Universal structures and universal homomorphisms

Christian Pech

25.07.2012

(joint work with Maja Pech)

Some Motivation

In

> [Rad64] R. Rado. Universal graphs and universal functions. *Acta Arith.*, 9:331–340, 1964.

Richard Rado used universal functions to explain his well-known construction of a universal countable graph:

$$K_{kl} := \left\{ f \mid f : \binom{\mathbb{N}}{l+1} \to \{0, \dots, k-1\} \right\}$$

Definition

$f^* \in K_{kl}$ is universal in K_{kl} if for every $f \in K_{kl}$ there exists a self-embedding φ of $\mathbb{N}$ such that

$$f(x_0, \dots, x_l) = f^*(\varphi(x_0), \dots, \varphi(x_l))$$

$K_{2,1}$ is essentially the class of countable graphs. Hence, a universal function $f^* \in K_{2,1}$ is a countable universal graph.

Outline

Universal homomorphisms

Universal polymorphisms

Cofinality of Menger algebras and clones

Retracts

Fraïssé-limits in comma categories

Outline

We propose to study universal homomorphisms

Definition

Let $\mathcal{K}$ be a class of structures, $\mathbf{T} \in \mathcal{K}$. $u : \mathbf{U} \to \mathbf{T}$ is called universal within $\mathcal{K}$ if $\mathbf{U} \in \mathcal{K}$, and

$$\forall \mathbf{A} \in \mathcal{K}, h : \mathbf{A} \to \mathbf{T} \quad \exists \iota : \mathbf{A} \hookrightarrow \mathbf{U} : u \circ \iota = h.$$

Note

u is a retraction: Consider $1_{\mathbf{T}} : \mathbf{T} \to \mathbf{T}$; by universality, there exists $\iota : \mathbf{T} \hookrightarrow \mathbf{U}$ such that $u \circ \iota = 1_{\mathbf{T}}$.

More general definition

A homomorphism $u : \mathbf{U}^n \to \mathbf{T}$ is called n-ary universal homomorphism to $\mathbf{T}$ within $\mathcal{K}$ if $\mathbf{U} \in \mathcal{K}$, and

$$\forall \mathbf{A} \in \mathcal{K}, h : \mathbf{A}^n \to \mathbf{T} \quad \exists \iota : \mathbf{A} \hookrightarrow \mathbf{U} : u \circ \iota^n = h.$$

Strict Fraïssé-classes

If $\mathcal{K}$ is an age, then $\overline{\mathcal{K}} := \{\mathbf{A} \mid \mathbf{A} \text{ countable}, \operatorname{Age}(\mathbf{A}) \subseteq \mathcal{K}\}$.

Definition (Dolinka)

A Fraïssé-class $\mathcal{K}$ of relational structures is called strict Fraïssé-class if every pair of morphisms in $(\mathcal{K}, \hookrightarrow)$ with the same domain has a pushout in $(\overline{\mathcal{K}}, \to)$.

Observation

Note that these pushouts will always be amalgams. Thus the strict amalgamation property postulates canonical amalgams.

Examples for strict Fraïssé-classes

- free amalgamation classes,
- the class of finite partial orders.

Homogeneous homomorphisms

Definition

Let $u : \mathbf{U}^n \to \mathbf{T}$ be an n-ary homomorphism. Let $\mathbf{A} \leq \mathbf{U}$ and let $\iota : \mathbf{A} \hookrightarrow \mathbf{U}$. We say that ι preserves u if the following diagram commutes:

$$\begin{tikzcd}
\mathbf{U}^n \arrow[r, "u"] & \mathbf{T} \\
\mathbf{A}^n \arrow[u, hook, "\subseteq"] \arrow[r, hook, "\iota^n"'] & \mathbf{U}^n \arrow[u, "u"']
\end{tikzcd}$$

Definition

Let $u : \mathbf{U}^n \to \mathbf{T}$ be an n-ary homomorphism. u is called homogeneous if for all finitely generated substructures $\mathbf{A}$ of $\mathbf{U}$, every u-preserving embedding $\iota : \mathbf{A} \to \mathbf{U}$ can be extended to a u-preserving automorphism of $\mathbf{U}$.

