Filters and topologies from the random graph

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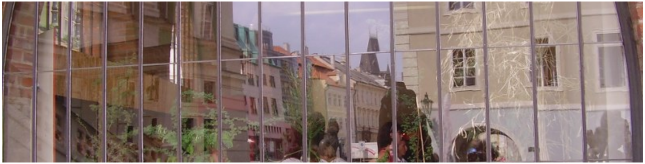

The random graph

This probably needs no introduction!
R is the unique countable graph which is

- universal (contains all finite and countable graphs) and
- homogeneous (all isomorphisms between finite subgroups extend to automorphisms).

If you choose edges independently with probability $1/2$ in a countable vertex set, you almost surely get the random graph.
Its automorphism group is simple, contains generic n-tuples of automorphisms, and embeds all finite or countable groups.
Its first-order theory is the first-order theory of almost all finite graphs.
And so on …

Recognizing R

Theorem

A countable graph X is isomorphic to R if and only if, given any two finite disjoint sets U and V of distinct vertices of R, there is a vertex z joined to everything in U and nothing in V.

The proof is probably familiar to you. Given two countable graphs satisfying the condition, we build an isomorphism between then by back-and-forth.

Theorem

Let X be a countable graph. Then R is isomorphic to a spanning subgraph of X if and only if any finite set of vertices of X has a common neighbour.

This will be needed later. The proof is very similar to that of the previous theorem. We build a bijection between R and X, but in the R to X direction we only insist that edges are preserved.

Almost automorphisms

Consider the following groups, described by Claude Laflamme in his talk:

- $\mathrm{Aut}_1(R)$ is the group of permutations which change only finitely many adjacencies (these are called almost-automorphisms, and Truss denotes the group by $\mathrm{AAut}(R)$);
- $\mathrm{Aut}_2(R)$ is the group of permutations which change only finitely many adjacencies at any vertex;
- $\mathrm{Aut}_3(R)$ is the group of permutations which change infinitely many adjacencies at only finitely many vertices.

If $C(g)$ denotes the set of pairs $\{v, w\}$ of vertices whose adjacency is changed by the permutation g, then $C(g^{-1}) = C(g)^{g^{-1}}$ and $C(gh) \subseteq C(g) \cup C(h)^{g^{-1}}$. It easily follows from this that $\mathrm{Aut}_i(R)$ is a group for $i = 1, 2, 3$.
It is an exercise to show that

$$\mathrm{Aut}(R) < \mathrm{Aut}_1(R) < \mathrm{Aut}_2(R) < \mathrm{Aut}_3(R) < \mathrm{Sym}(R).$$

The generic bipartite graph

Another character in our story is the graph B obtained (almost surely) by taking a countable vertex set partitioned into two parts and choosing at random edges between the two parts. B has properties resembling those of R. For example, it is the unique countable homogeneous universal graph-with-bipartition. [A bipartite graph cannot be homogeneous unless it is either complete bipartite or null; for otherwise a pair of vertices in the same bipartite block, and a non-adjacent pair in different blocks, are not equivalent under automorphisms of the graph.]

Recognizing B

The characteristic property of B is: if U and V are finite disjoint sets of vertices *in the same bipartite block*, then there is a vertex z in the other block joined to everything in U and nothing in V. The subgroup of $\mathrm{Aut}(B)$ fixing a bipartite block acts highly transitively on the points of that block. I will call this group $\mathrm{Aut}^{+}(B)$ (but note that it is not a closed subgroup of the symmetric group).

Three hypergraphs

Consider the following three hypergraphs:

$\mathcal{N}$: the vertex set is that of R; the edges are the neighbourhoods $N(v) = \{x : x \sim v\}$ of vertices v of R.

$\mathcal{N}^*$: the vertex set is that of R; the edges are the closed neighbourhoods $\overline{N}(v) = \{x : x = v \text{ or } x \sim v\}$ of vertices v of R.

