On the Riemann-Hurwitz formula for graphs

Alexander Mednykh

Sobolev Institute of Mathematics
Novosibirsk State University
Univerzita Mateja Bela

Winter School in Harmonic Functions on Graphs
and Combinatorial Designs

20 - 24 January, 2014

Riemann-Hurwitz formula for graphs

Recall the classical Riemann-Hurwitz formula. Given surjective holomorphic map $\varphi : S \to S'$ between Riemann surfaces of g and g', respectively, one has

$$2g - 2 = \deg(\varphi)(2g' - 2) + \sum_{x \in S}(r_\varphi(x) - 1), \tag{1}$$

where $r_\varphi(x)$ denotes the ramification index of φ at x. Let G be a finite group of conformal automorphisms acting on S and $\varphi : S \to S' = S/G$ is the canonical map induced by the group action. Then the above formula can be rewritten in the form

$$2g - 2 = |G|(2g' - 2) + \sum_{x \in S}(|G^x| - 1), \tag{2}$$

where G^x stands for the stabiliser of x in G and $|G^x| = r_\varphi(x)$ is the order of a stabiliser.

Remark that S has only finite number of points with non-trivial stabiliser.

Riemann-Hurwitz formula for graphs

The latter formula has a natural discrete analogue. By a graph we mean a finite connected multigraph without loops. We define genus of graph X as $g = |E(G)| - |V(G)| + 1$, that is as cyclomatic number of G. Let G be a finite group acting on graph X without fixed and invertible edges. Denote by g' genus of the factor graph $X' = X/G$. Then by [Baker-Norine, 2009] we have

$$g - 1 = |G|(g' - 1) + \sum_{x \in V(X)} (|G^x| - 1), \tag{3}$$

where $V(X)$ is the set of vertices of X.

The aim of present lecture is to extend this result to group actions with fixed and invertible edges.

We say that a group G acts on X if G is a subgroup of $\mathrm{Aut}(X)$.

Let G be a finite group acting on the graph X. An edge $\{x, \bar{x}\} \in E(X)$ consisting of two semi-edges x and $\bar{x}$ is said to be invertible (or reversible) by G if there is an element $g \in G$ such that g sends x to $\bar{x}$ and $\bar{x}$ to x.

An edge $\{x, \bar{x}\} \in E(X)$ is said to be fixed by G if there is a non-trivial element $g \in G$ that fixes x and $\bar{x}$.

We say that G acts on X without edge reversing if X has no edges invertible by G. Also, G acts on X without fixed edges if X has no edges fixed by G.

Groups acting on a graph without edge reversing

Our first result is the following theorem for groups acting on a graph without edge reversing.

Theorem 1 (M., 2013)

Let X be a graph of genus g and G is a finite group acting on X without edge reversing. Denote by $g(X/G)$ genus of the factor graph X/G. Then

$$g-1=|G|(g(X/G)-1)+\sum_{v\in V(X)}(|G^{v}|-1)-\sum_{e\in E(X)}(|G^{e}|-1),$$

where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of X, G^{x} stands for the stabiliser of $x\in V(X)\cup E(X)$ in G and $|G^{x}|$ is the order of a stabiliser.

Proof. Prescribe to every $\tilde{x} \in V(X/G) \cup E(X/G)$ a group $G_{\tilde{x}}$ isomorphic to G^x, where x is one of the preimages $\tilde{x}$ under the canonical map $\varphi : X \to X/G$. Since G acts transitively of fibres of φ the group $G_{\tilde{x}}$ is well defined. One can consider the graph X/G with prescribed groups G_v, $v \in V(X/G)$ and G_e, $e \in V(X/G)$ as a graph of groups in sense of the Bass-Serre theory. We note that the fibre $\varphi^{-1}(\tilde{x})$ of $\tilde{x}$ consists of $\frac{|G|}{|G_{\tilde{x}}|}$ elements. Hence,

$$|V(X)| = \sum_{v \in V(X)} 1 = \sum_{\tilde{v} \in V(X/G)} \frac{|G|}{|G_{\tilde{v}}|} \quad (4)$$

and

$$|E(X)| = \sum_{e \in E(X)} 1 = \sum_{\tilde{e} \in V(X/G)} \frac{|G|}{|G_{\tilde{e}}|}. \quad (5)$$

