Polish G-spaces similar to logic G-spaces of continuous structures

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Logic S_∞-space

Let $L = (R_i^{n_i})_{i\in I}$ be a countable relational language and

$$X_L = \Pi_{i\in I} 2^{\omega^{n_i}}$$

be the corresponding topol. space under the product topology τ.

We view X_L as the space of all L-structures on ω identifying every $\mathbf{x} = (...x_i...) \in X_L$ with the structure $(\omega, R_i)_{i\in I}$ where R_i is the n_i-ary relatn defined by the charactrstc functn $x_i : \omega^{n_i} \to 2$.

The **logic action** of the group S_∞ of all permutations of ω is defined on X_L by the rule:

$$g \circ \mathbf{x} = \mathbf{y} \Leftrightarrow \forall i \forall \bar{s} (y_i(\bar{s}) = x_i(g^{-1}(\bar{s}))).$$

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Canonical partition

Let $(\langle X, \tau\rangle, G)$ be a Polish G-space with a countable basis $\{C_j\}$.

H.Becker: there exists a unique partition of X,
$X = \bigcup\{Y_t : t \in T\}$
into invariant G_δ-sets Y_t s. t. every orbit from Y_t is dense in Y_t.

To construct it take $\{C_j\}$ and for any $t \in 2^{\mathbb{N}}$ define

$$Y_t = (\bigcap\{GC_j : t(j) = 1\}) \cap (\bigcap\{X \setminus GC_j : t(j) = 0\})$$

and take $T = \{t \in 2^{\mathbb{N}} : Y_t \neq \emptyset\}$.

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Vaught transforms

Let X be a Polish G-space, $B \subset X$ and $u \subset G$ is open.

Vaught transforms:

$$B^{\star u} = \{x \in X : \{g \in u : gx \in B\} \text{ is comeagre in } u\}$$

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In the case of the **logic action** of S_∞ on the space X_L if

$$B = \{M \in X_L : M \models \phi(s)\} \text{ with } s \in \omega$$

then

$$B^{*S_\infty} = \{M \in X_L : M \models \forall x \phi(x)\}.$$

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- If $B \in \mathbf{\Sigma}_\alpha$, then $B^{\Delta H} \in \mathbf{\Sigma}_\alpha$ and if $B \in \mathbf{\Pi}_\alpha$, then $B^{*H} \in \mathbf{\Pi}_\alpha$.
- For any open $B \subseteq X$ and any open $K < G$ we have $B^{\Delta K} = KB$, where $KB = \{gx : g \in K, x \in B\}$.

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Actions of closed subgroups of S_∞

Let G be a closed subgroup of S_∞.
Let $\mathcal{N}^G$ be the standard basis of the topology of G consisting of cosets of pointwise stabilisers of finite subsets of ω.

Let $(\langle X, \tau \rangle, G)$ be a Polish G-space with a countable basis $\mathcal{A}$.
Along with τ we shall consider another topology on X.

Nice topology:

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Nice topology

Definition (H.Becker) A topology $\mathbf{t}$ on X is **nice** for the G-space $(\langle X, \tau\rangle, G)$ if the following conditions are satisfied.
(A) $\mathbf{t}$ is a Polish topology, $\mathbf{t}$ is finer than τ and the G-action remains continuous with respect to $\mathbf{t}$.
(B) There exists a basis $\mathcal{B}$ for $\mathbf{t}$ (called **nice**) such that:

1. $\mathcal{B}$ is countable;
2. for all $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2 \in \mathcal{B}$;
3. for all $B \in \mathcal{B}$, $X \setminus B \in \mathcal{B}$;
4. for all $B \in \mathcal{B}$ and $u \in \mathcal{N}^G$, $B^{\Delta u}, B^{\star u} \in \mathcal{B}$;
5. for any $B \in \mathcal{B}$ there exists an open subgroup $H < G$ such that B is invariant under the corresponding H-action.

