A Katětov construction of the Gurarij space

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The isometry group of the Urysohn space...

Definition

The Urysohn space $\mathbf{U}$ is the unique separable, universal, homogeneous metric space. Equivalently, it is the unique limit of the Fraïssé class of finite metric spaces.
Similarly $\mathbf{U}_1$, for diameter ≤ 1.

Theorem (Uspenskij [Usp90])

The isometry groups $\mathrm{Iso}(\mathbf{U})$, $\mathrm{Iso}(\mathbf{U}_1)$, *equipped with the point-wise convergence, are universal Polish groups: every other Polish group embeds in either one homeomorphically.*

Sketch of proof.

1. We may assume that $G \subseteq \mathrm{Iso}(G)$.
2. Follow Katětov's construction of $\mathbf{U}$ as a functorial extension of any separable metric space X, and observe that $\mathrm{Iso}(X) \subseteq \mathrm{Iso}(\mathbf{U})$.

□

...and of the Gurarij space

Definition

The Gurarij space $\mathbf{G}$ is the unique separable, universal, approximately homogeneous Banach space. Equivalently, it is the unique limit of the Fraïssé class of finite-dimensional normed spaces.

Question (Uspenskij)

Is $\mathrm{Iso}_L(\mathbf{G})$, the linear isometry group, a universal Polish group as well?

Theorem (B. 2012?)

Yes.

Sketch of proof.

Same.

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Same. (More or less.) □

Outline

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Katětov functions

Let X, Y be a metric space. All embeddings are isometric.

- We say that a real-valued function ξ on X is Katětov if $\xi(x) \leq \xi(y) + d(x,y)$ and $d(x,y) \leq \xi(x) + \xi(y)$. Equivalently, if $\xi(x) = d(x,*)$ for some $* \in X' \supseteq X$.
- Let $K(X)$ denote the set of Katětov functions on X, equipped with the supremum distance. This is a complete metric space extension of X, where $x \in X$ corresponds to $d(x,\cdot) \in K(X)$.
- Functorial: $Y \hookrightarrow X$ gives rise to $K(Y) \hookrightarrow K(X)$, where ξ goes to $\hat{\xi}(x) = \inf_{y \in Y} d(x,y) + \xi(y)$, and everything composes and commutes as one would expect.
- Functoriality gives rise to a natural embedding $\mathrm{Iso}(X) \hookrightarrow \mathrm{Iso}\big(K(X)\big)$, where the image of $\varphi \in \mathrm{Iso}(X)$ (call it $K(\varphi)$) extends φ.

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Caveat: Katětov functions may "depend" on infinitely many points in X

- If X is separable, $K(X)$ need not be separable.
- The embedding $\mathrm{Iso}(X) \hookrightarrow \mathrm{Iso}\big(K(X)\big)$ need not be continuous.

"Local" Katětov functions

When $Y \subseteq X$, let us identify $K(Y)$ with its image in $K(X)$, and define

$$K_0(X) = \overline{\bigcup_{Y \subseteq X \text{ finite}} K(Y)} = \overline{\bigcup_{Y \subseteq X \text{ compact}} K(Y)} \subseteq K(X).$$

Then everything we said of $K(X)$ hold for $K_0(X)$, plus:

- $w\big(K_0(X)\big) = w(X)$. In particular, if X is separable, so is $K_0(X)$.
- The embedding $\mathrm{Iso}(X) \hookrightarrow \mathrm{Iso}\big(K_0(X)\big)$ is continuous, and therefore homeomorphic.

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- Katětov: construct $X = X_0 \subseteq X_1 = K_0(X_0) \subseteq \ldots$, and $X_\omega = \widehat{\bigcup X_n}$ is the Urysohn space.

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- Katětov: construct $X = X_0 \subseteq X_1 = K_0(X_0) \subseteq \ldots$, and $X_\omega = \widehat{\bigcup X_n}$ is the Urysohn space.
- Uspenskij: then $\mathrm{Iso}(X) = \mathrm{Iso}(X_0) \subseteq \mathrm{Iso}(X_1) \subseteq \ldots$, whence $\mathrm{Iso}(X) \subseteq \mathrm{Iso}(X_\omega) = \mathrm{Iso}(\mathbf{U})$.

