Chasing robbers on random graphs: zigzag theorem

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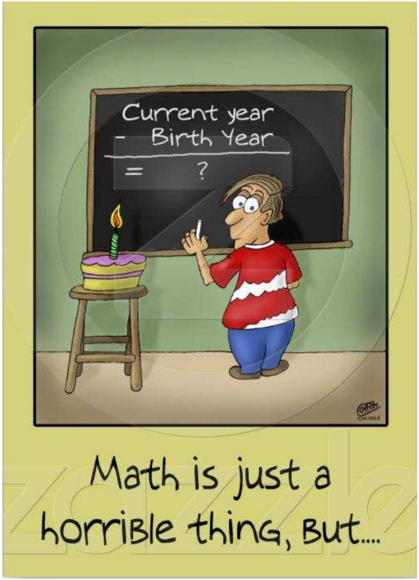
The 3rd Workshop on Graph Searching, Theory and Applications (GRASTA 09)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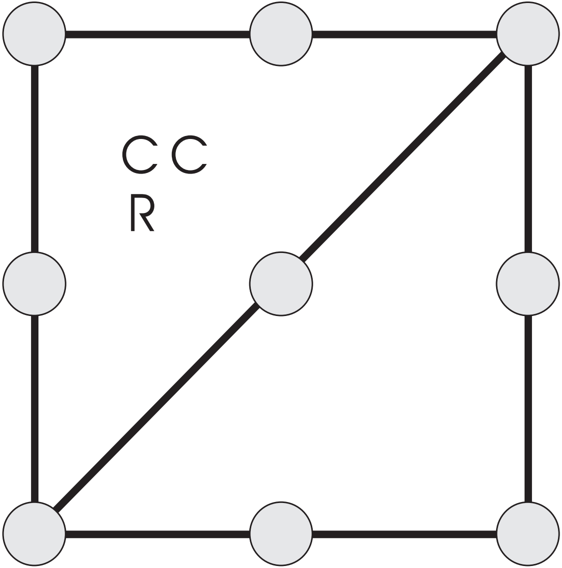
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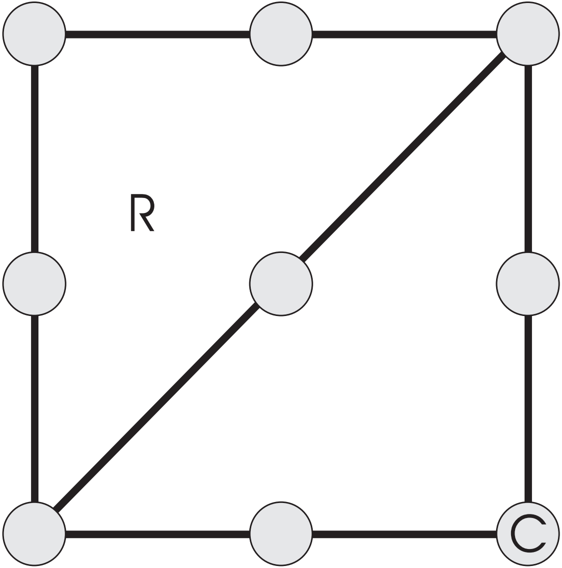
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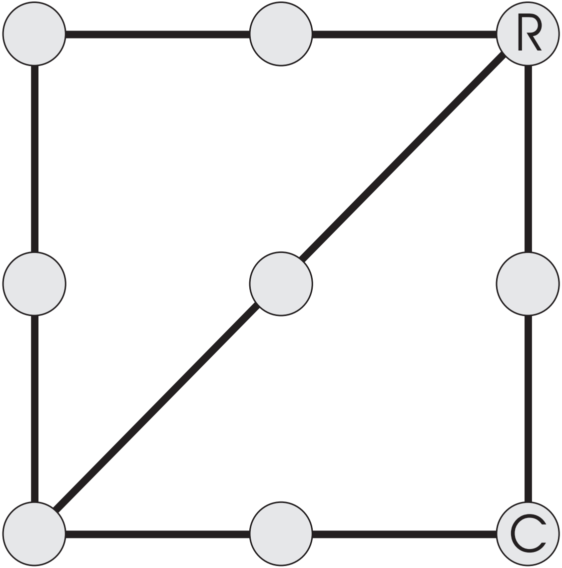
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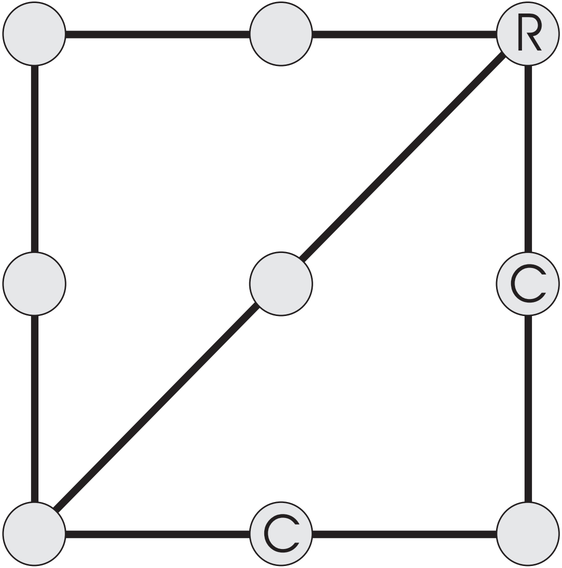