Example

Logic action

For any countable fragment F of $L_{\omega_1\omega}$, which is closed under quantifiers, all sets

$$Mod(\phi, \bar{s}) = \{M \in X_L : M \models \phi(\bar{s})\} \text{ with } \bar{s} \subset \omega$$

form a nice basis defining a nice topology (denoted by $\mathbf{t}_F$) of the S_∞-space X_L.

Each piece of the canonical partition corresponding to $\mathbf{t}_F$ consists of structures which satisfy the same F-sentences (without parameters).

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Example and illustration

Let G be a closed subgroup of S_∞ and (X, τ) be a Polish G-space. Let $\mathbf{t}$ be a nice topology for $(\langle X, \tau \rangle, G)$.

A generalized version of Lindström's model completeness theorem:

Theorem (B.M-I)

For any $x_1 \in X$ if $X_1 = Gx_1$ is a G_δ-subset of X, then both topologies τ and $\mathbf{t}$ coincide on X_1.

H.Becker: **J.Amer.Math.Soc**, 11(1998), 397 - 449 and **APAL**, 111(2001), 145 - 184

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Existence

Theorem

(H.Becker) Let G be a closed subgroup of S_∞ and (X,τ) be a Polish G-space.
Let $\mathbf{t}'$ be a topology on X finer than τ, such that the action remains continuous with respect to $\mathbf{t}'$.
Then there is a nice topology $\mathbf{t}$ for $(\langle X,\tau\rangle, G)$ such that $\mathbf{t}$ is finer than $\mathbf{t}'$.

Remark: All elements of $\mathbf{t}$ are τ-Borel.

Question

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Is it possible to extend the generalised model theory of H.Becker to actions of Polish groups (without the assumption $G \leq S_\infty$) ?

Continuous structures

A countable continuous signature:

$$L = \{d, R_1, ..., R_k, ..., F_1, ..., F_l, ...\}.$$

Definition

A **metric L-structure** is a complete metric space (M, d) with d bounded by 1, along with a family of uniformly continuous operations F_k on M and a family of predicates R_l, i.e. uniformly continuous maps from appropriate M^{k_l} to $[0, 1]$.

It is usually assumed that to a predicate symbol R_i a continuity modulus γ_i is assigned so that when $d(x_j, x_j') < \gamma_i(\varepsilon)$ with $1 \leq j \leq k_i$ the corresponding predicate of M satisfies

$$|R_i(x_1, ..., x_j, ..., x_{k_i}) - R_i(x_1, ..., x_j', ..., x_{k_i})| < \varepsilon.$$

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Canonical structure

Let (G, d) be a Polish group with a left invariant metric ≤ 1.
If $(\mathbf{X}, d)$ is its completion, then $G \leq Iso(\mathbf{X})$.
Let S be a countable dense subset of $\mathbf{X}$.
Enumerate all orbits of G of finite tuples of S.

For the closure of such an n-orbit C define a predicate $R_{\overline{C}}$ on $(\mathbf{X}, d)$ by

$$R_{\overline{C}}(y_1, ..., y_n) = d((y_1, ..., y_n), \overline{C}) \text{ (i.e. } inf\{d(\bar{y}, \bar{c}) : \bar{c} \in \overline{C}\}).$$

It is observed by J.Melleray that G is the automorphism group of the continuous structure M of all these predicates on $\mathbf{X}$, with continuous moduli $= id$.

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The space of continuous structures

Fix a relational continuous signature L and a Polish space $(\mathbf{Y}, d)$. Let S be a dense countable subset of $\mathbf{Y}$.

