Covering constructions of extremal graphs of given degree and diameter, or girth

Jozef Širáň

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Aim: To give a brief survey of lifting constructions in both problems.

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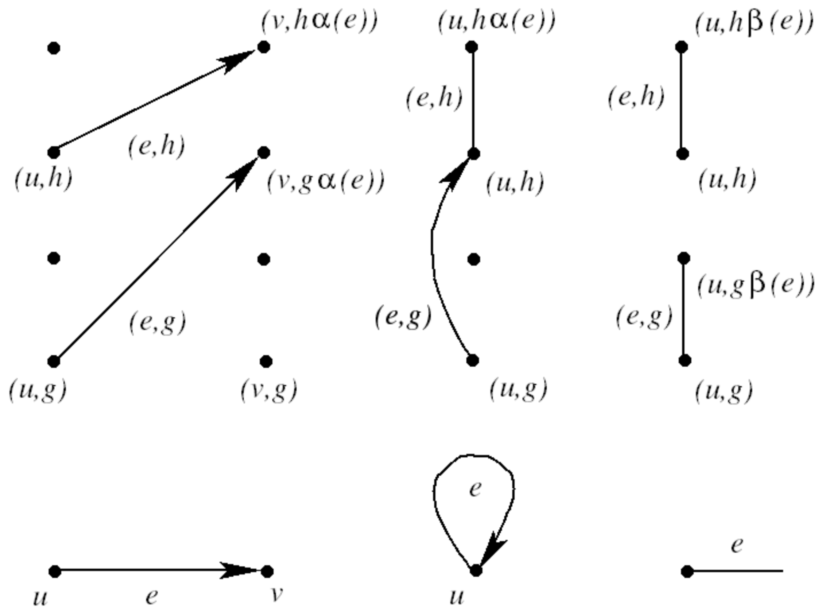

(v,hα(e))
(e,h)
(u,h)
(v,gα(e))
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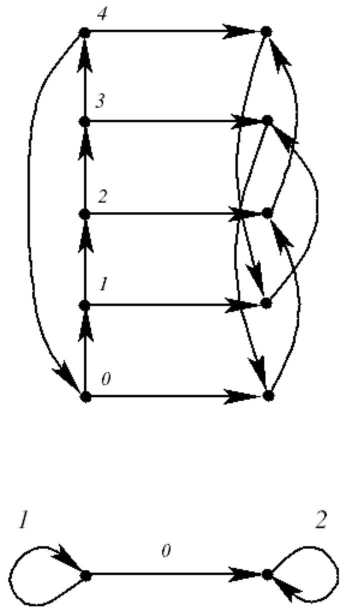
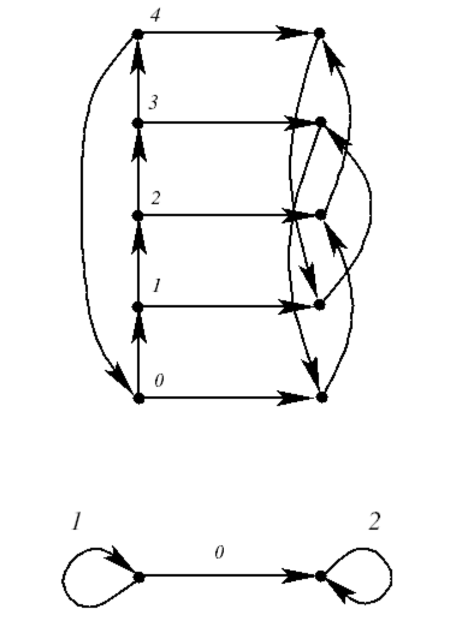
In particular, Cayley graphs are precisely lifts of one-vertex graphs.


4
3
2
1
0
1
0
2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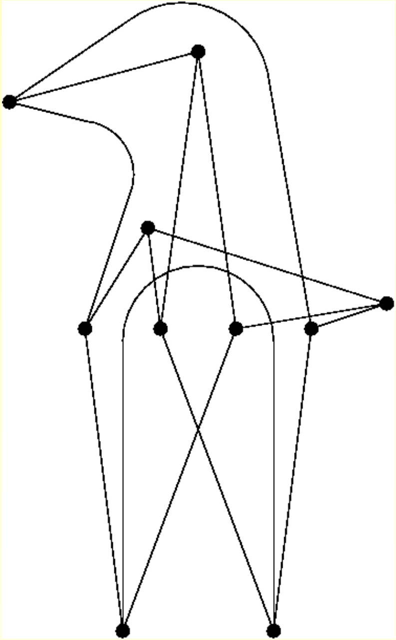