$\mathcal{N}^\dagger$: the vertex set is one bipartite block of B; the edges are the neighbourhoods of vertices in the other bipartite block.

Proposition

The three hypergraphs defined above are isomorphic.

Proof

We build two bipartite graphs from R as follows:

- B_1: the vertex set is $V(R) \times \{0,1\}$; $(v,a) \sim (w,b)$ if and only if $v \sim w$ and $a \neq b$.
- B_2: the vertex set is $V(R) \times \{0,1\}$; $(v,a) \sim (w,b)$ if and only if (either $v = w$ or $v \sim w$) and $a \neq b$.

These are the Levi graphs of the hypergraphs $\mathcal{N}$ and $\mathcal{N}^*$.
Then it is easy to show that $B_1 \cong B_2 \cong B$, by verifying that both B_1 and B_2 satisfy the characteristic property of B.

Two overgroups of $\mathrm{Aut}(R)$

The groups $\mathrm{Aut}(\mathcal{N})$ and $\mathrm{Aut}(\mathcal{N}^*)$ are both highly transitive subgroups of $\mathrm{Sym}(V(R))$ containing $\mathrm{Aut}(R)$.
They are isomorphic (indeed, conjugate) but not equal.
Moreover, we see later that $\langle \mathrm{Aut}(\mathcal{N}), \mathrm{Aut}(\mathcal{N}^*) \rangle < \mathrm{Sym}(V(R))$.
In the talk, I wondered whether $\mathrm{Aut}(\mathcal{N}) \cap \mathrm{Aut}(\mathcal{N}^*) > \mathrm{Aut}(R)$.
Afterwards, Michael Pinsker pointed out that these two groups are equal. For a permutation preserving $\mathcal{N}$ and $\mathcal{N}^*$ preserves the pairs (A, B) with $A \in \mathcal{N}$, $B \in \mathcal{N}^*$, $A \subset B$ with $|B \setminus A| = 1$, and hence the graph structure.
The rest of the talk will involve looking at two types of closure of these hypergraphs.

Filters

A filter on a set V is a family $\mathcal{F}$ of subsets of V satisfying

- $X, Y \in \mathcal{F}$ implies $X \cap Y \in F$;
- $X \in F$, $Y \supseteq X$ implies $Y \in \mathcal{F}$.

A filter $\mathcal{F}$ on a set V is trivial if it consists of all subsets of V. Given a family $\mathcal{A}$ of subsets of V, the filter generated by $\mathcal{A}$ is the upward closure of the set of finite intersections of sets in $\mathcal{A}$; that is, the set

$$\mathcal{F} = \{X \subseteq V : (\exists A_1, \ldots, A_n \in \mathcal{A})(A_1 \cap \cdots \cap A_n) \subseteq X\}.$$

Neighbourhood filters

Let Γ be a graph on a countable vertex set V. The neighbourhood filter of Γ is the filter $\mathcal{F}(\Gamma)$ generated by $\{\Gamma(v) : v \in V\}$, where $\Gamma(v)$ denotes the neighbourhood of v in Γ, the set of vertices adjacent to v.

Proposition

Suppose that Γ has the property that each vertex has a non-neighbour. Then the filter generated by the closed neighbourhoods $\overline{\Gamma}(v) = \Gamma(v) \cup \{v\}$ is equal to $\mathcal{F}(\Gamma)$.

For, if $w \not\sim v$, then $\overline{\Gamma}(v) \cap \overline{\Gamma}(w) \subseteq \Gamma(v) \subseteq \overline{\Gamma}(v)$.

Neighbourhood filters of R

Proposition

The following three conditions on a graph Γ are equivalent:

- $\mathcal{F}(\Gamma)$ *is nontrivial;*
- Γ *contains R as a spanning subgraph;*
- $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(R)$.

Proof.

A filter is trivial if and only if it contains the empty set. So $\mathcal{F}(\Gamma)$ is non-trivial if and only if any finitely many neighbourhoods have non-empty intersection. This is equivalent to the statement that R is a spanning subgraph of Γ, as we saw. So (a) and (b) are equivalent.