Define the spce $\mathbf{Y}_L$ of continuous L-strctres on $(\mathbf{Y}, d)$ as follows. **Metric on the set of L-structures**: Enumerate all tuples of the form $(j, \bar{s})$, where $\bar{s}$ is a tple $\in S$ of the lngth of the arity of R_j. For L-structures M and N on $\mathbf{Y}$ let

$$\delta(M, N) = \sum_{i=1}^{\infty} \{2^{-i} |R_j^M(\bar{s}) - R_j^N(\bar{s})| : i \text{ is the number of } (j, \bar{s})\}.$$

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- the space $\mathbf{Y}_L$ is Polish;
- the Polish group $Iso(\mathbf{Y})$ acts on $\mathbf{Y}_L$ continuously

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Universality

Theorem

For any Polish group G there is a Polish space $(\mathbf{Y}, d)$ and a continuous relational signature L such that

- $G < Iso(\mathbf{Y})$
- *for any Polish G-space $\mathbf{X}$ there is a Borel 1-1-map $\mathcal{M} : \mathbf{X} \to \mathbf{Y}_L$ such that for any $x, x' \in \mathbf{X}$ structures $\mathcal{M}(x)$ and $\mathcal{M}(x')$ are isomorphic if and only if x and x' are in the same G-orbit,*

The map $\mathcal{M}$ is a Borel G-invariant 1-1-reduction of the G-orbit equivalence relation on $\mathbf{X}$ to the $Iso(\mathbf{Y})$-orbit equivalence relation on the space $\mathbf{Y}_L$ of all L-stuctures.

Grades subsets and subgroups

A **graded subset** of $\mathbf{X}$, denoted $\phi \sqsubseteq \mathbf{X}$, is a function $\mathbf{X} \to [0,1]$.

It is **open (closed)**, $\phi \in \mathbf{\Sigma}_1$ (resp. $\phi \in \mathbf{\Pi}_1$), if it is upper (lower) semi-continuous, i.e. the set $\phi_{<r}$ (resp. $\phi_{>r}$) is open for all $r \in [0,1]$ (here $\phi_{<r} = \{z \in \mathbf{X} : \phi(z) < r\}$).

When G is a Polish group, then a graded subset $H \sqsubseteq G$ is called a **graded subgroup** if $H(1) = 0$, $\forall g \in G(H(g) = H(g^{-1}))$ and $\forall g, g' \in G(H(gg') \leq H(g) + H(g'))$.

We also define Borel classes $\mathbf{\Sigma}_\alpha$, $\mathbf{\Pi}_\alpha$ so that ϕ is $\mathbf{\Sigma}_\alpha$ if $\phi = inf\, \Phi$ for some countable $\Phi \subset \bigcup\{\mathbf{\Pi}_\gamma : \gamma < \alpha\}$ and $\mathbf{\Pi}_\alpha = \{1 - \phi : \phi \in \mathbf{\Sigma}_\alpha\}$.

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Illustration 1. Graded subsets of $\mathbf{Y}_L$

For $\bar{c}$ from $(\mathbf{Y}, d)$ and a linear δ with $\delta(0) = 0$
graded subgroup $H_{\delta,\bar{c}} \sqsubseteq Iso(\mathbf{Y})$:

$H_{\delta,\bar{c}}(g) = \delta(max(d(c_1, g(c_1)), ..., d(c_n, g(c_n))))$, where $g \in Iso(\mathbf{Y})$.

A **continuous formula** is an expression built from 0,1 and atomic formulas by applications of the following functions:

$$x/2 \; , \; x \dot{-} y = max(x - y, 0) \; , \; min(x, y) \; , \; max(x, y) \; , \; |x - y| \; ,$$

$$\neg(x) = 1 - x \; , \; x \dot{+} y = min(x + y, 1) \; , \; sup_x \text{ and } inf_x .$$

Any continuous sentence $\phi(\bar{c})$ defines a graded subset of $\mathbf{Y}_L$ which belongs to $\mathbf{\Sigma}_n$ for some n:

$$\phi(\bar{c}) \text{ takes } M \text{ to the value } \phi^M(\bar{c}).$$

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$$\phi(\bar{c}) \text{ takes } M \text{ to the value } \phi^M(\bar{c}).$$