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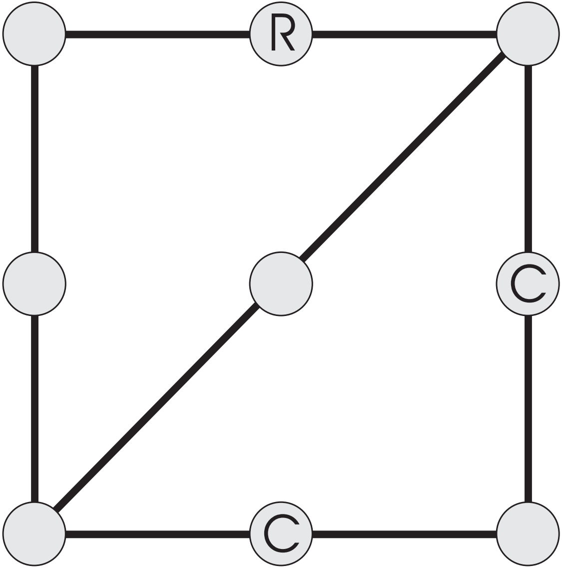
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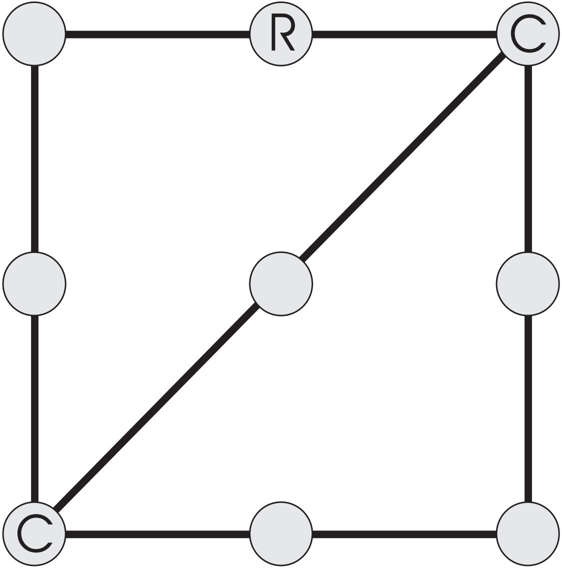
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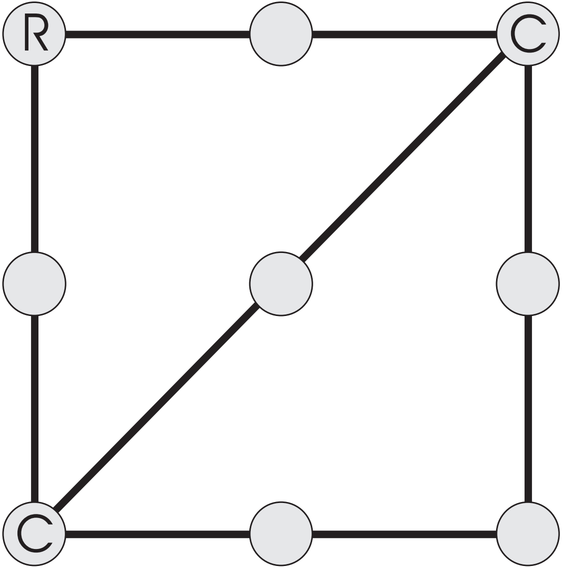
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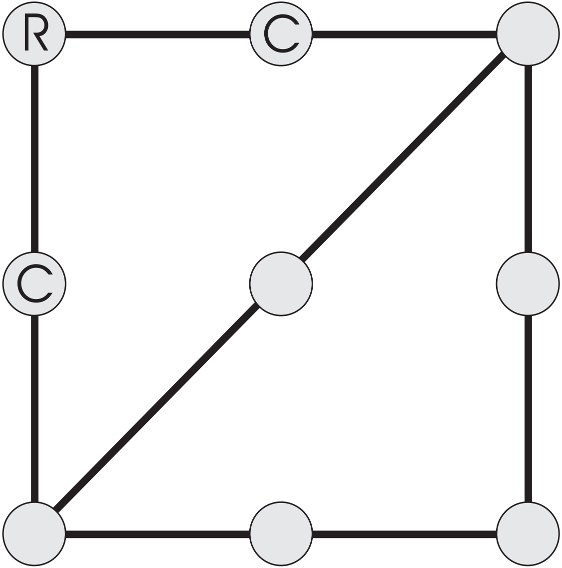
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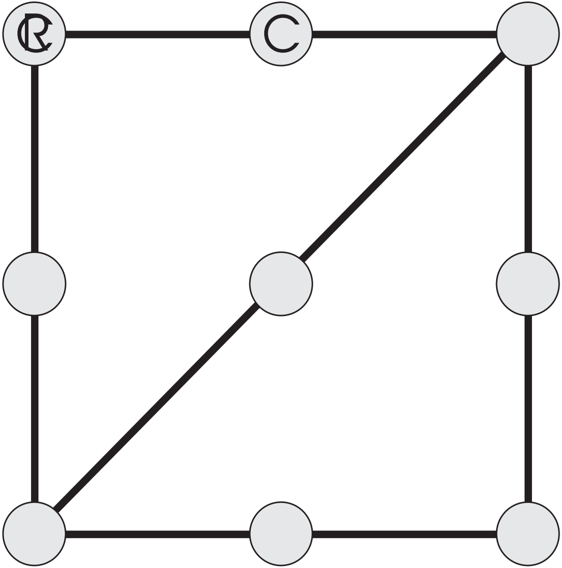
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Definition

