Injectivity and retracts of Fraïssé limits

Wiesław Kubiś

Czech Academy of Sciences (CZECH REPUBLIC)
and
Jan Kochanowski University, Kielce (POLAND)
http://www.math.cas.cz/~kubis/

2nd Workshop on Homogeneous Structures
Praha, 27 July 2012

Outline

We fix a pair of categories $\mathfrak{S} \subseteq \mathfrak{B}$.
The objects of $\mathfrak{S}$ will be called small, while the objects of $\mathfrak{B}$ will be called big.
We denote by $\mathfrak{S}^{\varepsilon}$ and $\mathfrak{B}^{\varepsilon}$ the same categories with restricted arrows, called embeddings.
We require that:

1. Every big object is the co-limit of a sequence of embeddings of small objects.
2. Every arrow between big objects is the co-limit of a sequence of arrows of small objects.

Small and big objects

We fix a pair of categories $\mathfrak{S} \subseteq \mathfrak{B}$.
The objects of $\mathfrak{S}$ will be called small, while the objects of $\mathfrak{B}$ will be called big.
We denote by $\mathfrak{S}^{\varepsilon}$ and $\mathfrak{B}^{\varepsilon}$ the same categories with restricted arrows, called embeddings.
We require that:

1. Every big object is the co-limit of a sequence of embeddings of small objects.
2. Every arrow between big objects is the co-limit of a sequence of arrows of small objects.

We fix a pair of categories $\mathfrak{S} \subseteq \mathfrak{B}$.
The objects of $\mathfrak{S}$ will be called small, while the objects of $\mathfrak{B}$ will be called big.
We denote by $\mathfrak{S}^\varepsilon$ and $\mathfrak{B}^\varepsilon$ the same categories with restricted arrows, called embeddings.
We require that:

1. Every big object is the co-limit of a sequence of embeddings of small objects.
2. Every arrow between big objects is the co-limit of a sequence of arrows of small objects.

Definition

We say that $\langle \mathfrak{S}^\varepsilon, \mathfrak{S} \rangle$ has the mixed pushout property if given a $\mathfrak{S}^\varepsilon$-arrow $e\colon c \to a$, and a $\mathfrak{S}$-arrow $f\colon c \to b$, there exist a $\mathfrak{S}^\varepsilon$-arrow $e'\colon b \to w$ and a $\mathfrak{S}$-arrow $f'\colon a \to w$ for which the diagram

$$
\begin{tikzcd}
b \arrow[r, hook, dashed, "e'"] & w \\
c \arrow[u, "f"] \arrow[r, hook, "e"'] & a \arrow[u, dashed, "f'"']
\end{tikzcd}
$$

is the pushout in $\mathfrak{S}$.

The pushout of $\langle f, g \rangle$

$$\begin{tikzcd} y \arrow[r, "g'"'] & w \\ z \arrow[u, "g"] \arrow[r, "f"'] & x \arrow[u, "f'"'] \end{tikzcd}$$

The pushout of $\langle f, g \rangle$

$$
\begin{tikzcd}
 & & v \\
y \arrow[r, "g'"'] \arrow[urr, bend left, "\bar{g}"] & w & \\
z \arrow[u, "g"] \arrow[r, "f"'] & x \arrow[u, "f'"] \arrow[uur, bend right, "\bar{f}"'] &
\end{tikzcd}
$$

The pushout of $\langle f, g\rangle$

$$
\begin{tikzcd}
 & & v \\
y \arrow[r, "g'"'] \arrow[urr, bend left, "\overline{g}"] & w \arrow[ur, dotted, "h"] & \\
z \arrow[u, "g"] \arrow[r, "f"'] & x \arrow[u, "f'"] \arrow[uur, bend right, "\overline{f}"'] &
\end{tikzcd}
$$

Definition

A big object U is the **Fraïssé limit** of $\mathfrak{S}^\varepsilon$ if

1. Every small object embeds into U.
2. Given an embedding of small objects $a \overset{e}{\hookrightarrow} b$, given an embedding $a \overset{i}{\hookrightarrow} U$, there exists an embedding $b \overset{j}{\hookrightarrow} U$ such that $j \circ e = i$.

Fact

The Fraïssé limit is unique, up to an isomorphism.