Outline

Convex Katětov functions

Katětov functions captures the isomorphism classes of a single point extensions of a metric space. Similarly, single point extensions of a Banach space E are captured by convex Katětov functions.

Lemma

Let E be a Banach space, $y \in F \supseteq E$ a point in an extension, and let $\xi \colon E \to \mathbf{R}$, $\xi(x) = \|y - x\|$. Then ξ is a convex Katětov function on E.

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Let E be a Banach space, $y \in F \supseteq E$ a point in an extension, and let $\xi \colon E \to \mathbf{R}$, $\xi(x) = \|y - x\|$. Then ξ is a convex Katětov function on E. Conversely, ξ determines the norm on $\langle E, x\rangle$, and every convex Katětov function on E arises in this manner.

Notation

Let X be a convex subset of a normed space E. We denote by $K_C(X)$ the space of convex Katětov functions on X.

Compatibility between convex functions and Katětov functions

Notation

Let X be a convex subset of a normed space E. We denote by $K_C(X)$ the space of convex Katětov functions on X.

- If $X \subseteq Y \subseteq E$ are convex, then the inclusion $K(X) \subseteq K(Y)$ restricts to $K_C(X) \subseteq K_C(Y)$. In other words, the natural Katětov extension of a convex Katětov function is again convex.
  In particular, K_C is functorial.
- Let $\xi \in K(X)$, and let ξ^C be the greatest convex function lying below ξ. Then $\xi^C \in K_C(E)$ is again Katětov.

"Local" convex Katětov functions

Definition

$$K_{C,0}(X) = K_C(X) \cap K_0(X).$$

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$$K_{C,0}(X) = \overline{\bigcup\{K_C(Y) \colon \text{convex compact } Y \subseteq X\}} = \{\xi^C : \xi \in K_0(X)\}$$

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$$K_{C,0}(X) \supseteq \overline{\bigcup\{K_C(Y) \colon \textit{convex compact } Y \subseteq X\}} \supseteq \{\xi^C : \xi \in K_0(X)\}$$
$$\supseteq K_{C,0}(X).$$

Proof.

Easy... □

Lemma

$$K_{C,0}(E) = \overline{\bigcup\{K_C(F) \colon \textit{finite-dimensional subspace } F \subseteq E\}}.$$

Proof.

- Enough to show: if $\dim E < \infty$ then $K_C(E) = K_{C,0}(E)$.
- Let $\xi \in K_C(E)$ and $\xi_n = \xi\restriction_{B(0,n)} \in K_{C,0}(E)$. Then $\xi_n \searrow \xi$ point-wise.

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- The Legendre dual ξ^* is continuous on B^*, the closed unit ball of E^*, and ∞ outside:

$$\xi^*(\lambda) = \sup_{x\in E} \lambda x - \xi(x).$$

- We have $\xi_n^* \nearrow \xi^*$ point-wise on B^*.
- Since B^* is compact and ξ^*, ξ_n^* are continuous, $\xi_n^* \nearrow \xi^*$ uniformly.
- It follows that $\xi_n = \xi_n^{**} \to \xi^{**} = \xi$ uniformly on E, so $\xi \in K_{C,0}$. □

Outline

Arens-Eells spaces

At this point, if E is a separable Banach space, then $E \subseteq K_{C,0}(E)$ isometrically, and $\mathrm{Iso}(E) \subseteq \mathrm{Iso}\big(K_{C,0}(E)\big)$ homeomorphically.
But $E \subseteq K_{C,0}(E)$ is not a normed space extension, and we may not define $E_0 = E$, $E_1 = K_{C,0}(E_0)$, ...

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But $E \subseteq K_{C,0}(E)$ is not a normed space extension, and we may not define $E_0 = E$, $E_1 = K_{C,0}(E_0)$, ...

Fact

Let $(X, 0)$ be a pointed metric space. Then there exists a Banach space $\mathrm{AE}(X)$, together with an isometric embedding $X \subseteq \mathrm{AE}(X)$ sending $0 \mapsto 0$, called the Arens-Eells space of X, having the following universal property: every $\theta \in \mathrm{Lip}_0(X, F)$, where F is a Banach space, admits a unique continuous linear extension $\theta' \colon \mathrm{AE}(X) \to F$, and this unique extension satisfies $\|\theta'\| \leq L(\theta)$.

