On multi-covers and their applications

Marek Tesař

Charles University (Prague, Czech Republic)

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Outline

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Definition (Homomorphism)

Let G and H be graphs. A ***homomorphism*** of G to H is a mapping $f : V(G) \to V(H)$ that preserves edges, i.e.

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Definition (Partial covering)

Let G and H be graphs. A homomorphism $f : V(G) \to V(H)$ is ***partial covering***, if mapping $f|_{N_G(v)} : N_G(v) \to N_H(f(v))$ is injective for every $v \in V(G)$.

Note that covering projection is also known as ***locally bijective homomorphism***.
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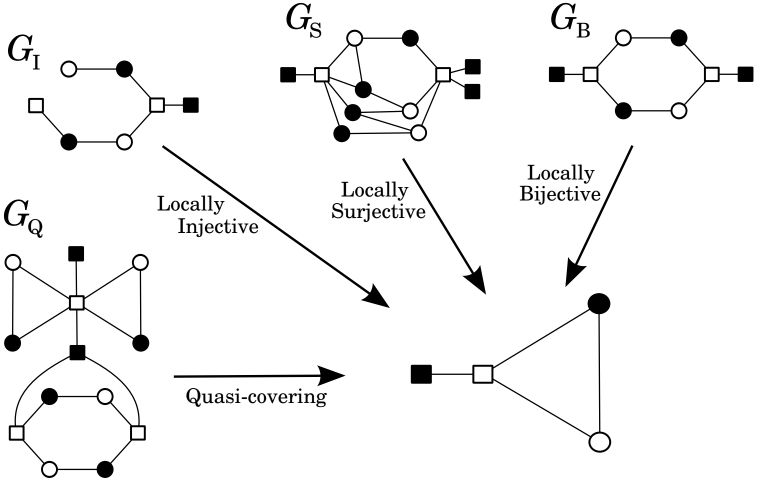

H-COVER problem

For every fixed graph H we can define the following decision problem:

Problem: H-COVER
Input: Graph G
Question: Does there exist a covering projection from G to H?

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1. Background
2. Known results
3. Generalization for non-regular graphs

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- Using Linear Algebra
- Degree Refinement Matrix (unique neighbor property)

DEGREE PARTITION of graph

Degree partition of graph G is a partition of vertices $V(G)$ into classes $B_1, B_2, \ldots, B_k$, s.t. there exist numbers $r_{i,j}$ s.t. $\forall i, j \in \{1, \ldots, k\}, \forall u \in B_i$ the number of edges incident with u and ending in B_j is $r_{i,j}$.

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Lemma

If G covers H then G and H have the same degree refinement matrix.

Theorem (J. Fiala, J. Kratochvíl, A. Proskurowski, J. A. Telle)

Let H be simple connected r-regular graph. If $r \leq 2$ then H-LBHOM is polynomially solvable. Otherwise it is NP-complete.

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Definition (of multi-cover)

Let H be r-regular graph. We say that r-regular graph G with specified vertex u is a multi-cover of H if for every vertex $x \in V(H)$ and every permutation $\varphi\colon N_G(u) \to N_H(x)$ there exists a covering projection $f\colon G \to H$ such that $f|_{N_G(u)} = \varphi$ and $f(u) = x$.

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Let G with specified vertex u is a multi-cover of r-regular graph H. Consider a partial covering $f\colon G_u \to H$. If all pendant vertices of G_u are mapped to the same vertex $x \in V(H)$ and all their (unique) neighbors in G_u are mapped to the different neighbors of x in H, we say that the partial covering f is ***good***, and ***bad*** otherwise.

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We are interested in gadgets G_u that do not allow any bad partial coverings. Kratochvíl et. al. use such a gadget in the proof that H-COVER is NP-complete for all r-regular graphs with $r \geq 3$. They reduce NP-hardness from the following problem:

Problem: Coloring of r-regular $(r-1)$-uniform hyper-graphs
Input: r-regular $(r-1)$-uniform hyper-graph F
Question: Does there exist a coloring of hyper-edges of F s.t. no vertex belongs to two or more hyper-edges of the same color?

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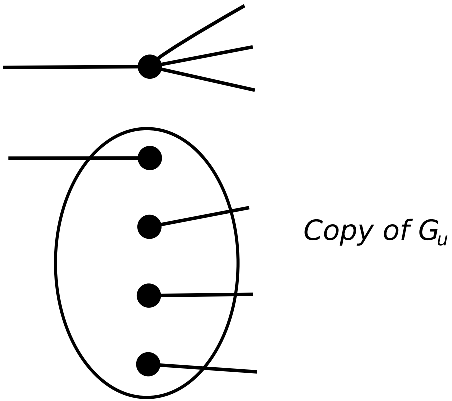
Let F be a r-regular $(r-1)$-uniform hyper-graph. We define r-regular graph G_F s.t. G_F covers H iff F is r-edge colorable.