Definition

A big object U is the Fraïssé limit of $\mathfrak{S}^{\varepsilon}$ if

1. Every small object embeds into U.
2. Given an embedding of small objects $a \xhookrightarrow{e} b$, given an embedding $a \xhookrightarrow{i} U$, there exists an embedding $b \xhookrightarrow{j} U$ such that $j \circ e = i$.

Fact

The Fraïssé limit is unique, up to an isomorphism.

Definition

A big object U is the Fraïssé limit of $\mathfrak{S}^{\varepsilon}$ if

1. Every small object embeds into U.
2. Given an embedding of small objects $a \overset{e}{\hookrightarrow} b$, given an embedding $a \overset{i}{\hookrightarrow} U$, there exists an embedding $b \overset{j}{\hookrightarrow} U$ such that $j \circ e = i$.

Fact

The Fraïssé limit is unique, up to an isomorphism.

Definition

A big object U is the Fraïssé limit of $\mathfrak{S}^{\varepsilon}$ if

1. Every small object embeds into U.
2. Given an embedding of small objects $a \xhookrightarrow{e} b$, given an embedding $a \xhookrightarrow{i} U$, there exists an embedding $b \xhookrightarrow{j} U$ such that $j \circ e = i$.

Fact

The Fraïssé limit is unique, up to an isomorphism.

Injectivity

Main definition

A big object X is $\mathfrak{S}$-injective if for every embedding of small objects $a \xhookrightarrow{e} b$, for every arrow $f\colon a \to X$, there exists an arrow $g\colon b \to X$ such that $f = g \circ e$, that is, the diagram

$$\begin{tikzcd} X & \\ a \arrow[u, "f"] \arrow[r, hook, "e"'] & b \arrow[ul, dashed, "g"'] \end{tikzcd}$$

is commutative.

Theorem

Assume

(h1) $\mathfrak{S}^{\varepsilon}$ *has a weakly initial object.*

(h2) $\langle \mathfrak{S}^{\varepsilon}, \mathfrak{S} \rangle$ *has the mixed pushout property.*

(h3) $\mathfrak{S}^{\varepsilon}$ *has the Fraïssé limit* U *in* $\mathfrak{B}$.

Let X *be a* $\mathfrak{B}$*-object. The following properties are equivalent:*

(a) X *is* $\mathfrak{S}$*-injective.*

(b) X *is a retract of* U*, that is, there exists an embedding* $e \colon X \to U$ *and a homomorphism* $r \colon U \to X$ *such that* $r \circ e = \mathrm{id}_X$.

Theorem

Assume

(h1) $\mathfrak{S}^\varepsilon$ *has a weakly initial object.*

(h2) $\langle \mathfrak{S}^\varepsilon, \mathfrak{S} \rangle$ *has the mixed pushout property.*

(h3) $\mathfrak{S}^\varepsilon$ *has the Fraïssé limit* U *in* $\mathfrak{B}$*.*

Let X *be a* $\mathfrak{B}$*-object. The following properties are equivalent:*

(a) X *is* $\mathfrak{S}$*-injective.*

(b) X *is a retract of* U*, that is, there exists an embedding* $e\colon X \to U$ *and a homomorphism* $r\colon U \to X$ *such that* $r \circ e = \mathrm{id}_X$*.*

Theorem

Assume

(h1) $\mathfrak{S}^\varepsilon$ *has a weakly initial object.*

(h2) $\langle \mathfrak{S}^\varepsilon, \mathfrak{S} \rangle$ *has the mixed pushout property.*

(h3) $\mathfrak{S}^\varepsilon$ *has the Fraïssé limit* U *in* $\mathfrak{B}$*.*

Let X *be a* $\mathfrak{B}$*-object. The following properties are equivalent:*

(a) X *is* $\mathfrak{S}$*-injective.*

(b) X *is a retract of* U*, that is, there exists an embedding* $e \colon X \to U$ *and a homomorphism* $r \colon U \to X$ *such that* $r \circ e = \mathrm{id}_X$*.*

Theorem

Assume

(h1) $\mathfrak{S}^{\varepsilon}$ *has a weakly initial object.*

(h2) $\langle \mathfrak{S}^{\varepsilon}, \mathfrak{S} \rangle$ *has the mixed pushout property.*

(h3) $\mathfrak{S}^{\varepsilon}$ *has the Fraïssé limit U in $\mathfrak{B}$.*

Let X be a $\mathfrak{B}$-object. The following properties are equivalent:

