On Split Liftings with Sectional Complements

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Regular covering projection of connected graphs

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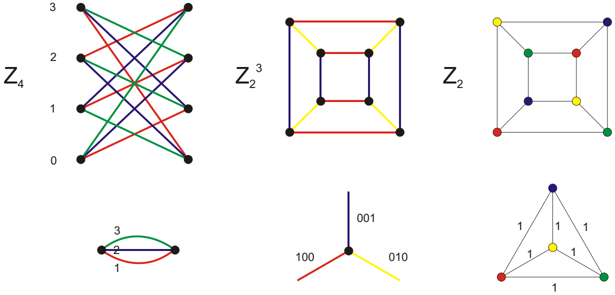

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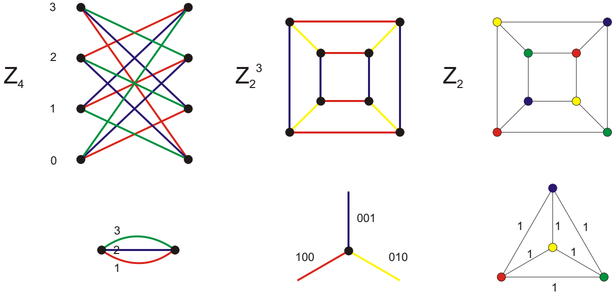

Construction/reconstruction
by a **voltage Cayley assignment** $\zeta\colon A(X) \to \Gamma \cong \mathrm{CT}_p$

Motivation in AGT: Studying symmetries of graphs

Lifting automorphisms along regular covering projections

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Applications

Construction of infinite families, compiling lists,
and classification of graphs with interesting symmetry properties.

Biggs, Algebraic Graph Theory, 1972

Thm.
Let $p\colon \tilde{X} \to X$ be a regular covering given in terms of Cayley voltages,

$$\zeta\colon A(X) \to \Gamma,$$

and let $G \leq \mathrm{Aut} X$. Suppose that the action of G on arcs is compatible with the assignment of voltages, that is, for each $g \in G$ there exists an automorphisms $g^{\#} \in \mathrm{Aut}\Gamma$ such that

$$\begin{array}{ccc} a & \xrightarrow{g} & g(a) \\ \zeta\downarrow & & \downarrow\zeta \\ \zeta_a & \xrightarrow{g^{\#}} & \zeta_{g(a)}. \end{array}$$

where $\#\colon g \mapsto g^{\#}$ is a homomorphism $G \to \mathrm{Aut}\Gamma$. Then G lifts along p as a split extension

$$\tilde{G} \cong \Gamma \rtimes_{\#} G.$$

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Biggs' compatibility condition implies that $\bar{G}$ preserves all vertices in $\tilde{X}$ that are labelled by 1. So $\bar{G}$ is a group, in fact, a complement to CT_p, and

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Consider p: Dodecahedron $\to$ Petersen
A_5 lifts to $\mathbb{Z}_2 \times A_5$. The unique copy of A_5 is transitive

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$\Omega = \{b\}$
Lifting the stabilizer $G_b \leq \mathrm{Aut}X$
Always lifts as $\mathrm{CT}_p \rtimes \tilde{G}_{\tilde{b}}$, where $\tilde{b} \in \mathrm{fib}_b$

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G lifts along a regular covering projection $p\colon \tilde{X} \to X$ as a split extension with a sectional complement over a G-invariant set Ω if and only if one of the two equivalent condition hold:

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1. The covering **can be reconstructed** by Cayley voltages $\zeta\colon X \to \Gamma$ that are $(1, G)$-**invariant** on Ω:

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Note.
Finding the right voltage assignment is difficult ! However, for **abelian** covers there is an efficient algorithm.

Abelian covers: Finding a sectional complement

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Note.
Computations can be carried out over $\mathbb{Z}$.

Define

$\mathrm{Cone}_X(\Omega) = X + *$, where $*$ adjacent to Ω

view G acting as a stabilzer of $*$

Finding all covers with sectional complements over Ω

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Thm. Let G lift along $p\colon Y \to \mathrm{Cone}_X(\Omega)$. If $Z = Y \setminus \mathrm{fib}_*$ is connected, then $\tilde{G}$ along $p_Z\colon Z \to X$ splits with an invariant section over Ω. Also, any $\tilde{X} \to X$ s.t. $\tilde{G}$ splits with an invariant section over Ω arises in this way.

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Note.
We can explicitly find all $\mathbb{Z}_p$-elementary abelian regular coverings along which G lifts in this manner. The problem is reduced to finding invariant subspaces of matrix group linearly representing the action of G on the first homology group $H_1(X, \mathbb{Z}_p)$.

Thm.
Let $p\colon \tilde{X} \to X$ be an abelian G-admissible regular covering projection. If $|\mathrm{CT}_p|$ is co-prime to the number of spanning trees in X, then G lifts as a sectional split extension over $V(X)$.

Thank you!