For every vertex v of F we add the following gadget to G_F:



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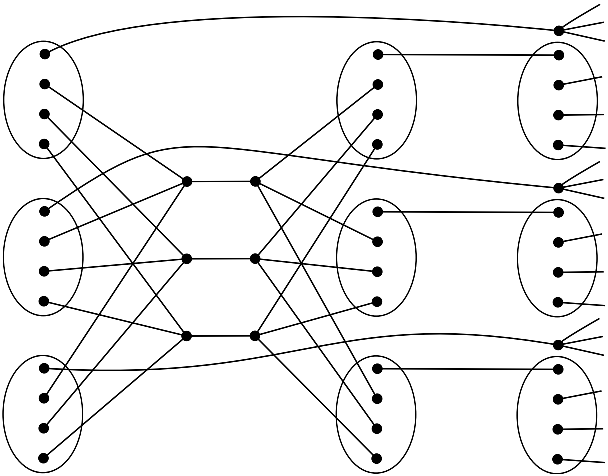

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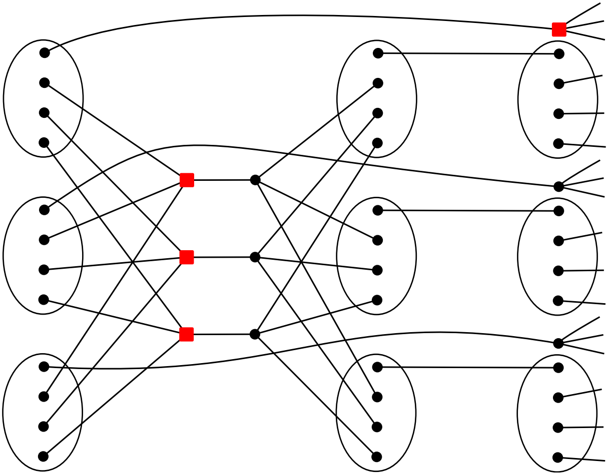

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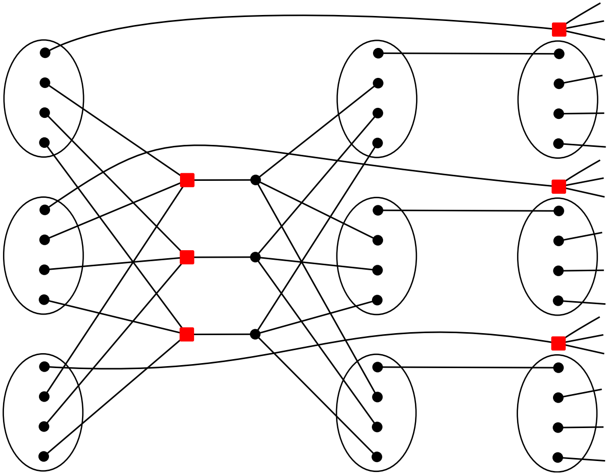

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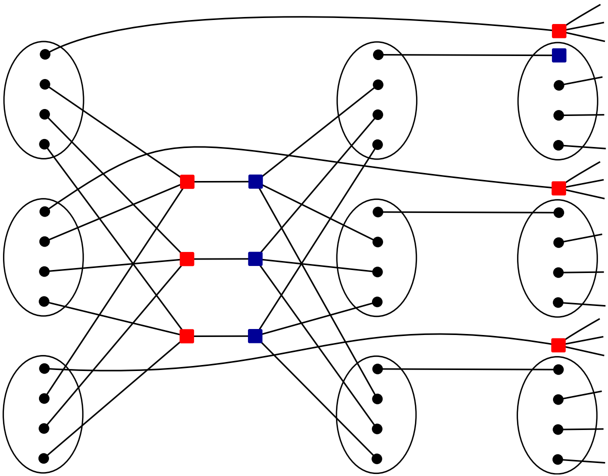

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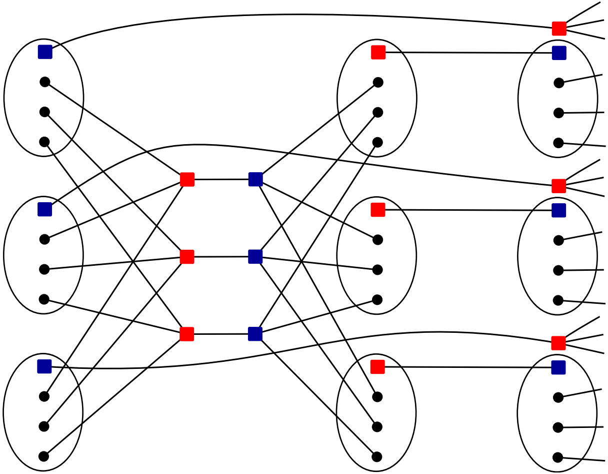