(a) *X is $\mathfrak{S}$-injective.*

(b) *X is a retract of U, that is, there exists an embedding $e \colon X \to U$ and a homomorphism $r \colon U \to X$ such that $r \circ e = \mathrm{id}_X$.*

Theorem

Assume

(h1) $\mathfrak{S}^\varepsilon$ *has a weakly initial object.*

(h2) $\langle \mathfrak{S}^\varepsilon, \mathfrak{S} \rangle$ *has the mixed pushout property.*

(h3) $\mathfrak{S}^\varepsilon$ *has the Fraïssé limit* U *in* $\mathfrak{B}$.

Let X *be a* $\mathfrak{B}$*-object. The following properties are equivalent:*

(a) X *is* $\mathfrak{S}$*-injective.*

(b) X *is a retract of* U*, that is, there exists an embedding* $e\colon X \to U$ *and a homomorphism* $r\colon U \to X$ *such that* $r \circ e = \mathrm{id}_X$.

Motivation

Theorem (Dolinka 2011)

Let $\mathfrak{M}$ be a nice Fraïssé class of finite models and let U be its Fraïssé limit. Given a countable model X, TFAE:

(a) *X is a retract of U.*

(b) *X is algebraically closed.*

Motivation

Theorem (Dolinka 2011)

Let $\mathfrak{M}$ be a nice Fraïssé class of finite models and let U be its Fraïssé limit. Given a countable model X, TFAE:

(a) *X is a retract of U.*

(b) *X is algebraically closed.*

Motivation

Theorem (Dolinka 2011)

Let $\mathfrak{M}$ be a nice Fraïssé class of finite models and let U be its Fraïssé limit. Given a countable model X, TFAE:

(a) *X is a retract of U.*

(b) *X is algebraically closed.*

Theorem

Let $\langle X, d\rangle$ be a separable complete metric space. TFAE:

1. *$\langle X, d\rangle$ is a non-expansive retract of the Urysohn space $\mathbb{U}$.*
2. *$\langle X, d\rangle$ is finitely hyperconvex, that is, given a finite family of closed balls*

$$\mathcal{F} = \{\overline{\mathrm{B}}(x_0, r_0), \ldots, \overline{\mathrm{B}}(x_{n-1}, r_{n-1})\}$$

with $\bigcap \mathcal{F} = \emptyset$, there exist $i < j < n$ such that

$$d(x_i, x_j) > r_i + r_j.$$

Theorem

Let $\langle X, d\rangle$ be a separable complete metric space. TFAE:

1. $\langle X, d\rangle$ *is a non-expansive retract of the Urysohn space* $\mathbb{U}$.
2. $\langle X, d\rangle$ *is finitely hyperconvex, that is, given a finite family of closed balls*

$$\mathcal{F} = \{\overline{\mathrm{B}}(x_0, r_0), \ldots, \overline{\mathrm{B}}(x_{n-1}, r_{n-1})\}$$

with $\bigcap \mathcal{F} = \emptyset$, *there exist* $i < j < n$ *such that*

$$d(x_i, x_j) > r_i + r_j.$$

Theorem (Wojtaszczyk 1972)

Let X be a separable Banach space. TFAE:

1. *X is linearly isometric to a 1-complemented subspace of the Gurariĭ space $\mathbb{G}$.*
2. *X is almost 1-injective for finite-dimensional spaces.*
3. *X is an isometric L^1 predual.*

Definition (Cameron & Nešetřil 2006)

A countable relational structure X is homomorphism homogeneous if every homomorphism between its finite substructures extends to an endomorphism of X.

Theorem

Let $\mathfrak{S} \subseteq \mathfrak{B}$ be a pair of categories of small – big objects, satisfying conditions (h1) – (h3) above. Let X be a big object. The following properties are equivalent:

(a) *X is homomorphism-homogeneous.*

(b) *X is a retract of the Fraïssé limit of some subcategory of $\mathfrak{S}^{\varepsilon}$ satisfying (h1) – (h3).*

Definition (Cameron & Nešetřil 2006)

A countable relational structure X is homomorphism homogeneous if every homomorphism between its finite substructures extends to an endomorphism of X.