As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*, written $c_0(G)$, which is the minimum number of cops needed to win on G.

The game of distance k Cops and Robbers is played in an analogous as is Cops and Robbers, except that the cops win if a cop is within distance at most k from the robber.

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Example

- $c_0(T) = 1$ for any tree T,
- $c_0(K_n) = 1$ for $n \geq 3$,
- $c_0(C_n) = 2$ for $n \geq 4$.

Theorem (Nowakowski, Winkler, 1983)

The cop-win graphs (that is, graphs G with $c_0(G) = 1$) are exactly those graphs which are dismantlable: there exists a linear ordering $(x_j : 1 \leq j \leq n)$ of the vertices so that for all $2 \leq j \leq n$, there is a $i < j$ such that

$$N[x_j] \cap \{x_1, x_2, \ldots, x_j\} \subseteq N[x_i] \cap \{x_1, x_2, \ldots, x_j\}.$$

Characterizations of k-cop-win graphs (Clarke, MacGillivray, 2009+)

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Our main results refer to the probability space $\mathcal{G}(n,p) = (\Omega, \mathcal{F}, \mathbb{P})$ of random graphs, where Ω is the set of all graphs with vertex set $[n] = \{1, 2, \dots, n\}$, $\mathcal{F}$ is the family of all subsets of Ω, and for every $G \in \Omega$

$$\mathbb{P}(G) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}.$$

It can be viewed as a result of $\binom{n}{2}$ independent coin flipping, one for each pair of vertices, with the probability of success (that is, drawing an edge) equal to p ($p = p(n)$ can tend to zero with n).

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Q: When does a random graph have constant cop number?