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Definition (of multi-cover for general graphs)

Let H be graph with blocks $B_1, B_2, \ldots, B_k$. We say that graph G with specified vertex u is a multi-cover of H for block B_i if for every vertex $x \in B_i$ and every permutation $\varphi\colon N_G(u) \to N_H(x)$ that respects block structure of H there exists a covering projection $f\colon G \to H$ such that $f|_{N_G(u)} = \varphi$ and $f(u) = x$.

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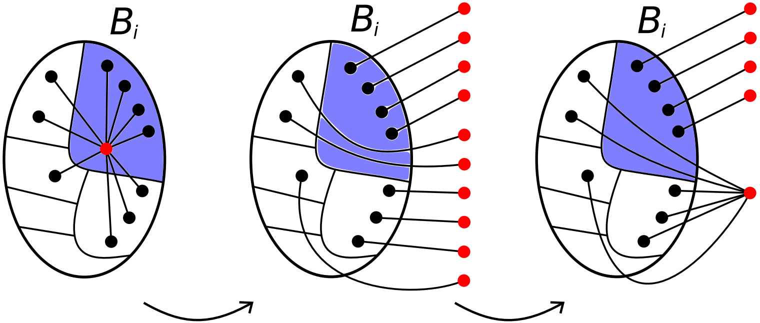
Does multi-cover exist for every graph H and every block B_i? - YES

Does there exist multi-cover G s.t. G_u only allows good partial coverings?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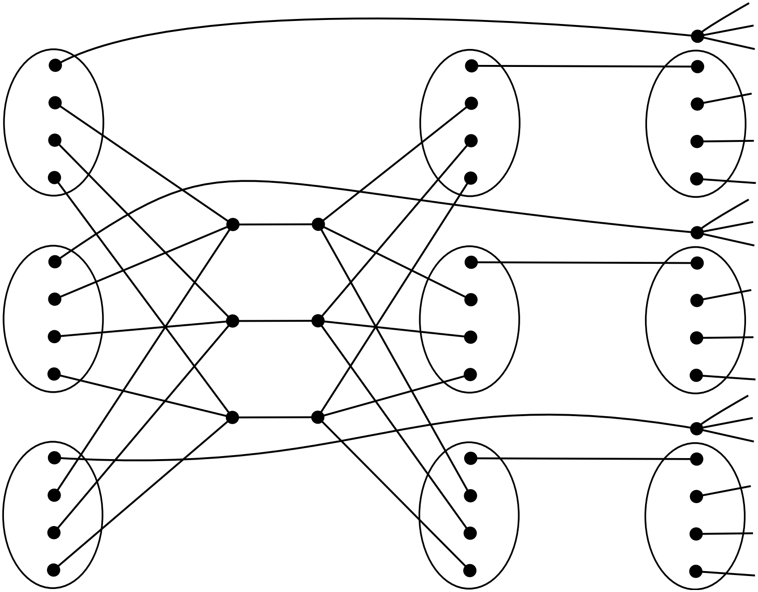
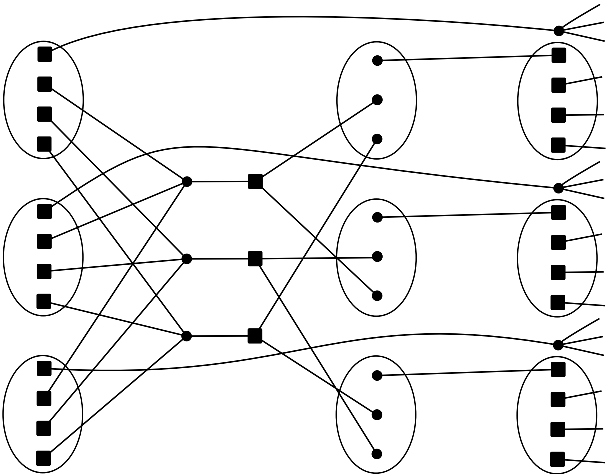


Similar reduction can also be used in case if for some $i \neq j$ are $r_{i,j} \geq 3$ and $r_{j,i} \geq 3$ (degrees of vertices in bipartite graph induced on blocks B_i and B_j).



Thank you!!