This universal property characterises the Arens-Eells space up to a unique isometric isomorphism. Its dual Banach space $\mathrm{AE}(X)^*$ is canonically isometrically isomorphic to $\mathrm{Lip}_0(X)$, the isomorphism consisting of sending a linear functional to its restriction to X.

In particular, AE is functorial.

Metric space extensions of normed spaces

Of course:

- If $(X, 0)$ is a Banach space, or contains a Banach space E (with $0 = 0_E$), then $E \subseteq \mathrm{AE}(X)$ is not linear.
- If $E \subseteq X \subseteq F$, where $E \subseteq F$ is a (linear) normed space inclusion, then for each $x \in X$, the map $y \mapsto d(x, y)$ is convex on E.

Definition

Let E be a fixed normed space. By a *metric space over* E we mean a metric space X containing E, such that for each $x \in X$ the function $y \mapsto d(y, x)$ is convex on E.

For a normed space F we then define $\mathrm{Lip}_E(X, F)$ to consist of all Lipschitz functions $\theta \colon X \to F$ which are linear on E.

Arens-Eells spaces relative to normed spaces

Theorem (B.)

Let X be a metric space over a normed space E. Then there exists a Banach space $\mathrm{AE}(X, E)$, together with an isometric embedding $X \subseteq \mathrm{AE}(X, E)$ which is linear on E, having the following universal property: every $\theta \in \mathrm{Lip}_E(X, F)$, where F is a Banach space, admits a unique continuous linear extension $\theta' \colon \mathrm{AE}(X, E) \to F$, and this unique extension satisfies $\|\theta'\| \leq L(\theta)$.

This universal property characterises $\mathrm{AE}(X, E)$ up to a unique isometric isomorphism, and we shall call it the Arens-Eells space of X over E. Its dual $\mathrm{AE}(X, E)^*$ is isometrically isomorphic to $\mathrm{Lip}_E(X)$ via restriction to X.

In particular, relative AE is functorial.

Proof.

Quotient $\mathrm{AE}(X, 0)$ by "E embeds linearly", and check some stuff. $\square$

Outline

A Katětov construction of G

Let E be any separable Banach (or normed) space, and define:

- $E_0 = E$.
- $X_n = K_{C,0}(E_n)$.
- $E_{n+1} = \mathrm{AE}(X_n, E_n)$.
- $G_0 = \bigcup E_n$, $G = \widehat{G_0}$.

A Katětov construction of $\mathbf{G}$

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- $E_{n+1} = \mathrm{AE}(X_n, E_n)$.
- $G_0 = \bigcup E_n$, $G = \widehat{G}_0$.

Then

1. G is separable, and for every finite dimensional $F \subseteq F'$, any isometric embedding of F into G_0 extends to one of F'.
2. It follows that $G = \mathbf{G}$, the Gurarij space.

Definition

A *Gurarij space* is a separable Banach space $\mathbf{G}$ having the property that for any $\varepsilon > 0$, finite-dimensional Banach space $F \subseteq F'$, and isometric embedding $\psi\colon F \to \mathbf{G}$, there is a linear embedding $\varphi\colon F' \to \mathbf{G}$ extending ψ such that in addition, for all $x \in F'$, $(1-\varepsilon)\|x\| \leq \|\varphi x\| \leq (1+\varepsilon)\|x\|$.

The Gurarij space exists (Gurarij [Gur66]) and is unique (Lusky [Lus76], and more recently Kubiś-Solecki, B., ...)

Conclusion

- For any Polish group H we may assume that $H \subseteq \mathrm{Iso}(\mathbb{H})$.
- Using ordinary Arens-Eells, we find separable $E \supseteq \mathbb{H}$ such that $\mathrm{Iso}(E) \supseteq \mathrm{Iso}(\mathbb{H})$.
- Now

$$H \subseteq \mathrm{Iso}(\mathbb{H}) \subseteq Iso(E) \subseteq \mathrm{Iso}(E_1) \subseteq \ldots \subseteq \mathrm{Iso}(\mathbf{G}).$$

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Thank you.

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