Theorem

Let $\mathfrak{S} \subseteq \mathfrak{B}$ be a pair of categories of small – big objects, satisfying conditions (h1) – (h3) above. Let X be a big object. The following properties are equivalent:

(a) *X is homomorphism-homogeneous.*

(b) *X is a retract of the Fraïssé limit of some subcategory of $\mathfrak{S}^{\varepsilon}$ satisfying (h1) – (h3).*

Definition (Cameron & Nešetřil 2006)

A countable relational structure X is homomorphism homogeneous if every homomorphism between its finite substructures extends to an endomorphism of X.

Theorem

Let $\mathfrak{S} \subseteq \mathfrak{B}$ be a pair of categories of small – big objects, satisfying conditions (h1) – (h3) above. Let X be a big object. The following properties are equivalent:

(a) *X is homomorphism-homogeneous.*

(b) *X is a retract of the Fraïssé limit of some subcategory of $\mathfrak{S}^{\varepsilon}$ satisfying (h1) – (h3).*

Main Lemma

Main Lemma

Assume $\mathfrak{S}^{\varepsilon} \subseteq \mathfrak{S}$ satisfy conditions (h1) – (h3) above.
For every big obect X there is an embedding $J\colon X \to U$ satisfying the following condition:

- Given a $\mathfrak{S}$-injective object Y, given a $\mathfrak{B}$-arrow $F\colon X \to Y$, there exists a $\mathfrak{B}$-arrow $G\colon U \to Y$ such that $G \circ J = F$.

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, tail, "J"'] & Y \\ U \arrow[ur, dashed, "G"'] & \end{tikzcd}$$

Main Lemma

Assume $\mathfrak{S}^{\varepsilon} \subseteq \mathfrak{S}$ satisfy conditions (h1) – (h3) above.
For every big obect X there is an embedding $J \colon X \to U$ satisfying the following condition:

- Given a $\mathfrak{S}$-injective object Y, given a $\mathfrak{B}$-arrow $F \colon X \to Y$, there exists a $\mathfrak{B}$-arrow $G \colon U \to Y$ such that $G \circ J = F$.

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, tail, "J"'] & Y \\ U \arrow[ur, dashed, "G"'] & \end{tikzcd}$$

Main Lemma

Assume $\mathfrak{S}^{\varepsilon} \subseteq \mathfrak{S}$ satisfy conditions (h1) – (h3) above.
For every big obect X there is an embedding $J\colon X \to U$ satisfying the following condition:

- Given a $\mathfrak{S}$-injective object Y, given a $\mathfrak{B}$-arrow $F\colon X \to Y$, there exists a $\mathfrak{B}$-arrow $G\colon U \to Y$ such that $G \circ J = F$.

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[d, tail, "J"'] & Y \\
U \arrow[ur, dashed, "G"'] &
\end{tikzcd}$$

About the proof

$$
\begin{tikzcd}
X_0 \arrow[r, tail] \arrow[d, tail] & U_{\ell_0} \arrow[d, tail] \arrow[dr, tail] & & & & \\
X_1 \arrow[r, tail] \arrow[d, tail] & W_{1,1} \arrow[r, tail] \arrow[d, tail] & U_{\ell_1} \arrow[d, tail] \arrow[dr, tail] & & & \\
X_2 \arrow[r, tail] \arrow[d, tail] & W_{2,1} \arrow[r, tail] \arrow[d, tail] & W_{2,2} \arrow[r, tail] \arrow[d, tail] & U_{\ell_2} \arrow[d, tail] \arrow[dr, tail] & & \\
X_3 \arrow[r, tail] \arrow[d, tail] & W_{3,1} \arrow[r, tail] \arrow[d, tail] & W_{3,2} \arrow[r, tail] \arrow[d, tail] & W_{3,3} \arrow[r, tail] \arrow[d, tail] & U_{\ell_3} \arrow[d, tail] \arrow[dr, tail] & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}
$$

THE END

Selected bibliography

- DOLINKA, I., *A characterization of retracts in certain Fraïssé limits*, MLQ. Mathematical Logic Quarterly, **58** (2012) 46–54
- KUBIŚ, W., *Fraïssé sequences: category-theoretic approach to universal homogeneous structures*, preprint, http://arxiv.org/abs/0711.1683
- KUBIŚ, W., *Injective objects and retracts of Fraïssé limits*, preprint, http://arxiv.org/abs/1107.4620