Existence of universal homogeneous homomorphisms

Theorem

Let $\mathcal{K}$ be a strict Fraïssé-class, $\mathbf{T} \in \overline{\mathcal{K}}$. Then there exists a universal homogeneous n-ary homomorphism $u : \mathbf{U}^n \to \mathbf{T}$ within $\overline{\mathcal{K}}$. Moreover, if $\hat{u} : \hat{\mathbf{U}}^n \to \mathbf{T}$ is another such homomorphism, then there exists an isomorphism $h : \hat{\mathbf{U}} \to \mathbf{U}$ such that

$$\begin{tikzcd}
\mathbf{U}^n \arrow[r, "u"] & \mathbf{T} \\
\hat{\mathbf{U}}^n \arrow[u, "h^n"', "\cong"] \arrow[ur, "\hat{u}"'] &
\end{tikzcd}$$

commutes.

Let $\mathbf{K}$ be the Fraïssé-limit of $\mathcal{K}$.
If all structures from $\mathcal{K}$ are finite, $\mathrm{Aut}(\mathbf{K})$ is oligomorphic, and if $\mathrm{Aut}(\mathbf{T})$ is oligomorphic, then $\mathrm{Aut}(\mathbf{U})$ is oligomorphic, too.

Corollary

For every $\mathbf{T} \in \overline{\mathcal{K}}$, the homomorphism-equivalence class $\overline{\mathcal{K}}_{\mathbf{T}}$ has a universal element.

Countable universal well-founded posets

Let α be a countable ordinal number.
Let $\mathcal{C}$ be the class of all countable well-founded strict posets of height $\leq \alpha$.

Question

Does $\mathcal{C}$ have a universal object?

Note

$\mathcal{C}$ is not elementary.

Countable universal well-founded posets

Let α be a countable ordinal number.
Let $\mathcal{C}$ be the class of all countable well-founded strict posets of height $\leq \alpha$.

Question

Does $\mathcal{C}$ have a universal object?

Note

$\mathcal{C}$ is not elementary.

Answer to the question

Yes, $\mathcal{C}$ has a universal element **U**.

- Note that $\overline{\mathcal{C}}$ consists of all countable strict posets that have a homomorphism to $(\alpha, \in)$.
- Take $\mathcal{K}$ as the class of all finite posets. Set **T** $:= (\alpha, \in)$. Then there exists a universal homogeneous homomorphism u : **U** $\to$ **T**.
- Observe that **U** is universal in $\mathcal{C}$.

A countable universal directed acyclic graph

- $\mathcal{K}$ be the class of all finite structures with one binary relation,
- $\mathbf{T} := (\mathbb{Q}, <)$,
- $u : \mathbf{U} \to \mathbf{T}$ be a universal homogeneous homomorphism within $\overline{\mathcal{K}}$.

Then **U** is a countable universal directed acyclic graph.

Note

- The signature is finite. Hence the Fraïssé-limit of $\mathcal{K}$ is $\aleph_0$-categorical.
- $(\mathbb{Q}, <)$ is homogeneous over a finite signature. Hence it is $\aleph_0$-categorical.

Hence **U** is $\aleph_0$-categorical, too.

Outline

Universal homogeneous polymorphisms

Let R denote the countable random graph.
Let $u : \mathbf{U}^n \to R$ be an n-ary universal homogeneous homomorphism within the class of all countable simple graphs.

Question

What is $\mathbf{U}$? (clearly, $\mathrm{Age}(\mathbf{U}) = \mathrm{Age}(R)$)

Universal homogeneous polymorphisms

Let R denote the countable random graph.
Let $u : \mathbf{U}^n \to R$ be an n-ary universal homogeneous homomorphism within the class of all countable simple graphs.

Question

What is $\mathbf{U}$? (clearly, $\mathrm{Age}(\mathbf{U}) = \mathrm{Age}(R)$)

Answer

$\mathbf{U} \cong R$. That is, we can assume w.l.o.g., that $\mathbf{U} = R$.
Hence u is an n-ary polymorphism of R.

In other words:

The countable random graph has universal homogeneous polymorphisms of every arity.