By definition of genus from (4) and (5) we obtain

$$
\begin{aligned}
g-1 &= |E(X)|-|V(X)| = \sum_{\bar{e}\in V(X/G)} \frac{|G|}{|G_{\tilde{e}}|} - \sum_{\tilde{v}\in V(X/G)} \frac{|G|}{|G_{\tilde{v}}|} \\
&= |G|\Big(\sum_{\tilde{e}\in E(X/G)} 1 - \sum_{\tilde{v}\in V(X/G)} 1\Big) \\
&+ \sum_{\tilde{e}\in E(X/G)} \frac{|G|}{|G_{\tilde{e}}|}(1-|G_{\bar{e}}|) - \sum_{\tilde{v}\in V(X/G)} \frac{|G|}{|G_{\tilde{v}}|}(1-|G_{\tilde{v}}|) \\
&= |G|(g(X/G)-1) + \sum_{e\in E(X)} (1-|G^{e}|) - \sum_{v\in V(X)} (1-|G^{v}|) \\
&= |G|(g(X/G)-1) + \sum_{v\in V(X)} (|G^{v}|-1) - \sum_{e\in E(X)} (|G^{e}|-1).
\end{aligned}
$$

Let now G be a finite group acting on a graph X, possibly with invertible edges. An edge $e \in V(X)$ with endpoints $\{u, v\}$ is invertible by G if there is an element $g \in G$ that sends e to e, u to v and v to u. We say the group G acts on a graph X with inversions (or with edge reversing), if X has an invertible edge.

In this case, there are three different ways to define the factor graph X/G.

1°. *The factor graph with loops* $(X/G)_{loop}$.

2°. *The factor graph with semi-edges* $(X/G)_{tail}$

3°. *The factor graph without semi-edges* $(X/G)_{free}$.

We have the following result.

Theorem 2 (M., 2013)

Let X be a graph of genus g and G is a finite group acting on X, possibly with edge reversing. Denote by $\gamma = g(X/G)_{tail}$ genus of the factor graph $(X/G)_{tail}$. Then

$$g-1=|G|(\gamma-1)+\sum_{v\in V(X)}(|G^{v}|-1)-\sum_{e\in E(X)}(|G^{e}|-1)+\sum_{e\in E^{inv}(X)}|G^{e}|,$$

where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of X, G^{x} is the stabiliser of $x\in V(X)\cup E(X)$ in G, and $E^{inv}(X)$ is the set of invertibile edges of X.

Harmonic group action on graphs

Let finite group G acts *harmonically* on a graph X, that is it acts free on the set of drecyed edges of X. Then $|G^e| = 1$ for each $e \in E(X)$. We have the following corollary from the previous theorem (See also Baker-Norine, 2009 and Corry, 2011).

Corollary

Let X be a graph of genus g and G is a finite group acting on X harmonically, possibly with edge reversing. Denote by $g(X/G)_{free}$ genus of the factor graph $(X/G)_{free}$. Then

$$g - 1 = |G|(g(X/G)_{free} - 1) + \sum_{v \in V(X)} (|G^v| - 1) + |E^{inv}(X)|,$$

where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of X, G^v is the stabiliser of $v \in V(X)$ in G, and $E^{inv}(X)$ is the set of invertible edges of X.

Harmonic group action on graphs

Recall some classical results for Riemann surface theory. For each $g \geq 2$ define

$$N(g) := \max\{|\mathrm{Aut}(S_g)| : \ S_g \text{ is a compact Riemann surface of genus } g\}.$$

Then

$$8(g+1) \leq N(g) \leq 84(g-1),$$

and these bounds are sharp in the sense that both the upper and lower bound are attained for infinitely many values of g. The upper bound was found by Hurwitz (1893). The lower bound was independently obtained by R. Accola (1968) and C. Maclachlan (1969).

Harmonic group action on graphs

Denote by $N(g)$ maximum size of a finite group acting harmonically on a graph of genus $g \geq 2$.

Theorem (Scott Corry, 2011)

For $g \geq 2$ we have

$$4(g-1) \leq N(g) \leq 6(g-1).$$

The upper and lower bound are attained for infinitely many values of g.

Recent paper by Scott Corry (2013) states that maximal graph groups G with $|G| = 6(g-1)$ are exactly the finite quotients of the modular group $\Gamma = < x, y | x^2 = y^3 = 1 >$ of size at least 6.

Oikawa theorem

In 1956 Kotaro Oikawa proved the following theorem.

Theorem (Oikawa, 1956)

Let S_g be a closed Riemann surface of genus g and A is a finite subset of S_g consisting of $|A| \geq 1$ elements. Suppose that $2g - 2 + |A| > 0$ and G is a group of conformal automorphisms of S_g leaving the set A invariant. Then

$$|G| \leq 12(g-1) + 6|A|.$$

In the next section we find a discrete version of the Oikawa's. Again, the key point of the proof is the Riemann-Hurwitz relation.

Oikawa's theorem for graphs

Our result for graphs is the following theorem.

