

Universal structures and universal homomorphisms

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(joint work with Maja Pech)

Some Motivation

In

[Rad64] R. Rado. [Universal graphs and universal functions.](#)
Acta Arith., 9:331–340, 1964.

Richard Rado used universal functions to explain his well-known construction of a universal countable graph:

$$K_{kl} := \left\{ f \mid f : \binom{\mathbb{N}}{l+1} \rightarrow \{0, \dots, k-1\} \right\}$$

Definition

$f^* \in K_{kl}$ is universal in K_{kl} if for every $f \in K_{kl}$ there exists a self-embedding φ of \mathbb{N} such that

$$f(x_0, \dots, x_l) = f^*(\varphi(x_0), \dots, \varphi(x_l))$$

$K_{2,1}$ is essentially the class of countable graphs. Hence, a universal function $f^* \in K_{2,1}$ is a countable universal graph.

Outline

Universal homomorphisms

Universal polymorphisms

Cofinality of Menger algebras and clones

Retracts

Fraïssé-limits in comma categories

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We propose to study universal homomorphisms

Definition

Let \mathcal{K} be a class of structures, $\mathbf{T} \in \mathcal{K}$. $u : \mathbf{U} \rightarrow \mathbf{T}$ is called universal within \mathcal{K} if $\mathbf{U} \in \mathcal{K}$, and

$$\forall \mathbf{A} \in \mathcal{K}, h : \mathbf{A} \rightarrow \mathbf{T} \quad \exists \iota : \mathbf{A} \hookrightarrow \mathbf{U} : u \circ \iota = h.$$

Note

u is a retraction: Consider $1_{\mathbf{T}} : \mathbf{T} \rightarrow \mathbf{T}$; by universality, there exists $\iota : \mathbf{T} \hookrightarrow \mathbf{U}$ such that $u \circ \iota = 1_{\mathbf{T}}$.

More general definition

A homomorphism $u : \mathbf{U}^n \rightarrow \mathbf{T}$ is called n -ary universal homomorphism to \mathbf{T} within \mathcal{K} if $\mathbf{U} \in \mathcal{K}$, and

$$\forall \mathbf{A} \in \mathcal{K}, h : \mathbf{A}^n \rightarrow \mathbf{T} \quad \exists \iota : \mathbf{A} \hookrightarrow \mathbf{U} : u \circ \iota^n = h.$$

Strict Fraïssé-classes

If \mathcal{K} is an age, then $\overline{\mathcal{K}} := \{\mathbf{A} \mid \mathbf{A} \text{ countable, } \text{Age}(\mathbf{A}) \subseteq \mathcal{K}\}$.

Definition (Dolinka)

A Fraïssé-class \mathcal{K} of relational structures is called **strict Fraïssé-class** if every pair of morphisms in $(\mathcal{K}, \hookrightarrow)$ with the same domain has a pushout in $(\overline{\mathcal{K}}, \rightarrow)$.

Observation

Note that these pushouts will always be amalgams. Thus the strict amalgamation property postulates canonical amalgams.

Examples for strict Fraïssé-classes

- ▶ free amalgamation classes,
- ▶ the class of finite partial orders.

Homogeneous homomorphisms

Definition

Let $u : \mathbf{U}^n \rightarrow \mathbf{T}$ be an n -ary homomorphism. Let $\mathbf{A} \leq \mathbf{U}$ and let $\iota : \mathbf{A} \hookrightarrow \mathbf{U}$. We say that ι preserves u if the following diagram commutes:

$$\begin{array}{ccc} \mathbf{U}^n & \xrightarrow{u} & \mathbf{T} \\ \uparrow \subseteq & & \uparrow u \\ \mathbf{A}^n & \xleftarrow{\iota^n} & \mathbf{U}^n \end{array}$$

Definition

Let $u : \mathbf{U}^n \rightarrow \mathbf{T}$ be an n -ary homomorphism. u is called homogeneous if for all finitely generated substructures \mathbf{A} of \mathbf{U} , every u -preserving embedding $\iota : \mathbf{A} \rightarrow \mathbf{U}$ can be extended to a u -preserving automorphism of \mathbf{U} .