Illustration 2. Invariant graded subsets

Assuming that continuity moduli of L-symbols are **id**
for any $\phi(\bar{x})$ as above we find a linear function δ such that the graded subgroup

$$H_{\delta,\bar{c}}(g) = \delta(max(d(c_1, g(c_1)), ..., d(c_n, g(c_n)))), \text{ where } g \in Iso(\mathbf{Y}).$$

and the graded subset $\phi(\bar{c}) \sqsubseteq \mathbf{Y}_L$ satisfy

$$\phi^{g(M)}(\bar{c}) \leq \phi^M(\bar{c}) \dot{+} H_{\delta,\bar{c}}(g).$$

Definition

Let $\mathbf{X}$ be a continuous G-space. A graded subset $\phi \sqsubseteq \mathbf{X}$ is called invariant with respect to a graded subgroup $H \sqsubseteq G$ if for any $g \in G$ we have $\phi(g(x)) \leq \phi(x) \dot{+} H(g)$.

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Vaught transforms 3

For any non-empty open $J \sqsubseteq G$ let

$$\phi^{\Delta J}(x) = inf\{r \dot{+} s : \{h : \phi(h(x)) < r\} \text{ is not meagre in } J_{<s}\}.$$

$$\phi^{*J}(x) = sup\{r \dot{-} s : \{h : \phi(h(x)) \leq r\} \text{ is not comeagre in } J_{<s}\},$$

Theorem

- $\phi^{*J}(x) = 1 - (1-\phi)^{\Delta J}(x)$ *for all* $x \in \mathbf{X}$.
- $\phi^{\Delta J}(x) \leq \phi^{*J}(x)$ *for all* $x \in \mathbf{X}$.
- *If* ϕ *is a graded* $\mathbf{\Sigma}_\alpha$*-subset, then* $\phi^{\Delta J}$ *is also* $\mathbf{\Sigma}_\alpha$.
  If ϕ *is a graded* $\mathbf{\Pi}_\alpha$*-subset, then* $\phi^{*J}(x)$ *is also* $\mathbf{\Pi}_\alpha$.
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Invariantness of Vaught transforms

Theorem

If H is a graded subgroup of G, then both $\phi^{*H}(x)$ and $\phi^{\Delta H}(x)$ are H-invariant:

$$\phi^{*H}(x) \dot{-} H(h) \le \phi^{*H}(h(x)) \le \phi^{*H}(x) \dot{+} H(h) \text{ and}$$

$$\phi^{\Delta H}(x) \dot{-} H(h) \le \phi^{\Delta H}(h(x)) \le \phi^{\Delta H}(x) \dot{+} H(h).$$

Moreover if $\phi(x) \le \phi(h(x)) \dot{+} H(h)$ for all x and h, then

$$\phi^{*H}(x) = \phi(x) = \phi^{\Delta H}(x).$$

Graded bases

We consider G together with a distinguished countable family of open graded subsets $\mathcal{R}$ so that all $\rho_{<r}$ for $\rho \in \mathcal{R}$ and $r \in \mathbb{Q}$, form a basis of the topology of G.

We usually assume that $\mathcal{R}$ consists of **graded cosets**, i.e. for such $\rho \in \mathcal{R}$ there is a graded subgroup $H \in \mathcal{R}$ and an element $g_0 \in G$ so that for any $g \in G$, $\rho(g) = H(gg_0^{-1})$.
(For every Polish group G there is a countable family of open graded subsets $\mathcal{R}$ as above.)

Considering a $(G, \mathcal{R})$-space $\mathbf{X}$ we distinguish a similar family too: a cntble family $\mathcal{U}$ of open graded sbsts of $\mathbf{X}$ generating the topol.