Questions:

1. Which structures have universal homogeneous endomorphisms?
2. Which structures have universal homogeneous polymorphisms?

Existence of universal homogeneous polymorphisms

Homo amalgamation property (HAP)

$\mathcal{K}$ has the (HAP) if for all $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$, for all homomorphisms $f_1 : \mathbf{A} \to \mathbf{B}_1$, $f_2 : \mathbf{A} \hookrightarrow \mathbf{B}_2$ there exist $\mathbf{C} \in \mathcal{K}$, $g_1 : \mathbf{B}_1 \hookrightarrow \mathbf{C}$, and $g_2 : \mathbf{B}_2 \to \mathbf{C}$, such that the following diagram commutes:

$$\begin{tikzcd} \mathbf{A} \arrow[r, "f_1"] \arrow[d, hook, "f_2"'] & \mathbf{B}_1 \arrow[d, hook, "g_1"] \\ \mathbf{B}_2 \arrow[r, "g_2"] & \mathbf{C} \end{tikzcd}$$

Theorem

Let $\mathcal{K}$ be a strict Fraïssé-class with the HAP. Let $\mathbf{U}$ be a Fraïssé-limit of $\mathcal{K}$. If $\mathcal{K}$ is closed with respect to n-th powers, then $\mathbf{U}$ has an n-ary universal homogeneous polymorphism.

Some examples

The following structures have universal homogeneous polymorphisms of every arity:

- the countable random graph R (here $\mathcal{K}$ is the class of all finite simple graphs),
- the countable generic poset $\mathbb{P} = (P, \leq)$ (here $\mathcal{K}$ is the class of all finite posets),
- the countable atomless Boolean algebra $\mathbb{A}$ (here $\mathcal{K}$ is the class of finite Boolean algebras),
- the countable universal homogeneous semilattice Ω (here $\mathcal{K}$ is the class of all finite semilattices),
- the countable universal homogeneous distributive lattice $\mathbb{D}$ (here $\mathcal{K}$ is the class of all finite distributive lattices).

Outline

Clones and Menger algebras

Given a set A.

$$O_A^{(n)} := \{f \mid f : A^n \to A\}, \quad O_A := \bigcup_{n \in \mathbb{N}\setminus\{0\}} O_A^{(n)}$$

$$f \in O_A^{(n)}, g_1, \ldots, g_n \in O_A^{(m)} : f \circ \langle g_1, \ldots, g_n \rangle \in O_A^{(m)}$$

$$f \circ \langle g_1, \ldots, g_n \rangle : \bar{x} \mapsto f(g_1(\bar{x}), \ldots, g_n(\bar{x})).$$

$$e_i^n \in O_A^{(n)} : (x_1, \ldots, x_n) \mapsto x_i \quad \text{(projections)}$$

Definition

A clone on A is a subset of O_A that contains all projections and is closed with respect to composition.

Definition

An n-ary pre-Menger algebra on A is a subset of $O_A^{(n)}$ that is closed with respect to composition.

Definition

An n-ary pre-Menger algebra is called Menger algebra if it contains all e_i^n

Clones and Menger algebras of Polymorphisms

Given a structure **A** (with carrier A).

$$\mathrm{Pol}^{(n)}(\mathbf{A}) := \{f \in O_A^{(n)} \mid f : \mathbf{A}^n \to \mathbf{A}\}$$
$$\mathrm{Pol}(\mathbf{A}) = \bigcup_{n \in \mathbb{N}\setminus\{0\}} \mathrm{Pol}^{(n)}(\mathbf{A})$$

Note:

1. $\mathrm{Pol}^{(n)}(\mathbf{A})$ is an n-ary Menger algebra,
2. $\mathrm{Pol}(\mathbf{A})$ is a clone.

Cofinality of clones and Menger algebras

- The notions subclone, and pre-Menger subalgebra are defined in the obvious way.
- Let C be a clone, M be an n-ary pre-Menger algebra

Definition
C is said to have uncountable cofinality if it can not be written as the union of a countable chain of proper subclones.

Definition
M is said to have uncountable cofinality if it can not be written as the union of a countable chain of proper pre-Menger subalgebras.