Existence of universal homogeneous homomorphisms

Theorem

Let \mathcal{K} be a strict Fraïssé-class, $\mathbf{T} \in \overline{\mathcal{K}}$. Then there exists a universal homogeneous n -ary homomorphism $u : \mathbf{U}^n \rightarrow \mathbf{T}$ within $\overline{\mathcal{K}}$. Moreover, if $\hat{u} : \hat{\mathbf{U}}^n \rightarrow \mathbf{T}$ is another such homomorphism, then there exists an isomorphism $h : \hat{\mathbf{U}} \rightarrow \mathbf{U}$ such that

$$\begin{array}{ccc} \mathbf{U}^n & \xrightarrow{u} & \mathbf{T} \\ \uparrow h^n \cong & \nearrow \hat{u} & \\ \hat{\mathbf{U}}^n & & \end{array}$$

commutes.

Let \mathbf{K} be the Fraïssé-limit of \mathcal{K} .

If all structures from \mathcal{K} are finite, $\text{Aut}(\mathbf{K})$ is oligomorphic, and if $\text{Aut}(\mathbf{T})$ is oligomorphic, then $\text{Aut}(\mathbf{U})$ is oligomorphic, too.

Corollary

For every $\mathbf{T} \in \overline{\mathcal{K}}$, the homomorphism-equivalence class $\overline{\mathcal{K}}_{\mathbf{T}}$ has a universal element.

Countable universal well-founded posets

Let α be a countable ordinal number.

Let \mathcal{C} be the class of all countable well-founded strict posets of height $\leq \alpha$.

Question

Does \mathcal{C} have a universal object?

Note

\mathcal{C} is not elementary.

Countable universal well-founded posets

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Answer to the question

Yes, \mathcal{C} has a universal element \mathbf{U} .

- ▶ Note that $\overline{\mathcal{C}}$ consists of all countable strict posets that have a homomorphism to (α, \in) .
- ▶ Take \mathcal{K} as the class of all finite posets. Set $\mathbf{T} := (\alpha, \in)$. Then there exists a universal homogeneous homomorphism $u : \mathbf{U} \rightarrow \mathbf{T}$.
- ▶ Observe that \mathbf{U} is universal in \mathcal{C} .

A countable universal directed acyclic graph

- ▶ \mathcal{K} be the class of all finite structures with one binary relation,
- ▶ $\mathbf{T} := (\mathbb{Q}, <)$,
- ▶ $u : \mathbf{U} \rightarrow \mathbf{T}$ be a universal homogeneous homomorphism within $\overline{\mathcal{K}}$.

Then \mathbf{U} is a countable universal directed acyclic graph.

Note

- ▶ The signature is finite. Hence the Fraïssé-limit of \mathcal{K} is \aleph_0 -categorical.
- ▶ $(\mathbb{Q}, <)$ is homogeneous over a finite signature. Hence it is \aleph_0 -categorical.

Hence \mathbf{U} is \aleph_0 -categorical, too.

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Universal polymorphisms

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Universal homogeneous polymorphisms

Let R denote the countable random graph.

Let $u : \mathbf{U}^n \rightarrow R$ be an n -ary universal homogeneous homomorphism within the class of all countable simple graphs.

Question

What is \mathbf{U} ? (clearly, $\text{Age}(\mathbf{U}) = \text{Age}(R)$)

Universal homogeneous polymorphisms

Let R denote the countable random graph.

Let $u : \mathbf{U}^n \rightarrow R$ be an n -ary universal homogeneous homomorphism within the class of all countable simple graphs.

Question

What is \mathbf{U} ? (clearly, $\text{Age}(\mathbf{U}) = \text{Age}(R)$)

Answer

$\mathbf{U} \cong R$. That is, we can assume w.l.o.g., that $\mathbf{U} = R$.
Hence u is an n -ary polymorphism of R .

In other words:

The countable random graph has universal homogeneous polymorphisms of every arity.

Questions:

1. Which structures have universal homogeneous endomorphisms?
2. Which structures have universal homogeneous polymorphisms?

Existence of universal homogeneous polymorphisms

Homo amalgamation property (HAP)

\mathcal{K} has the (HAP) if for all $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$, for all homomorphisms $f_1 : \mathbf{A} \rightarrow \mathbf{B}_1$, $f_2 : \mathbf{A} \hookrightarrow \mathbf{B}_2$ there exist $\mathbf{C} \in \mathcal{K}$, $g_1 : \mathbf{B}_1 \hookrightarrow \mathbf{C}$, and $g_2 : \mathbf{B}_2 \rightarrow \mathbf{C}$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{f_1} & \mathbf{B}_1 \\ \uparrow & & \uparrow \\ f_2 \downarrow & & g_1 \downarrow \\ \mathbf{B}_2 & \xrightarrow{g_2} & \mathbf{C} \end{array}$$

Theorem

Let \mathcal{K} be a strict Fraïssé-class with the HAP. Let \mathbf{U} be a Fraïssé-limit of \mathcal{K} . If \mathcal{K} is closed with respect to n -th powers, then \mathbf{U} has an n -ary universal homogeneous polymorphism.