If Γ contains R as a spanning subgraph, then $R(v) \subseteq \Gamma(v)$ for all v. So (b) implies (c). Conversely, $\mathcal{F}_R$ is non-trivial (by our proof that (b) implies (a)), so (c) implies (a). □

A remark

So, in some sense, $\mathcal{F}(R)$ is the unique maximal neighbourhood filter; but this is only up to isomorphism, since there are countable chains of neighbourhood filters all isomorphic to $\mathcal{F}(R)$.

Here is a simple example. Let T be the random 3-edge colouring of the countable complete graph, with colours red, green and blue. Let R_1 be the graph consisting of red edges, and R_2 the graph consisting of red and green edges, in T. Then both R_1 and R_2 are isomorphic to R. Since $R_1(v) \subseteq R_2(v)$, we have $\mathcal{F}(R_2) \subseteq \mathcal{F}(R_1)$. It is not hard to show that the inequality is strict.

Another remark

We can now see why $\langle \mathrm{Aut}(\mathcal{N}), \mathrm{Aut}(\mathcal{N}^*) \rangle < \mathrm{Sym}(V)$. For the automorphism groups of both the hypergraphs $\mathcal{N}$ and $\mathcal{N}^*$ are contained in $\mathrm{Aut}(\mathcal{F}(R))$.

Problem

Is it true that $\langle \mathrm{Aut}(\mathcal{N}), \mathrm{Aut}(\mathcal{N}^*) \rangle < \mathrm{Aut}(\mathcal{F}(R))$*?*

Relations

Claude Laflamme said more about the relations between these groups (and some others) in his talk. Here is one simple fact.

Proposition

$\mathrm{Aut}_2(R) \leq \mathrm{Aut}(\mathcal{F}(R))$, *but* $\mathrm{Aut}_3(R)$ *and* $\mathrm{Aut}(\mathcal{F}_R)$ *are incomparable.*

Problem

Is it true that $\mathrm{Aut}_2(R) < \mathrm{Aut}_3(R) \cap \mathrm{Aut}(\mathcal{F}_R)$*?*

Topologies

A topology is a family of sets which is closed under finite intersections and arbitrary unions.
Given a family $\mathcal{A}$ of sets, the topology generated by $\mathcal{A}$ consists of all unions of finite intersections of the sets in $\mathcal{A}$.
If $\mathcal{F}$ is a filter, then $\mathcal{T} = \mathcal{F} \cup \{\emptyset\}$ is a topology, having the same automorphism group as $\mathcal{F}$.

Topologies from R

There are two rather more interesting topologies associated with R.
In the first topology $\mathcal{T}$, a sub-basis for the open sets consists of the neighbourhoods of vertices. Thus the open sets are all unions of sets which are finite intersections of neighbourhoods.
The topology $\mathcal{T}$ is not Hausdorff: in fact, any two open sets have non-empty intersection. However, this topology does satisfy the T1 separation condition.
What about using closed neighbourhoods?

Other constructions

Consider the three topologies $\mathcal{T}$, $\mathcal{T}^*$ and $\mathcal{T}^\dagger$ generated by the three families of sets $\mathcal{N}$, $\mathcal{N}^*$ and $\mathcal{N}^\dagger$ defined earlier.

Proposition

- *The three topologies defined above are all homeomorphic.*
- *The homeomorphism groups of these topologies are highly transitive.*

This follows from the isomorphism of the corresponding neighbourhood hypergraphs.

A remark

Note that $\mathcal{T}$ and $\mathcal{T}^*$ are not identical: the identity map is a continuous bijection from $\mathcal{T}^*$ to $\mathcal{T}$ but is not a homeomorphism.

Problem

Is it true that $\mathrm{Aut}(\mathcal{T}^\dagger) > \mathrm{Aut}(\mathcal{N}^\dagger)$*?*

The right-hand group is the group of permutations induced on a bipartite block by its stabiliser in $\mathrm{Aut}(B)$.