Uncountable cofinality for clones

- For $M \subseteq O_A$ the smallest clone on A containing M is denoted by $\langle M \rangle_{O_A}$,
- For a clone C on A, we define $C^{(n)} := C \cap O_A^{(n)}$

Proposition

A clone C has uncountable cofinality if and only if there exists some $k \in \mathbb{N} \setminus \{0\}$ such that

1. $C = \langle C^{(k)} \rangle_{O_A}$,
2. $C^{(k)}$*, considered as a pre-Menger algebra, has uncountable cofinality*

Uncountable cofinality for Menger algebras

Proposition

Let $\mathbf{A}$ *be a structure such that* $\mathrm{End}(\mathbf{A})$ *has uncountable cofinality. If* $\mathbf{A}$ *has a universal n-ary polymorphism, then* $\mathrm{Pol}^{(n)}(\mathbf{A})$ *has uncountable cofinality, too.*

Remark

- *In the proposition above, uncountable cofinality can be replaced by strong uncountable cofinality,*
- *strong uncountable cofinality is equivalent to uncountable cofinality + Bergman property.*

Here a pre-Menger algebra M has the Bergman property if for every generating set T there exists a k_T such that every element of M can be obtained by a term over T of depth $\leq k$.

Examples

The following Menger algebras have uncountable cofinality and the Bergman property:

- $\mathrm{Pol}^{(n)}(R)$, where R is the countable random graph,
- $\mathrm{Pol}^{(n)}(\mathbb{P})$ where $\mathbb{P} = (P, \leq)$ is the countable generic poset,
- $\mathrm{Pol}^{(n)}(\mathbb{A})$, where $\mathbb{A}$ is the countable atomless Boolean algebra,
- $\mathrm{Pol}^{(n)}(\Omega)$, where Ω is the countable universal homogeneous semilattice,
- $\mathrm{Pol}^{(n)}(\mathbb{D})$, where $\mathbb{D}$ is the countable universal homogeneous distributive lattice,
- $O_A^{(n)}$.

Corollary

The clone O_A has uncountable cofinality (since it is generated by $O_A^{(2)}$).

Outline

Universal homogeneous retractions

Definition

A retraction $r : \mathbf{U} \twoheadrightarrow \mathbf{T}$ is called universal homogeneous retraction if it is a universal homogeneous homomorphism to $\mathbf{T}$ within $\overline{\mathrm{Age}(\mathbf{U})}$

Theorem

Let $\mathcal{C}$ be a Fraïssé-class with Fraïssé-limit $\mathbf{U}$, and let $\mathbf{T} \in \overline{\mathcal{C}}$. Then there exists a universal homogeneous retraction $r : \mathbf{U} \twoheadrightarrow \mathbf{T}$ if and only if

$$\begin{tikzcd}
 & \mathbf{T} \\
\mathbf{B}_1 \arrow[ur, bend left, "h_1"] & \\
\mathbf{A} \arrow[u, hook, "f_1"] \arrow[r, hook, "f_2"'] & \mathbf{B}_2 \arrow[uu, bend right, "h_2"']
\end{tikzcd}$$

Universal homogeneous retractions

Definition

A retraction $r : \mathbf{U} \twoheadrightarrow \mathbf{T}$ is called universal homogeneous retraction if it is a universal homogeneous homomorphism to $\mathbf{T}$ within $\overline{\mathrm{Age}(\mathbf{U})}$

Theorem

Let $\mathcal{C}$ be a Fraïssé-class with Fraïssé-limit $\mathbf{U}$, and let $\mathbf{T} \in \overline{\mathcal{C}}$. Then there exists a universal homogeneous retraction $r : \mathbf{U} \twoheadrightarrow \mathbf{T}$ if and only if

$$\begin{tikzcd}
 & & \mathbf{T} \\
\mathbf{B}_1 \arrow[r, hook, "g_1"'] \arrow[urr, bend left, "h_1"] & \mathbf{C} \arrow[ur, "h"] & \\
\mathbf{A} \arrow[u, hook, "f_1"] \arrow[r, hook, "f_2"] & \mathbf{B}_2 \arrow[u, hook, "g_2"] \arrow[uur, bend right, "h_2"'] &
\end{tikzcd}$$