Some examples

The following structures have universal homogeneous polymorphisms of every arity:

- ▶ the countable random graph R (here \mathcal{K} is the class of all finite simple graphs),
- ▶ the countable generic poset $\mathbb{P} = (P, \leq)$ (here \mathcal{K} is the class of all finite posets),
- ▶ the countable atomless Boolean algebra \mathbb{A} (here \mathcal{K} is the class of finite Boolean algebras),
- ▶ the countable universal homogeneous semilattice Ω (here \mathcal{K} is the class of all finite semilattices),
- ▶ the countable universal homogeneous distributive lattice \mathbb{D} (here \mathcal{K} is the class of all finite distributive lattices).

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Clones and Menger algebras

Given a set A .

$$O_A^{(n)} := \{f \mid f : A^n \rightarrow A\}, \quad O_A := \bigcup_{n \in \mathbb{N} \setminus \{0\}} O_A^{(n)}$$

$$f \in O_A^{(n)}, g_1, \dots, g_n \in O_A^{(m)} : f \circ \langle g_1, \dots, g_n \rangle \in O_A^{(m)}$$
$$f \circ \langle g_1, \dots, g_n \rangle : \bar{x} \mapsto f(g_1(\bar{x}), \dots, g_n(\bar{x})).$$

$$e_i^n \in O_A^{(n)} : (x_1, \dots, x_n) \mapsto x_i \quad (\text{projections})$$

Definition

A **clone** on A is a subset of O_A that contains all projections and is closed with respect to composition.

Definition

An n -ary **pre-Menger algebra** on A is a subset of $O_A^{(n)}$ that is closed with respect to composition.

Definition

An n -ary pre-Menger algebra is called **Menger algebra** if it contains all e_i^n

Clones and Menger algebras of Polymorphisms

Given a structure \mathbf{A} (with carrier A).

$$\text{Pol}^{(n)}(\mathbf{A}) := \{f \in O_A^{(n)} \mid f : \mathbf{A}^n \rightarrow \mathbf{A}\}$$

$$\text{Pol}(\mathbf{A}) = \bigcup_{n \in \mathbb{N} \setminus \{0\}} \text{Pol}^{(n)}(\mathbf{A})$$

Note:

1. $\text{Pol}^{(n)}(\mathbf{A})$ is an n -ary Menger algebra,
2. $\text{Pol}(\mathbf{A})$ is a clone.

Cofinality of clones and Menger algebras

- ▶ The notions **subclone**, and **pre-Menger subalgebra** are defined in the obvious way.
- ▶ Let C be a clone, M be an n -ary pre-Menger algebra

Definition

C is said to have **uncountable cofinality** if it can not be written as the union of a countable chain of proper subclones.

Definition

M is said to have uncountable cofinality if it can not be written as the union of a countable chain of proper pre-Menger subalgebras.

Uncountable cofinality for clones

- ▶ For $M \subseteq O_A$ the smallest clone on A containing M is denoted by $\langle M \rangle_{O_A}$,
- ▶ For a clone C on A , we define $C^{(n)} := C \cap O_A^{(n)}$

Proposition

A clone C has uncountable cofinality if and only if there exists some $k \in \mathbb{N} \setminus \{0\}$ such that

1. $C = \langle C^{(k)} \rangle_{O_A}$,
2. $C^{(k)}$, considered as a pre-Menger algebra, has uncountable cofinality

Uncountable cofinality for Menger algebras

Proposition

Let \mathbf{A} be a structure such that $\text{End}(\mathbf{A})$ has uncountable cofinality. If \mathbf{A} has a universal n -ary polymorphism, then $\text{Pol}^{(n)}(\mathbf{A})$ has uncountable cofinality, too.

Remark

- ▶ In the proposition above, uncountable cofinality can be replaced by **strong uncountable cofinality**,
- ▶ strong uncountable cofinality is equivalent to uncountable cofinality + **Bergman property**.

Here a pre-Menger algebra M has the **Bergman property** if for every generating set T there exists a k_T such that every element of M can be obtained by a term over T of depth $\leq k$.