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Nice basis

Definition. A family $\mathcal{B}$ of Borel graded subsets of the G-space $\mathbf{X}$ is a **nice basis** w.r.to $\mathcal{R}$ if:

- $\mathcal{B}$ is countable and generates the topol. finer than τ;
- for all $\phi_1, \phi_2 \in \mathcal{B}$, the functions $min(\phi_1, \phi_2)$, $max(\phi_1, \phi_2)$, $|\phi_1 - \phi_2|$, $\phi_1 \dot{-} \phi_2$ $\phi_1 \dot{+} \phi_2$ belong to $\mathcal{B}$;
- for all $\phi \in \mathcal{B}$ and rational $r \in [0, 1]$, $r\phi$ and $1 - \phi \in \mathcal{B}$;
- for all $\phi \in \mathcal{B}$ and $\rho \in \mathcal{R}$, $\phi^{*\rho}, \phi^{\Delta\rho} \in \mathcal{B}$;
- for any $\phi \in \mathcal{B}$ there exists an open graded subgroup $H \in \mathcal{R}$ such that ϕ is invariant under the corresponding H-action.

A topology $\mathbf{t}$ on $\mathbf{X}$ is $\mathcal{R}$-**nice** for the G-space $\langle \mathbf{X}, \tau \rangle$ if:
(a) $\mathbf{t}$ is a Polish topology, $\mathbf{t}$ is finer than τ and the G-action remains continuous with respect to $\mathbf{t}$;
(b) there exists a nice basis $\mathcal{B}$ so that $\mathbf{t}$ is generated by all $\phi_{<q}$ with $\phi \in \mathcal{B}$ and $q \in \mathbb{Q} \cap (0, 1]$.

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Example. The case of $\mathbf{U}_L$

Let $\mathbf{U}$ be **Urysohn** spce of diameter 1: This is the unique Polish mtrc space which is universal and ultrahomogeneous, i.e. every isometry between fnte substs of $\mathbf{U}$ extends to an isometry of $\mathbf{U}$.

There is a ctble family $\mathcal{R}$ consisting of cosets of clopen graded subgroups of $Iso(\mathbf{U})$ of the form

$$H_{\mathbf{s}} : g \to d(g(\mathbf{s}), \mathbf{s}), \text{ where } \mathbf{s} \subset S \text{ (ctble,dense) },$$

which generates the topology of $Iso(\mathbf{U})$.

Let L be a continuous signature of continuity moduli id. Then the family of all continuous L-sentences

$$\phi(\mathbf{s}) : M \to \phi^M(\mathbf{s}), \text{ where } \bar{s} \in S,$$

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Existence 2

Theorem. Let $(G, \mathcal{R})$ be a Polish group with $\mathcal{R}$ satisfying
(i) for every graded subgroup $H \in \mathcal{R}$ if $gH \in \mathcal{R}$, then $H^g \in \mathcal{R}$;
(ii) $\mathcal{R}$ is closed under **max** and multiplying by rationals.
Let $\langle \mathbf{X}, \tau \rangle$ be a G-space and $\mathcal{U}$ be a countable family of Borel graded subsets of $\mathbf{X}$ generating a topology finer than τ, so that each $\phi \in \mathcal{U}$ is invariant with respect to some graded subgroup $H \in \mathcal{R}$.
Then there is an $\mathcal{R}$-nice topology for $(\langle \mathbf{X}, \tau \rangle, G)$ so that $\mathcal{U}$ consists of open graded subsets.

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Lindström, abstract form

G is a Polish group with a graded basis $\mathcal{R}$ consisting of graded cosets,
$\langle \mathbf{X}, \tau \rangle$ is a Polish G-space, ect.

Theorem

Let $\mathbf{t}$ be $\mathcal{R}$-nice.
Let $X = Gx_0$ for some (any) $x_0 \in X$ and X be a G_δ-subset of $\mathbf{X}$.
Then both topologies τ and $\mathbf{t}$ are equal on X.