Problem

What is the relationship between the automorphism group of $\mathcal{F}(R)$ *and the homeomorphism group of* $\mathcal{T}$*?*

They cannot be equal since then $\mathrm{Aut}(\mathcal{T})$ would contain $\mathrm{Aut}(T^*)$.

Another topology

The second topology $\mathcal{U}$ is obtained by symmetrising this one with respect to the graph R and its complement R^c; in other words, we also take closed neighbourhoods in R^c to be open sets. So a basis for the open sets consists of all sets of the form

$$Z(U,V) = \{z \in V(R) : (\forall u \in U)(z \sim u) \wedge (\forall v \in V)(z \not\sim v)\}$$

for finite disjoint sets U and V: that is, the sets of witnesses for the characteristic property of R.

Again it holds that all the non-empty open sets are infinite.

This time the topology is totally disconnected. For given $u \neq v$, there is a point $z \in Z(\{u\},\{v\})$; then the open neighbourhood of z is open and closed in the topology and contains u but not v.

By Sierpiński's Theorem, this topology is homeomorphic to $\mathbb{Q}$.

So R as a countable topological space is homeomorphic to $\mathbb{Q}$.

k-neighbourhood graphs

Given a connected graph X, let $X_i(v)$ denote the set of vertices at distance i from v, and let $\overline{X}_k(v)$ denote the set of vertices at distance at most k from v (called the k-neighbourhood of v in the graph X).

Problem

What can be said about graphs X for which the k-neighbourhoods generate a non-trivial filter? In particular, which graphs are "maximal" with this property (in the sense that R is when $k = 1$)?

Clearly a graph with non-trivial k-neighbourhood filter has diameter at most $2k$ (else there are two vertices whose k-neighbourhoods are disjoint) and at least $k + 1$ (else every k-neighbourhood is the entire vertex set).

Integral metric spaces

A metric space is integral if all distances are integers.
There is a unique countable homogeneous universal integral metric space M_∞. Moreover, for any $m > 1$, there is a unique countable homogeneous universal integal metric space of diameter m, say M_m. These spaces arise from the path metrics in certain interesting graphs R_∞ and R_m. Here R_2 is the random graph R.

Proposition

An integral metric space is isometric to M_m if and only if, for any finite set A of vertices, and any Katětov function on A, that is, any function f from A to $\{0, 1, \ldots, m\}$ which satisfies

$$|f(a) - f(b)| \leq d(a, b) \leq f(a) + f(b)$$

for all $a, b \in A$, there exists a point z satisfying $d(z, a) = f(a)$ for all $a \in A$.

k-neighbourhoods

Proposition

Suppose that $k+1 \leq m \leq 2k$. Form a graph on M_m by joining two points if their distance is at most k. Then the resulting graph is isomorphic to R.

Proof.

Let U and V be finite disjoint sets in M_m. Define f on $U \cup V$ to take the value k on U and $k+1$ on V. The inequality $m \leq 2k$ shows that f is a Katětov function. So there exists a point z with $d(z,a) = f(a)$ for all $a \in U \cup V$. Then z is joined to all vertices in U and to none in V, in the distance-$\leq k$ graph defined in the Proposition. $\square$

And finally . . .

Here are two problems which arose in the talk. The first was a question of Jarik Nešetřil; the second is a refinement of a question of Greg Cherlin.

Problem

- *We saw that the topology $\mathcal{U}$ constructed from R in a natural way is homeomorphic to $\mathbb{Q}$. Make this explicit; that is, find an explicit bijection between the vertex set of R (in your favourite description) and $\mathbb{Q}$ which is a homeomorphism.*
- *Let G be a highly-transitive permutation group of countable degree which contains no non-trivial finitary permutations. Does G contain a subgroup H such that the closure of H (in the topology of pointwise convergence) is isomorphic to $\mathrm{Aut}(R)$ (as permutation group)?*