Examples

The following Menger algebras have uncountable cofinality and the Bergman property:

- ▶ $\text{Pol}^{(n)}(R)$, where R is the countable random graph,
- ▶ $\text{Pol}^{(n)}(\mathbb{P})$ where $\mathbb{P} = (P, \leq)$ is the countable generic poset,
- ▶ $\text{Pol}^{(n)}(\mathbb{A})$, where \mathbb{A} is the countable atomless Boolean algebra,
- ▶ $\text{Pol}^{(n)}(\Omega)$, where Ω is the countable universal homogeneous semilattice,
- ▶ $\text{Pol}^{(n)}(\mathbb{D})$, where \mathbb{D} is the countable universal homogeneous distributive lattice,
- ▶ $O_A^{(n)}$.

Corollary

The clone O_A has uncountable cofinality (since it is generated by $O_A^{(2)}$).

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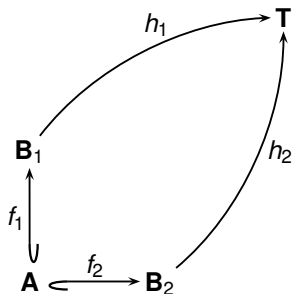
Universal homogeneous retractions

Definition

A retraction $r : \mathbf{U} \rightarrow \mathbf{T}$ is called **universal homogeneous retraction** if it is a universal homogeneous homomorphism to \mathbf{T} within $\overline{\text{Age}(\mathbf{U})}$

Theorem

Let \mathcal{C} be a Fraïssé-class with Fraïssé-limit \mathbf{U} , and let $\mathbf{T} \in \overline{\mathcal{C}}$. Then there exists a universal homogeneous retraction $r : \mathbf{U} \rightarrow \mathbf{T}$ if and only if



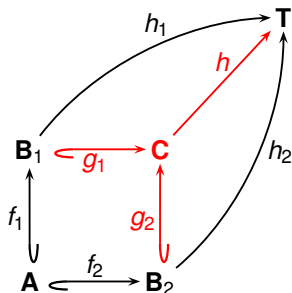
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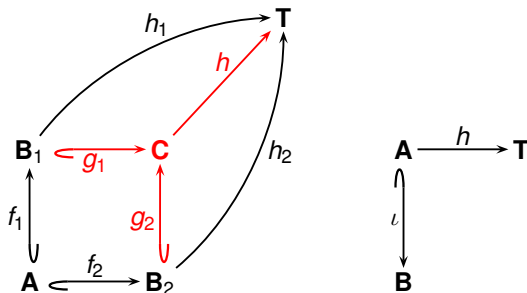
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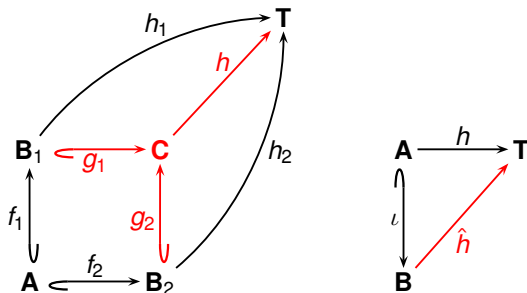
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Theorem

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Subretracts of universal homogeneous retracts

Proposition

Let \mathcal{C} be a Fraïssé-class with Fraïssé-limit \mathbf{U} and let $\mathbf{V}, \mathbf{W} \in \overline{\mathcal{C}}$. Let $r : \mathbf{U} \twoheadrightarrow \mathbf{V}$ be a universal homogeneous retraction. Let $s : \mathbf{V} \twoheadrightarrow \mathbf{W}$ be any retraction. Then there is a universal homogeneous retraction $\hat{s} : \mathbf{U} \twoheadrightarrow \mathbf{W}$.

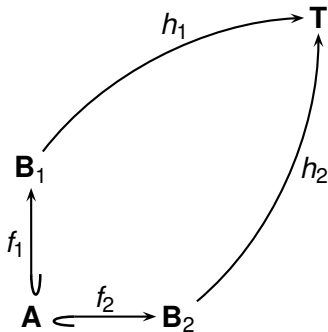
Corollary

If \mathbf{U} has a universal homogeneous endomorphism, then every retract of \mathbf{U} is induced by a universal homogeneous retraction.

