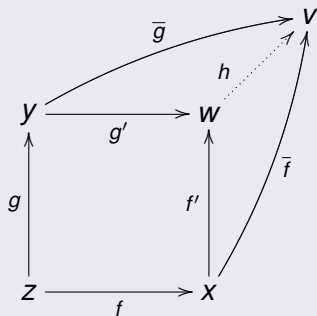
is the pushout in  $\mathfrak{G}$ .

The pushout of  $\langle f, g \rangle$ 

$$\begin{array}{ccc} y & \xrightarrow{g'} & w \\ \uparrow g & & \uparrow f' \\ z & \xrightarrow{f} & x \end{array}$$

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## Definition

A big object  $U$  is the **Fraïssé limit** of  $\mathcal{G}^\varepsilon$  if

- ① Every small object embeds into  $U$ .
- ② Given an embedding of small objects  $a \xrightarrow{e} b$ , given an embedding  $a \xrightarrow{i} U$ , there exists an embedding  $b \xrightarrow{j} U$  such that  $j \circ e = i$ .

## Fact

*The Fraïssé limit is unique, up to an isomorphism.*

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# Injectivity

## Main definition

A big object  $X$  is  **$\mathcal{G}$ -injective** if for every embedding of small objects  $a \hookrightarrow b$ , for every arrow  $f: a \rightarrow X$ , there exists an arrow  $g: b \rightarrow X$  such that  $f = g \circ e$ , that is, the diagram

$$\begin{array}{ccc} & X & \\ & \uparrow f & \\ a & \xrightarrow{e} & b \\ & \nwarrow g & \end{array}$$

is commutative.

## Theorem

### *Assume*

- (h1)  $\mathcal{G}^\varepsilon$  has a weakly initial object.
- (h2)  $\langle \mathcal{G}^\varepsilon, \mathcal{G} \rangle$  has the mixed pushout property.
- (h3)  $\mathcal{G}^\varepsilon$  has the Fraïssé limit  $U$  in  $\mathfrak{B}$ .

Let  $X$  be a  $\mathfrak{B}$ -object. The following properties are equivalent:

- (a)  $X$  is  $\mathcal{G}$ -injective.
- (b)  $X$  is a retract of  $U$ , that is, there exists an embedding  $e: X \rightarrow U$  and a homomorphism  $r: U \rightarrow X$  such that  $r \circ e = \text{id}_X$ .

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# Motivation

## Theorem (Dolinka 2011)

Let  $\mathfrak{M}$  be a *nice* Fraïssé class of finite models and let  $U$  be its Fraïssé limit. Given a countable model  $X$ , TFAE:

- (a)  $X$  is a retract of  $U$ .
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## Theorem

Let  $\langle X, d \rangle$  be a separable complete metric space. TFAE:

- 1  $\langle X, d \rangle$  is a non-expansive retract of the Urysohn space  $\mathbb{U}$ .
- 2  $\langle X, d \rangle$  is *finitely hyperconvex*, that is, given a finite family of closed balls

$$\mathcal{F} = \{\bar{B}(x_0, r_0), \dots, \bar{B}(x_{n-1}, r_{n-1})\}$$

with  $\bigcap \mathcal{F} = \emptyset$ , there exist  $i < j < n$  such that

$$d(x_i, x_j) > r_i + r_j.$$

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## Theorem (Wojtaszczyk 1972)

Let  $X$  be a separable Banach space. TFAE:

- 1  $X$  is linearly isometric to a 1-complemented subspace of the Gurarii space  $\mathbb{G}$ .
- 2  $X$  is almost 1-injective for finite-dimensional spaces.
- 3  $X$  is an isometric  $L^1$  predual.

## Definition (Cameron & Nešetřil 2006)

A countable relational structure  $X$  is **homomorphism homogeneous** if every homomorphism between its finite substructures extends to an endomorphism of  $X$ .

## Theorem

*Let  $\mathfrak{G} \subseteq \mathfrak{B}$  be a pair of categories of small – big objects, satisfying conditions (h1) – (h3) above. Let  $X$  be a big object. The following properties are equivalent:*

- (a)  *$X$  is homomorphism-homogeneous.*
- (b)  *$X$  is a retract of the Fraïssé limit of some subcategory of  $\mathfrak{G}^\varepsilon$  satisfying (h1) – (h3).*

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## Main Lemma

Assume  $\mathfrak{G}^\varepsilon \subseteq \mathfrak{G}$  satisfy conditions (h1) – (h3) above.

For every big object  $X$  there is an embedding  $J: X \rightarrow U$  satisfying the following condition:

- Given a  $\mathfrak{G}$ -injective object  $Y$ , given a  $\mathfrak{B}$ -arrow  $F: X \rightarrow Y$ , there exists a  $\mathfrak{B}$ -arrow  $G: U \rightarrow Y$  such that  $G \circ J = F$ .



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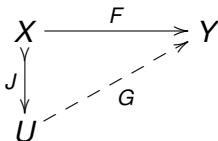


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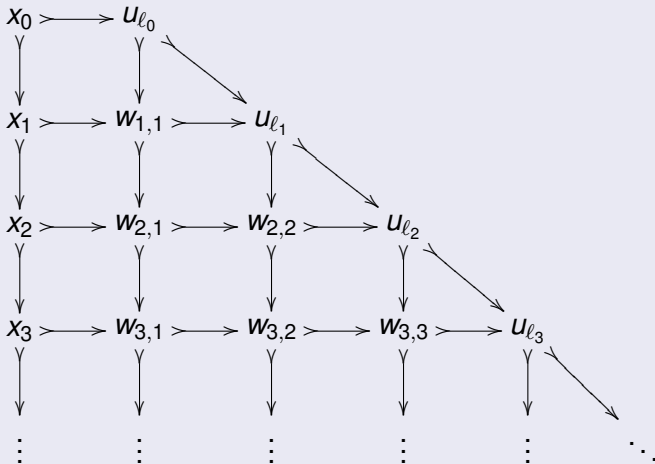
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## About the proof





THE END

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-  DOLINKA, I., *A characterization of retracts in certain Fraïssé limits*, MLQ. Mathematical Logic Quarterly, **58** (2012) 46–54
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