Theorem 3 (R. Nedela, A. Mednykh, 2013)

Let X be a graph of genus g and A is a subset of vertices of X consisting of $|A| \geq 1$ elements. Suppose that $g - 1 + |A| > 0$ and G is a finite group acting on X harmonically and leaving the set A invariant. Then

$$|G| \leq 2(g-1) + 2|A|.$$

The upper bound is sharp and is attained for arbitrary large values of g and $|A|$. So, at least infinitely many often.

Two Arakawa's theorems

Now our aim is to find discrete versions of two Arakawa's theorems (2000).

The first one states that if G be a finite group of automorphisms of a compact Riemann surface X of genus $g \geq 2$ and A and B are two disjoint G-invariant subsets of X of the orders $|A| \geq |B| \geq 1$ then $|G| \leq 8(g-1) + |A| + 4|B|$.

The second theorem asserts that if A, B and C are three disjoint the G-invariant subsets of X with $|A| \geq |B| \geq |C| \geq 1$ then $|G| \leq 2(g-1) + |A| + |B| + |C|$.

We present a discrete version of the first Arakawa's theorem by the following theorem.

Theorem 4 (R. Nedela, A. Mednykh and I. Mednykh 2013)

Let X be a graph of genus $g \geq 2$ and A and B are two disjoint subsets of vertices of X of the orders $|A| \geq |B| \geq 1$. Suppose that G is a finite group acting harmonically on X and leaving the sets A and B invariant. Then

$$|G| \leq \frac{3(g-1) + |A| + 3|B|}{2}.$$

Again, the upper bound is sharp and is attained for arbitrary large values of g and s.

Two Arakawa's theorems

A discrete version of the second Arakawa's theorem is given by the following theorem.

Theorem 5 (R. Nedela, A. Mednykh and I. Mednykh, 2013)

Let X be a graph of genus $g \geq 2$ and A, B and C are three disjoint subsets of vertices of X of the orders $|A| \geq |B| \geq |C| \geq 1$. Suppose that G is a finite group acting harmonically on X and leaving the sets A, B and C invariant. Then

$$|G| \leq \frac{g - 1 + |A| + |B| + |C|}{2}.$$

As in the two previous theorems, the upper bound is sharp and is attained for arbitrary large values of g and s.

Klein's quartic curve, $x^3y + y^3z + z^3x = 0$, admits the group $\mathrm{PSL}_2(7)$ as its full group of conformal automorphisms. It is characterised as the curve of smallest genus realising the upper bound $84(g-1)$ on the order of a group of conformal automorphisms of a curve of genus $g > 1$, given by A. Hurwitz in 1893. Around the same time, A. Wiman (1895) characterised the curves $w^2 = z^{2g+1} - 1$ and $w^2 = z(z^{2g} - 1), g > 1$, as the unique curves of genus g admitting cyclic automorphism groups of the largest and the second largest possible order ($4g + 2$ and $4g$, respectively). The modern proof of these and similar results is contained in the paper by K. Nakagawa (1984).

Wiman's theorem

The aim of the present section is to find a discrete version of the Wiman theorem.

Theorem 6 (A. Mednykh and I. Mednykh, 2013)

Let X be a graph of genus $g \geq 2$ and $\mathbb{Z}_N$ is a cyclic group acting harmonically on X. Then $N \leq 2g+2$. The upper bound $N = 2g+2$ is attained for any even g. In this case, the signature of orbifold $X/\mathbb{Z}_N$ is $(0; 2, g+1)$, that is, $X/\mathbb{Z}_N$ is a tree with two branch points of order 2 and $g+1$, respectively.

Wiman's theorem

Theorem 7 (A. Mednykh and I. Mednykh, 2013)

Let X be a graph of genus $g \geq 2$ and $\mathbb{Z}_N$ is a cyclic group acting harmonically on X. Let $N < 2g + 2$ then $N \leq 2g$. The upper bound $N = 2g$ is attained only in the following cases:

(i) $N = 2g$ *and* $X/\mathbb{Z}_N$ *is an orbifold of the signature* $(0; 2, 2g), g \geq 2$;

(ii) $N = 12$ *and* $X/\mathbb{Z}_N$ *is an orbifold of the signature* $(0; 3, 4)$, $g = 6$.

Also, if $N < 2g$ then $N \leq 2g - 1$. The upper bound $N = 2g - 1$ is attained only in two cases:

(iii) $N = 3$ *and* $X/\mathbb{Z}_N$ *is an orbifold of the signature* $(0; 3, 3)$, $g = 2$;

(iv) $N = 15$ *and* $X/\mathbb{Z}_N$ *is an orbifold of the signature* $(0; 3, 5)$, $g = 8$.