Universal homogeneous endomorphisms revisited

Kubiś's amalgamated extension property

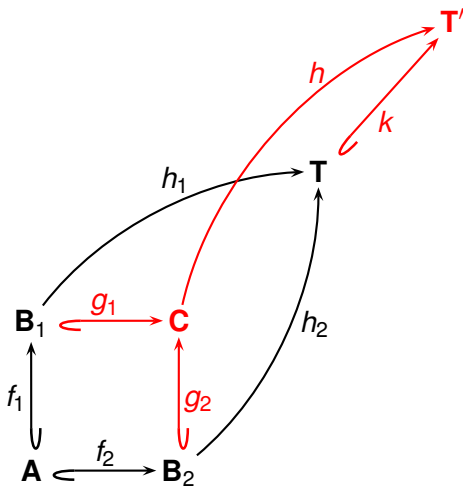
Let \mathcal{C} be a class of countable, finitely generated structures. We say that \mathcal{C} has the **amalgamated extension property** if



Universal homogeneous endomorphisms revisited

Kubiś's amalgamated extension property

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Universal homogeneous endomorphisms revisited II

Proposition

Let \mathcal{C} be a Fraïssé-class. Let \mathbf{U} be its Fraïssé-limit. Then \mathbf{U} has a universal homogeneous endomorphism if and only if

- 1. \mathcal{C} has the amalgamated extension property, and*
- 2. \mathcal{C} has the homo amalgamation property.*

Proposition

Let \mathbf{U} be a countable structure that has a universal homogeneous endomorphism. Then \mathbf{U} is homogeneous if and only if \mathbf{U} is homomorphism homogeneous.

Corollary

Let \mathcal{C} be a Fraïssé-class with Fraïssé-limit \mathbf{U} . Then \mathcal{C} has the HAP and the amalgamated extension property if and only if every retract of \mathbf{U} is induced by a universal homogeneous retraction.

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Universal homogeneous objects in categories

Definition

We call a category \mathcal{C} a λ -amalgamation category if

1. all morphisms of \mathcal{C} are monomorphisms,
2. \mathcal{C} is λ -algebroidal,
3. $\mathcal{C}_{<\lambda}$ has the joint embedding property,
4. $\mathcal{C}_{<\lambda}$ has the amalgamation property.

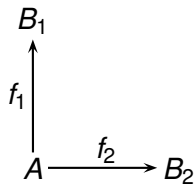
Theorem (Droste, Göbel '92)

Let λ be a regular cardinal, and let \mathcal{C} be a λ -algebroidal category in which all morphisms are monomorphisms. Then there exists a \mathcal{C} -universal, $\mathcal{C}_{<\lambda}$ -homogeneous object in \mathcal{C} if and only if \mathcal{C} is a λ -amalgamation category. Moreover, any two \mathcal{C} -universal, $\mathcal{C}_{<\lambda}$ -homogeneous objects in \mathcal{C} are isomorphic.

(F, G) -amalgamation property

Given categories \mathfrak{A} , \mathfrak{B} , \mathfrak{C} , functors $F : \mathfrak{A} \rightarrow \mathfrak{C}$, $G : \mathfrak{B} \rightarrow \mathfrak{C}$.

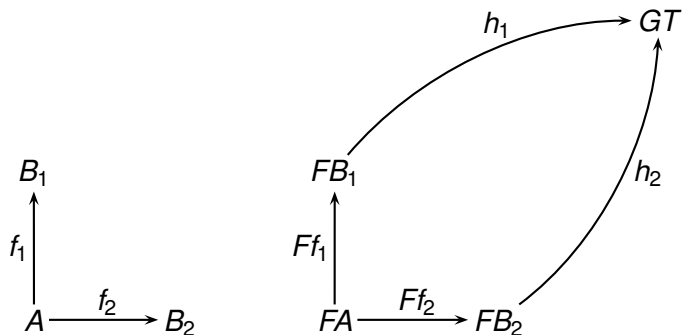
\mathfrak{A} has the (F, G) -amalgamation property if



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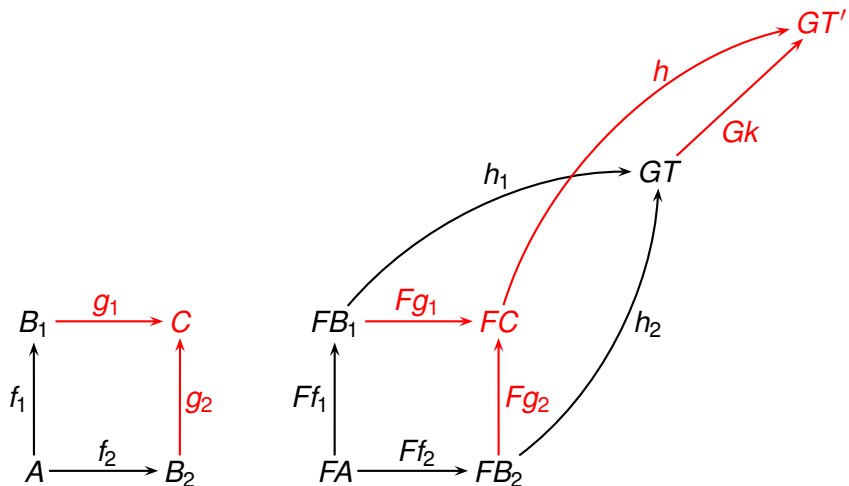
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\mathfrak{A} has the (F, G) -amalgamation property if