Universal homogeneous retractions

Definition

A retraction $r : \mathbf{U} \twoheadrightarrow \mathbf{T}$ is called universal homogeneous retraction if it is a universal homogeneous homomorphism to $\mathbf{T}$ within $\overline{\mathrm{Age}(\mathbf{U})}$

Theorem

Let $\mathcal{C}$ be a Fraïssé-class with Fraïssé-limit $\mathbf{U}$, and let $\mathbf{T} \in \overline{\mathcal{C}}$. Then there exists a universal homogeneous retraction $r : \mathbf{U} \twoheadrightarrow \mathbf{T}$ if and only if

$$\begin{tikzcd}
& & \mathbf{T} \\
\mathbf{B}_1 \arrow[urr, bend left, "h_1"] \arrow[r, hook, "g_1"'] & \mathbf{C} \arrow[ur, "h"] & \\
\mathbf{A} \arrow[u, hook, "f_1"] \arrow[r, hook, "f_2"] & \mathbf{B}_2 \arrow[u, hook, "g_2"] \arrow[uur, bend right, "h_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{A} \arrow[r, "h"] \arrow[d, hook, "\iota"'] & \mathbf{T} \\
\mathbf{B} &
\end{tikzcd}$$

Universal homogeneous retractions

Definition

A retraction $r : \mathbf{U} \twoheadrightarrow \mathbf{T}$ is called universal homogeneous retraction if it is a universal homogeneous homomorphism to $\mathbf{T}$ within $\overline{\mathrm{Age}(\mathbf{U})}$

Theorem

Let $\mathcal{C}$ be a Fraïssé-class with Fraïssé-limit $\mathbf{U}$, and let $\mathbf{T} \in \overline{\mathcal{C}}$. Then there exists a universal homogeneous retraction $r : \mathbf{U} \twoheadrightarrow \mathbf{T}$ if and only if

$$
\begin{tikzcd}
 & & \mathbf{T} \\
\mathbf{B}_1 \arrow[r, hook, "g_1"'] \arrow[urr, bend left, "h_1"] & \mathbf{C} \arrow[ur, "h"] & \\
\mathbf{A} \arrow[u, hook, "f_1"] \arrow[r, hook, "f_2"] & \mathbf{B}_2 \arrow[u, hook, "g_2"'] \arrow[uur, bend right, "h_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{A} \arrow[r, "h"] \arrow[d, hook, "\iota"'] & \mathbf{T} \\
\mathbf{B} \arrow[ur, "\hat{h}"'] &
\end{tikzcd}
$$

Subretracts of universal homogeneous retracts

Proposition

Let $\mathcal{C}$ be a Fraïssé-class with Fraïssé-limit $\mathbf{U}$ *and let* $\mathbf{V}, \mathbf{W} \in \overline{\mathcal{C}}$ *Let* $r : \mathbf{U} \twoheadrightarrow \mathbf{V}$ *be a universal homogeneous retraction. Let* $s : \mathbf{V} \twoheadrightarrow \mathbf{W}$ *be any retraction. Then there is a universal homogeneous retraction* $\hat{s} : \mathbf{U} \twoheadrightarrow \mathbf{W}$.

Corollary

If $\mathbf{U}$ *has a universal homogeneous endomorphism, then every retract of* $\mathbf{U}$ *is induced by a universal homogeneous retraction.*

Universal homogeneous endomorphisms revisited

Kubiś's amalgamated extension property

Let $\mathcal{C}$ be a class of countable, finitely generated structures. We say that $\mathcal{C}$ has the amalgamated extension property if

$$
\begin{tikzcd}
 & & \mathbf{T} \\
\mathbf{B}_1 \arrow[urr, bend left, "h_1"] & & \\
\mathbf{A} \arrow[u, hook, "f_1"] \arrow[r, hook, "f_2"] & \mathbf{B}_2 \arrow[uur, bend right, "h_2"'] &
\end{tikzcd}
$$