A: $p(n)$ has to tend to 1 with n.

Theorem (Prałat, 2009+)

Let $k \in \mathbb{Z}_+$ and

$$p = p(n) = 1 - \left(\frac{k \log n + a_n}{n} \right)^{\frac{1}{k}}.$$

Then the following holds:

- *if $a_n \to -\infty$, then a.a.s. $c_0(G(n,p)) \leq k$,*
- *if $a_n \to a \in \mathbb{R}$, then the probability that $c_0(G(n,p)) = k$ tends to $1 - e^{-e^{-a}/k!}$; $c_0(G(n,p)) = k+1$ otherwise,*
- *if $a_n \to \infty$, then a.a.s. $c_0(G(n,p)) \geq k+1$.*

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For $p = p(n) < 1$, define $\mathbb{L}n = \log_{\frac{1}{1-p}} n$.

Theorem (Bonato, Prałat, Wang, 2009)

If $d = pn \geq 2\sqrt{n}\log n$ and $\omega(n)$ is any function tending to infinity, then a.a.s.

$$\mathbb{L}n - \mathbb{L}((p^{-1}\mathbb{L}n)(\log n)) \leq c_0(G(n,p)) \leq \mathbb{L}n + \mathbb{L}(\omega(n)).$$

Upper bound: any set of $K = \mathbb{L}n + \mathbb{L}(\omega(n))$ vertices is a dominating set a.a.s.

Lower bound: graph is $(1,k)$-*existentially closed* for $k = \mathbb{L}n - \mathbb{L}((p^{-1}\mathbb{L}n)(\log n))$. (For each k-set $S \subset V$ and vertex $u \notin S$, there is a vertex $z \notin S$ not joint to a vertex in S and joined to u.)

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Upper bound for $p = o(1)$: the probability that **any** fixed set of

$$k = \lceil \mathbb{L}n + \mathbb{L}(\omega(n)) \rceil = (1+o(1))\frac{\log n}{-\log(1-p)} = (1+o(1))\frac{\log n}{p}$$

vertices is a dominating set is equal to

$$\begin{aligned}\left(1-(1-p)^k\right)^{n-k} &\geq 1-(n-k)(1-p)^k \\ &\geq 1-n(1-p)^k \\ &\geq 1-n(1-p)^{\mathbb{L}n+\mathbb{L}(\omega(n))} \\ &= 1-\frac{1}{\omega(n)} \\ &= 1-o(1).\end{aligned}$$

Lower bound for $p = o(1)$: graph is $(1, k)$-*existentially closed* for $k = \mathbb{L}n - \mathbb{L}\big((p^{-1}\mathbb{L}n)(\log n)\big)$. (For each k-set $S \subset V$ and vertex $u \notin S$, there is a vertex $z \notin S$ not joint to a vertex in S and joined to u.)

Let X be the random variable counting the number of S and u for which no suitable z can be found. We then have that

$$\begin{aligned}
\mathbb{E}(X) &= \binom{n}{k}(n-k)\left(1-p(1-p)^k\right)^{n-k-1} \\
&\leq n^{k+1}\left(1-\frac{(\mathbb{L}n)(\log n)}{n}\right)^{n(1-(\mathbb{L}n)/n)} \\
&= n^{k+1}\exp\left(-(\mathbb{L}n)(\log n)(1-(\mathbb{L}n)/n)\right)(1+o(1)) \\
&\leq n^{k+1}\exp\left(-\left(k+\frac{2\log\log n}{p}\right)(\log n)(1+o(1))\right) = o(1),
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and the bound follows from Markov's inequality.

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Corollary (Bonato, Prałat, Wang, 2009)

If $d = np = n^{\alpha+o(1)}$, *where* $1/2 < \alpha \leq 1$, *then a.a.s.*

$$c_0(G(n,p)) = \Theta(\log n/p) = n^{1-\alpha+o(1)}$$

and $c_0(G(n, n^{-1/2+o(1)})) = n^{1/2+o(1)}$ *a.a.s.*

Let us define the function $f : (0,1) \to \mathbb{R}$ as

$$f(\alpha) = \frac{\log \bar{c}_0(G(n, n^{\alpha-1}))}{\log n},$$

where $\bar{c}_0(G(n,p))$ denotes the most likely value of the cop number for $G(n,p)$.