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Categoricity

Let $\mathcal{B}$ be a nice basis defining $\mathcal{R}$-nice $\mathbf{t}$,
H be an open graded subgroup from $\mathcal{R}$,
X be an invariant G_δ-subset of $\mathbf{X}$ with respect to $\mathbf{t}$.
(1) A family $\mathcal{F}$ of subsets of the form $\phi_{<r}$ with H-invariant $\phi \in \mathcal{B}$ is called an H-**type** in X, if it is maximal w.r. to the condition $X \cap \bigcap \mathcal{F} \neq \emptyset$.
(2) An H-type $\mathcal{F}$ is called **principal** if there is an H-invariant graded basic set $\phi \in \mathcal{B}$ and there is r such that $\phi_{<r} \in \mathcal{F}$ and $\bigcap \{\overline{B} : B \in \mathcal{F}\} \cap X$ coincides with the closure of $\phi_{<r} \cap X$. Then we say that $\phi_{<r}$ **defines** $\mathcal{F}$.

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Ryll-Nardzewski, abstract form

Let $\mathcal{R}$ consist of clopen graded cosets.
Let $\mathcal{B}$ be an $\mathcal{R}$-nice basis of a G-space $\langle \mathbf{X}, \tau \rangle$ and
$\mathbf{t}$ be the corresponding nice topology,

Theorem

Assume that the action satisfies the **approximation property** *for graded subgroups.*
A piece X of the canonical partition with respect to the topology $\mathbf{t}$ *is a G-orbit if and only if for any basic open graded subgroup $H \sqsubset G$ any H-type of X is principal.*

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Approximation property for graded subgroups

Definition

The $(G, \mathcal{R})$-space $(\mathbf{X}, \mathcal{U})$ has the **approximation property for graded subgroups** if for any $\varepsilon > 0$
for any grded subgrp $H \in \mathcal{R}$, any c and $c' \in \mathbf{X}$ of the same G-orbt

if c, c' belong to the same subsets of the form $\phi_{\leq t}$ for H-invarnt $\phi \in \mathcal{U}$, then
c' can be approximated by the values $g(c)$ with $H(g) < \varepsilon$.

When $G = Aut(M)$, where M is an approximately ultrahomogeneous separably categorical structure on $\mathbf{Y}$, then this holds in the space of all L-expansions of M.

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Ultrahomogenity

A relational continuous structure M is **approximately ultrahomogeneous** if for any n-tuples $(a_1, .., a_n)$ and $(b_1, ..., b_n)$ with the same quantifier-free type (i.e. with the same values of predicates for corresponding subtuples) and any $\varepsilon > 0$ there exists $g \in Aut(M)$ such that

$$max\{d(g(a_j), b_j) : 1 \leq j \leq n\} \leq \varepsilon.$$

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Complexity

Let $(\mathbf{Y}, d)$ be a Polish space.

Theorem

- *There is a Borel subset $\mathcal{SC} \subset \mathbf{Y}_L$ of separably categorical continuous L-structures on $(\mathbf{Y}, d)$ so that any separably categorical continuous structure from $\mathbf{Y}_L$ is isomorphic to a structure from $\mathcal{SC}$.*
- *There is a Borel subset $\mathcal{SCU} \subset \mathbf{Y}_L$ of separably categorical approximately ultahomogeneous continuous structures on $\mathbf{Y}$ so that any sep.cat., appr. ultrhom. structure from $\mathbf{Y}_L$ is isomorphic to a structure from $\mathcal{SCU}$.*

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CLI metrics

Observation.
Let M be a Polish approximately ultrahomogeneous continuous str. Then $Aut(M)$ admits a compatible complete left-invariant metric if and only if there is no proper embeding of M into itself.

The subset of $\mathbf{Y}_L$ consisting of structures M so that $Aut(M)$ admits compatible complete left-invariant metric, is coanalityc in any Borel subset of appr. ultr. structures of $\mathbf{Y}_L$.
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