Universal homogeneous endomorphisms revisited

Kubiś's amalgamated extension property

Let $\mathcal{C}$ be a class of countable, finitely generated structures. We say that $\mathcal{C}$ has the amalgamated extension property if

$$
\begin{tikzcd}
 & & & \mathbf{T}' \\
 & & \mathbf{T} \arrow[ur, hook, "k"'] & \\
\mathbf{B}_1 \arrow[r, hook, "g_1"] \arrow[urr, bend left, "h_1"] & \mathbf{C} \arrow[uurr, bend left, "h"] & & \\
\mathbf{A} \arrow[u, hook, "f_1"] \arrow[r, hook, "f_2"] & \mathbf{B}_2 \arrow[u, hook, "g_2"'] \arrow[uur, bend right, "h_2"'] & &
\end{tikzcd}
$$

Universal homogeneous endomorphisms revisited II

Proposition

Let $\mathcal{C}$ be a Fraïssé-class. Let $\mathbf{U}$ be its Fraïssé-limit. Then $\mathbf{U}$ has a universal homogeneous endomorphism if and only if

1. *$\mathcal{C}$ has the amalgamated extension property, and*
2. *$\mathcal{C}$ has the homo amalgamation property.*

Proposition

Let $\mathbf{U}$ be a countable structure that has a universal homogeneous endomorphism. Then $\mathbf{U}$ is homogeneous if and only if $\mathbf{U}$ is homomorphism homogeneous.

Corollary

Let $\mathcal{C}$ be a Fraïssé-class with Fraïssé-limit $\mathbf{U}$. Then $\mathcal{C}$ has the HAP and the amalgamated extension property if and only if every retract of $\mathbf{U}$ is induced by a universal homogeneous retraction.

Outline

Universal homogeneous objects in categories

Definition

We call a category $\mathfrak{C}$ a λ-amalgamation category if

1. all morphisms of $\mathfrak{C}$ are monomorphisms,
2. $\mathfrak{C}$ is λ-algebroidal,
3. $\mathfrak{C}_{<\lambda}$ has the joint embedding property,
4. $\mathfrak{C}_{<\lambda}$ has the amalgamation property.

Theorem (Droste, Göbel ’92)

Let λ be a regular cardinal, and let $\mathfrak{C}$ be a λ-algebroidal category in which all morphisms are monomorphisms. Then there exists a $\mathfrak{C}$-universal, $\mathfrak{C}_{<\lambda}$-homogeneous object in $\mathfrak{C}$ if and only if $\mathfrak{C}$ is a λ-amalgamation category. Moreover, any two $\mathfrak{C}$-universal, $\mathfrak{C}_{<\lambda}$-homogeneous objects in $\mathfrak{C}$ are isomorphic.

(F, G)-amalgamation property

Given categories $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, functors $F : \mathfrak{A} \to \mathfrak{C}$, $G : \mathfrak{B} \to \mathfrak{C}$.

$\mathfrak{A}$ has the (F, G)-amalgamation property if

$$\begin{array}{ccc} B_1 & & \\ \uparrow{\scriptstyle f_1} & & \\ A & \xrightarrow{f_2} & B_2 \end{array}$$

(F, G)-amalgamation property

Given categories $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, functors $F : \mathfrak{A} \to \mathfrak{C}$, $G : \mathfrak{B} \to \mathfrak{C}$.

$\mathfrak{A}$ has the (F, G)-amalgamation property if

$$\begin{tikzcd} B_1 & \\ A \arrow[u, "f_1"] \arrow[r, "f_2"'] & B_2 \end{tikzcd} \qquad \begin{tikzcd} & & GT \\ FB_1 \arrow[urr, bend left, "h_1"] & & \\ FA \arrow[u, "Ff_1"] \arrow[r, "Ff_2"'] & FB_2 \arrow[uur, bend right, "h_2"'] & \end{tikzcd}$$

(F, G)-amalgamation property

Given categories $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, functors $F : \mathfrak{A} \to \mathfrak{C}$, $G : \mathfrak{B} \to \mathfrak{C}$.