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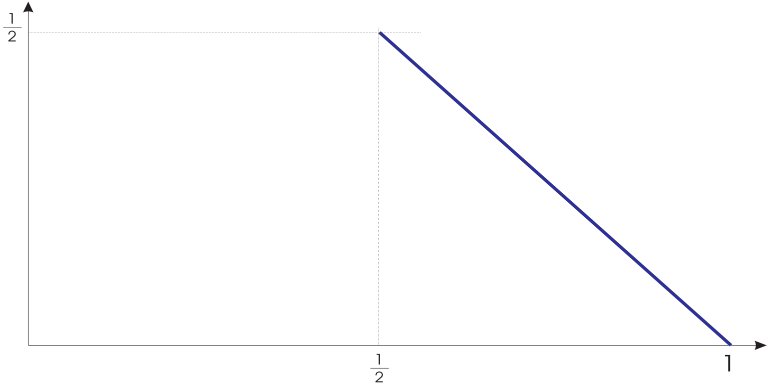

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Outline

Theorem (Bollobás, Kun, Leader, 2009+)

If $p(n) \geq 2.1 \log n/n$, then a.a.s.

$$\frac{1}{(np)^2} n^{\frac{1}{2} \frac{\log\log(np)-9}{\log\log(np)}} \leq c_0(G(n,p)) \leq 160000\sqrt{n}\log n\,.$$

Since if either $np = n^{o(1)}$ or $np = n^{1/2+o(1)}$ then a.a.s. $c_0(G(n,p)) = n^{1/2+o(1)}$, it would be natural to assume that the cops number of $G(n,p)$ is close to $\sqrt{n}$ also for $np = n^{\alpha+o(1)}$, where $0 < \alpha < 1/2$.

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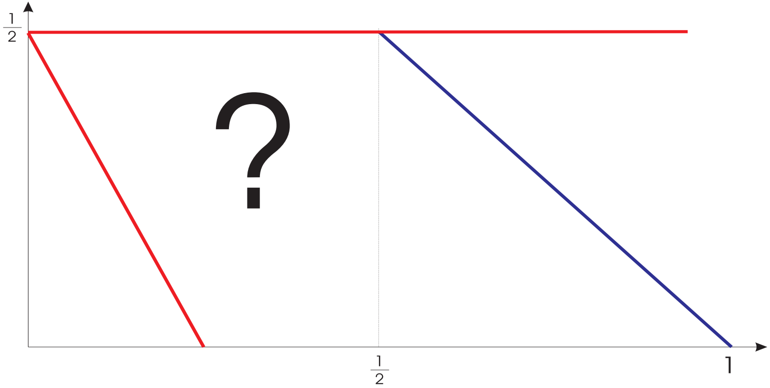
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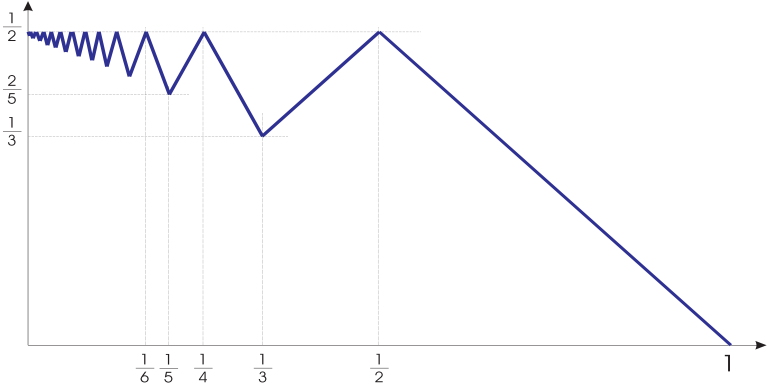

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Theorem (Łuczak, Prałat, 2009+)

Let $0 < \alpha < 1$ *and* $d = d(n) = np = n^{\alpha+o(1)}$.