Theorem

Let \mathfrak{A} be a λ -algebroidal category all of whose morphisms are monos, and let \mathfrak{B} be a λ -amalgamation category. Let \mathfrak{C} be any category. Let $F : \mathfrak{A} \rightarrow \mathfrak{C}$, $G : \mathfrak{B} \rightarrow \mathfrak{C}$. Further suppose that

1. F is λ -continuous,
2. F preserves λ -smallness with respect to G ,
3. G preserves monomorphisms,
4. for every $A \in \mathfrak{A}_{<\lambda}$ and for every $B \in \mathfrak{B}_{<\lambda}$ there are at most λ morphisms in $\mathfrak{C}(FA \rightarrow GB)$.

Then $(F \downarrow G)$ is a λ -amalgamation category if and only if

- a. $(F|_{\mathfrak{A}_{<\lambda}} \downarrow G|_{\mathfrak{B}_{<\lambda}})$ has the joint embedding property, and
- b. \mathfrak{A} has the $(F|_{\mathfrak{A}_{<\lambda}}, G|_{\mathfrak{B}_{<\lambda}})$ -amalgamation property.

Question

Let (U, u, T) be universal homogeneous in $(F \downarrow G)$. When is U universal homogeneous in \mathfrak{A} ?

Mixed amalgamation

Definition

Let \mathfrak{A} , \mathfrak{B} , \mathfrak{C} be categories and let $F : \mathfrak{A} \rightarrow \mathfrak{C}$ and $G : \mathfrak{B} \rightarrow \mathfrak{C}$. We say that F and G have the **mixed amalgamation property** if for all $A, B \in \mathfrak{A}$, $S \in \mathfrak{B}$, $g : A \rightarrow B$, $a : FA \rightarrow GS$,

$$\begin{array}{ccc} FA & \xrightarrow{a} & GS \\ \downarrow Fg & & \\ FB & & \end{array}$$

Mixed amalgamation

Definition

Let \mathfrak{A} , \mathfrak{B} , \mathfrak{C} be categories and let $F : \mathfrak{A} \rightarrow \mathfrak{C}$ and $G : \mathfrak{B} \rightarrow \mathfrak{C}$. We say that F and G have the **mixed amalgamation property** if for all $A, B \in \mathfrak{A}$, $S \in \mathfrak{B}$, $g : A \rightarrow B$, $a : FA \rightarrow GS$, there exists $T \in \mathfrak{B}$, $h : S \rightarrow T$, and $b : FB \rightarrow GT$ such that the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{a} & GS \\ Fg \downarrow & & \downarrow Gh \\ FB & \xrightarrow{b} & GT \end{array}$$

At last...

Let $(\widehat{\mathfrak{A}}, \mathfrak{A})$ be a λ -amalgamation pair, \mathfrak{B} be a λ -amalgamation category, and let \mathfrak{C} be a category. Let $\widehat{F} : \widehat{\mathfrak{A}} \rightarrow \mathfrak{C}$, $G : \mathfrak{B} \rightarrow \mathfrak{C}$ and let F be the restriction of \widehat{F} to \mathfrak{A} . Further suppose that

1. F and G are λ -continuous,
2. F preserves λ -smallness with respect to G ,
3. G preserves monomorphisms,
4. for every $A \in \mathfrak{A}_{<\lambda}$ and for every $B \in \mathfrak{B}_{<\lambda}$ there are at most λ morphisms in $\mathfrak{C}(FA \rightarrow GB)$.

Finally, suppose that F is faithful, and that $(F \downarrow G)$ is a λ -amalgamation category.

Let (U, u, T) be universal and homogeneous in $(F \downarrow G)$

Proposition

U is $\mathfrak{A}_{<\lambda}$ -saturated in \mathfrak{A} if and only if $F|_{\mathfrak{A}_{<\lambda}}$ and $G|_{\mathfrak{B}_{<\lambda}}$ have the mixed amalgamation property.

Thank you for your attention!