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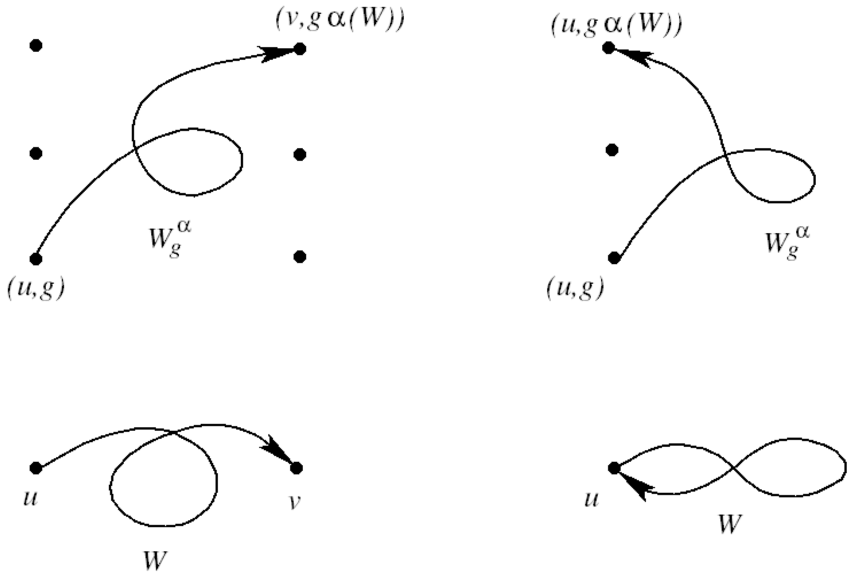
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W_g^α
(u,g)
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v
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u
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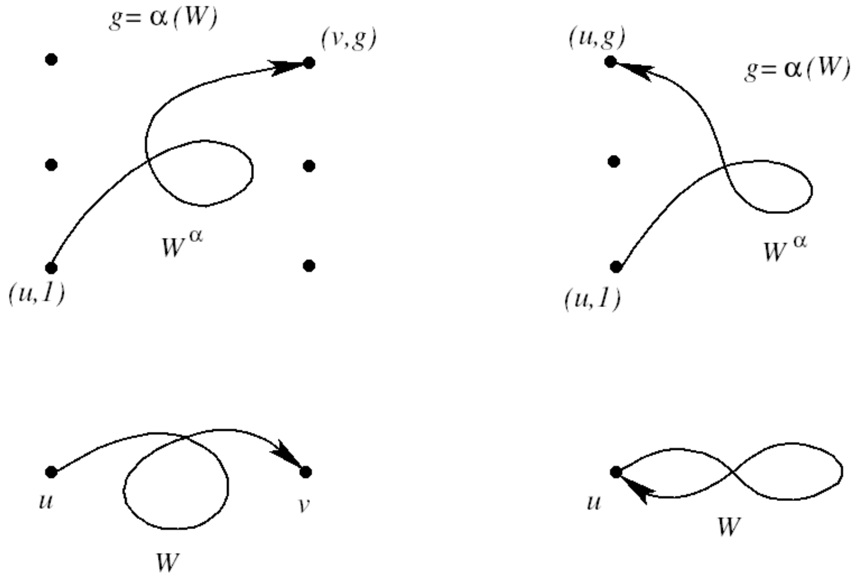
- Γ^α *has girth $\geq \ell$ iff every noncontractible closed walk of length $< \ell$ has nonidentity voltage.*



$g = \alpha(W)$
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Best bound on $Cay(d,k)$; also on $vt(d,k)$ for $c(d+3)/2 \leq k \leq (d+1)/2$ for $c \in (0,1)$ s.t. $(2e/3)^c(1-c)^{1-c} = 1$; $c \approx 0.61834\ldots$ MŠŠV 2010

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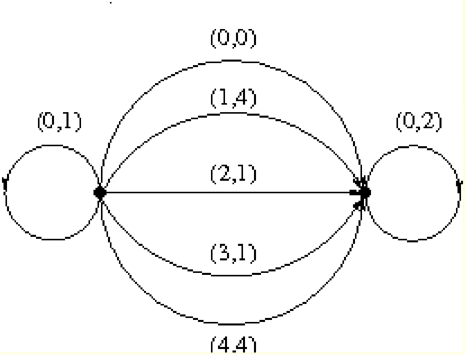
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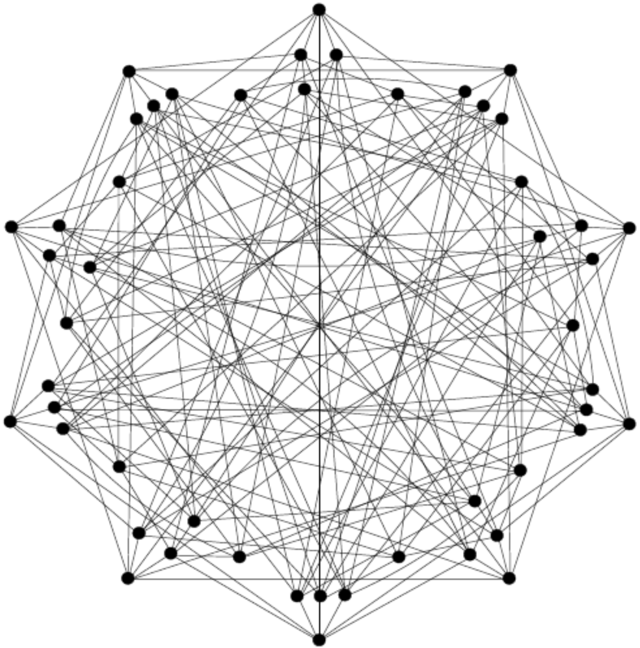

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The lift $(\Delta_q^*)^\alpha$ gives $vt(d, 2) \geq \frac{8}{9}(d + \frac{1}{2})^2$ for $d = (3q-1)/2$.

Šiagiová 2001; McKay, Miller, Š 1998

Γ - a complete graph, $V = F^\times$ for even q, a semiedge σ_r at every $r \in V$

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The lift Γ^α of degree q and order $q(q-1) \approx d^2$ just 'narrowly fails' to have diameter two... can be fixed by increasing the degree by $O(\sqrt{q})$

If done carefully, this results in Cayley graphs of order $d^2 - o(d^2)$ for $d = q + O(\sqrt{q})$, q even.

Large vt and Cayley graphs of diameter two

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Computational results on lifts

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Fact. *Let α be a voltage assignment in a group G on a bouquet B_t such that the lift B_t^α has diameter k. Further, let the set of voltages in α be preserved by b automorphisms of G. The probability that a randomly chosen voltage assignment on B_t gives a lift of diameter k is at least*

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THANK YOU.