1. *If* $\frac{1}{2j+1} < \alpha < \frac{1}{2j}$ *for some* $j \geq 1$*, then a.a.s.*

$$c_0(G(n,p)) = \Theta(d^j)\,.$$

2. *If* $\frac{1}{2j} < \alpha < \frac{1}{2j-1}$ *for some* $j \geq 1$*, then a.a.s.*

$$\Omega\left(\frac{n}{d^j}\right) = c_0(G(n,p)) = O\left(\frac{n}{d^j}\log n\right)\,.$$

We get a good upper estimate for $c_0(G(n,p))$ also for $d = n^{1/k+o(1)}$ ($k = 2, 3, \ldots$), and our argument for lower bound can be repeated in this case to determine $c_0(G(n,p))$ up to $\log^{O(1)} n$ factor in the whole range of p, provided $n^{\varepsilon-1} \leq p \leq n^{-\varepsilon}$ for some $\varepsilon > 0$.

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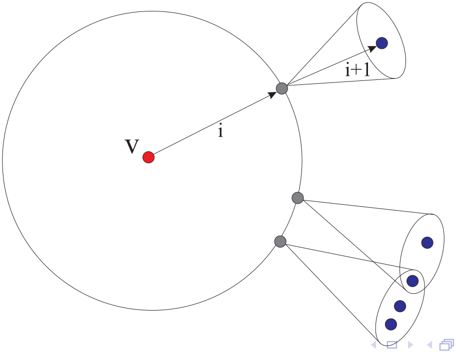
Upper bound

Assume first that $(n \log n)^{1/(2i+1)} \leq d \leq n^{1/(2i)}$. We place $5000(10d)^i$ cops uniformly at random on vertices of $G(n,p)$. Then, the robber selects his vertex v. Now, we assign to each vertex u in $N_i(v) \setminus N_{i-1}(v)$ the unique cop that occupies a vertex in $N_{i+1}(u)$.



If this can be done, then cops assigned to vertices are moving into their destinations and after $i + 1$ steps the robber is surrounded. Finally, the cops move towards the robber eventually capturing him.

In order to show that the above strategy is a.a.s. winning, we use Hall's theorem for matchings in bipartite graphs.

One can mimic the above argument to show that if $n^{1/(2i+2)} \leq d \leq n^{1/(2i+1)}$, then a.a.s. $5000 n \log n / (0.1 d)^{i+1}$ cops can win the game.

The difference in the estimates of the cop number follows from the fact that in the immediate pursuit strategy only cops who are within distance $2i+1$ from the robber are 'active', that is, they can take part in the chase. In the previous case all but a small fraction of cops were active.

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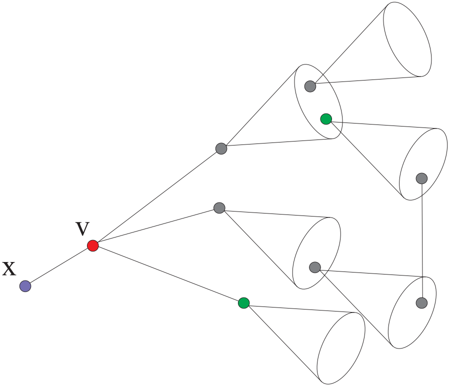
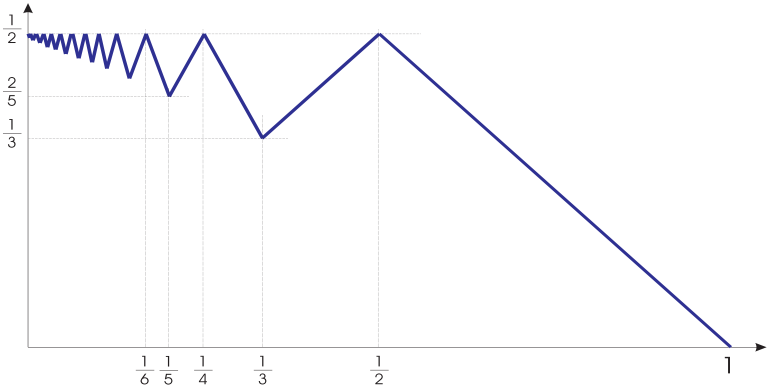
Lower bound: vertex v is safe, if for some 'deadly' neighbour x of v we have $C_0^x(v) = 0$, and for every $i = 1, 2, \ldots, j$