$\mathfrak{A}$ has the (F, G)-amalgamation property if

$$
\begin{tikzcd}
B_1 \arrow[r, "g_1"] & C \\
A \arrow[u, "f_1"] \arrow[r, "f_2"] & B_2 \arrow[u, "g_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & GT' \\
 & & GT \arrow[ur, "Gk"'] & \\
FB_1 \arrow[r, "Fg_1"] \arrow[urr, bend left, "h_1"] & FC \arrow[uurr, bend left, "h"] & & \\
FA \arrow[u, "Ff_1"] \arrow[r, "Ff_2"] & FB_2 \arrow[u, "Fg_2"'] \arrow[uur, bend right, "h_2"'] & &
\end{tikzcd}
$$

Theorem

Let $\mathfrak{A}$ be a λ-algebroidal category all of whose morphisms are monos, and let $\mathfrak{B}$ be a λ-amalgamation category. Let $\mathfrak{C}$ be any category. Let $F : \mathfrak{A} \to \mathfrak{C}$, $G : \mathfrak{B} \to \mathfrak{C}$. Further suppose that

1. *F is λ-continuous,*
2. *F preserves λ-smallness with respect to G,*
3. *G preserves monomorphisms,*
4. *for every $A \in \mathfrak{A}_{<\lambda}$ and for every $B \in \mathfrak{B}_{<\lambda}$ there are at most λ morphisms in $\mathfrak{C}(FA \to GB)$.*

Then $(F \downarrow G)$ is a λ-amalgamation category if and only if

a. *$(F|_{\mathfrak{A}_{<\lambda}} \downarrow G|_{\mathfrak{B}_{<\lambda}})$ has the joint embedding property, and*

b. *$\mathfrak{A}$ has the $(F|_{\mathfrak{A}_{<\lambda}}, G|_{\mathfrak{B}_{<\lambda}})$-amalgamation property.*

Question

Let (U, u, T) be universal homogeneous in $(F \downarrow G)$. When is U universal homogeneous in $\mathfrak{A}$?

Mixed amalgamation

Definition

Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be categories and let $F : \mathfrak{A} \to \mathfrak{C}$ and $G : \mathfrak{B} \to \mathfrak{C}$. We say that F and G have the mixed amalgamation property if for all $A, B \in \mathfrak{A}$, $S \in \mathfrak{B}$, $g : A \to B$, $a : FA \to GS$,

$$\begin{tikzcd} FA \arrow[r, "a"] \arrow[d, "Fg"'] & GS \\ FB & \end{tikzcd}$$

Mixed amalgamation

Definition

Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be categories and let $F : \mathfrak{A} \to \mathfrak{C}$ and $G : \mathfrak{B} \to \mathfrak{C}$. We say that F and G have the mixed amalgamation property if for all $A, B \in \mathfrak{A}$, $S \in \mathfrak{B}$, $g : A \to B$, $a : FA \to GS$,
there exists $T \in \mathfrak{B}$, $h : S \to T$, and $b : FB \to GT$ such that the following diagram commutes:

$$\begin{tikzcd}
FA \arrow[r, "a"] \arrow[d, "Fg"'] & GS \arrow[d, "Gh"'] \\
FB \arrow[r, "b"] & GT
\end{tikzcd}$$

At last. . .

Let $(\widehat{\mathfrak{A}}, \mathfrak{A})$ be a λ-amalgamation pair, $\mathfrak{B}$ be a λ-amalgamation category, and let $\mathfrak{C}$ be a category. Let $\widehat{F} : \widehat{\mathfrak{A}} \to \mathfrak{C}$, $G : \mathfrak{B} \to \mathfrak{C}$ and let F be the restriction of $\widehat{F}$ to $\mathfrak{A}$. Further suppose that

1. F and G are λ-continuous,
2. F preserves λ-smallness with respect to G,
3. G preserves monomorphisms,
4. for every $A \in \mathfrak{A}_{<\lambda}$ and for every $B \in \mathfrak{B}_{<\lambda}$ there are at most λ morphisms in $\mathfrak{C}(FA \to GB)$.

Finally, suppose that F is faithful, and that $(F \downarrow G)$ is a λ-amalgamation category.
Let (U, u, T) be universal and homogeneous in $(F \downarrow G)$

Proposition

U is $\mathfrak{A}_{<\lambda}$-saturated in $\mathfrak{A}$ if and only if $F|_{\mathfrak{A}_{<\lambda}}$ and $G|_{\mathfrak{B}_{<\lambda}}$ have the mixed amalgamation property.

Thank you for your attention!