$$C_{2i-1}^x(v), C_{2i}^x(v) \leq \left[\frac{d}{3cj}\right]^i.$$




1/2
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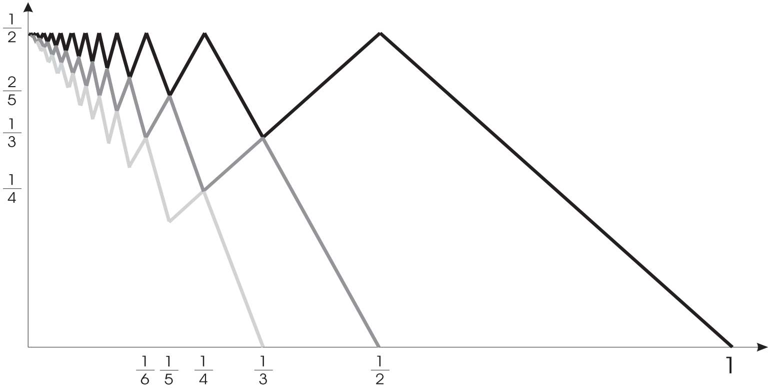
1

Distance k Cops and Robbers



(joint work with Bonato and Chiniforooshan)

Outline

It follows from the Zigzag Theorem that the cop number of $G(n,p)$ is always $O(\sqrt{n}\log n)$ provided the graph is connected. It supports then the conjecture of Meyniel that $c_0(G) = O(\sqrt{|V|})$ for any connected graph G which would be best possible.

Theorem (Frankl, 1987)

$$c_0(n) = O\left(\frac{n \log\log n}{\log n}\right)$$

Theorem (Bonato, Chiniforooshan, Prałat, 2009+)

For integers $n > 0$ and $k \geq 0$ (where k can be a function of n)

$$c_k(n) = O\left(\frac{n}{\log\left(\frac{2n}{k+1}\right)}\frac{\log(k+2)}{k+1}\right)$$

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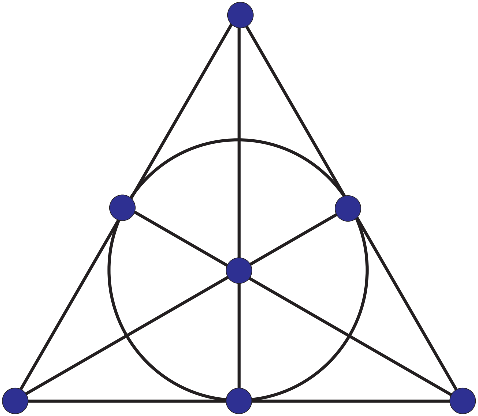
Theorem (Lu, Peng, 2009+)

$$c_0(n) \le n2^{-(1-o(1))\sqrt{\log_2 n}} = n^{1-o(1)}$$

According to the combinatorial definition, a *projective plane* consists of a set of *lines* and a set of *points* with the following properties:

- Given any two distinct points, there is exactly one line incident with both of them.
- Given any two distinct lines, there is exactly one point incident with both of them.
- There are four points such that no line is incident with more than two of them.

Below, we present the Fano plane, the projective plane with the least number of points and lines: 7 each.



One can show that a projective plane has the same number of lines as it has points. A finite projective plane has $q^2 + q + 1$ points, where q is an integer called the *order* of the projective plane.

It has been shown that there exists a finite projective plane of order q, if q is a prime power (that is, $q = p^a$ for a prime number p and $a \geq 1$), and for all known finite projective planes, the order q is a prime power. The existence of finite projective planes of other orders is an open question.

Finally, for a fixed prime power q, let $G_q = (P, L, E)$ be a bipartite graph with bipartition P, L where P and L denote the set of points and, respectively, lines in the projective plane. A point is joined to a line if it is contained in it. Then G_q has $2(q^2 + q + 1)$ many vertices and is $(q+1)$-regular.

Theorem (Prałat, 2009+)

$c_0(G_q) = q + 1.$

Corollary (Prałat, 2009+)

There is an infinite family of graphs $\{\hat{G}_n = ([n], E)\}$ with $c_0(\hat{G}_n) > \sqrt{n/8}$ and $c_0(\hat{G}_n) > \sqrt{n/2} - n^{0.2625}$ for n sufficiently large.

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