Harmonic Maps

Alexander Mednykh

Sobolev Institute of Mathematics
Novosibirsk State University

Winter School in Harmonic Functions on Graphs and Combinatorial Designs

20 - 24 January, 2014

Definitions and basic properties

Let G, G' be graphs. A function $\varphi : V(G) \cup E(G) \to V(G') \cup E(G')$ is said to be a *morphism* from G to G' if $\varphi(V(G)) \subseteq V(G')$, and for every edge $e \in E(G)$ with endpoints x and y, either $\varphi(e) \in E(G')$ and $\varphi(x), \varphi(y)$ are the endpoints of $\varphi(e)$, or $\varphi(e) \in V(G')$ and $\varphi(e) = \varphi(x) = \varphi(y)$. We write $\varphi : G \to G'$ for brevity. If $\varphi(E(G)) \subseteq E(G')$ then we say that φ is a *homomorphism.* A bijective homomorphism is called an *isomorphism*, and an isomorphism $\varphi : G \to G$ is called an *automorphism.*

Basic definition:

A morphism $\varphi : G \to G'$ is said to be *harmonic* if, for all $x \in V(G)$, $y \in V(G')$ such that $y = \varphi(x)$, the quantity $|e \in E(G)|x \in e, \varphi(e) = e'|$ is the same for all edges $e' \in E(G')$ such that $y \in e'$.

In recent papers harmonic maps are called also as *quasi-covering, branched coverings of graphs*. Another, not so popular names, are *wrapped quasi-coverings* and *horizontally conformal* maps. Harmonic maps are generalisation of graph coverings. The simplest examples are given by the following list

1. Any covering of graphs is a harmonic map
2. A natural projection of the wheel graph W_6 onto the wheel graph W_2 is a harmonic map

Harmonic Maps

We say that a group G acts on X if G is a subgroup of $\mathrm{Aut}(X)$.
A group G acts harmonically if G acts fixed point free on the set of directed edges $D(X)$ of a graph X.
In the latter case, the group G acts pure harmonically if G has no invertible edges on X.

Scott Corry and Roman Nedela made the following useful observation

If a group G acts pure harmonically on a graph X then the canonical projection $X \to X/G$ is a harmonic map.

That gives us a lot of non-trivial examples of harmonic maps.

Harmonic Maps

Let $\varphi : G \to G'$ be a morphism and let $x \in V(G)$. Define the *vertical multiplicity* of φ at x by

$$v_\varphi(x) = |e \in E(G)| x \in e, \, \varphi(e) = \varphi(x)|.$$

This is simply the number of *vertical edges* incident to x, where an edge e is called *vertical* if $\varphi(e) \in V(G')$ (and is called *horizontal* otherwise).
If φ is harmonic and $|V(G')| > 1$, we define the *horizontal multiplicity* of φ at x by

$$m_\varphi(x) = |e \in E(G)| x \in e, \, \varphi(e) = e'|$$

for any edge $e' \in E(G)$ such that $\varphi(x) \in e'$. By the definition of a harmonic morphism, $m_\varphi(x)$ is independent of the choice of e'.

Define the degree of a harmonic morphism $\varphi : G \to G'$ by the formula

$$\deg(\varphi) := |e \in E(G)|\varphi(e) = e'|$$

for any edge $e' \in E(G')$. By virtue of the following lemma $\deg(\varphi)$ does not depend on the choice of e' (and therefore is well defined):

Lemma 1.

The quantity $|e \in E(G) : \varphi(e) = e'|$ is independent of the choice of $e' \in E(G')$.

Proof of Lemma 1.

Let $y \in V(G')$, and suppose there are two edges e', $e'' \in E(G')$ incident to y. Since φ is harmonic, for each $x \in V(G)$ with $\varphi(x) = y$ we have

$$|\{e \in E(G) |\, x \in e, \varphi(e) = e'\}| = |\{\tilde{e} \in E(G) | x \in \tilde{e}, \varphi(\tilde{e}) = e''\}|.$$

Therefore

$$\begin{aligned} |\{e \in E(G) |\, \varphi(e) = e'\}| &= \sum_{x \in \varphi^{-1}(y)} |\{e \in E(G) |\, x \in \ e,\ \varphi(e) = e'\}| \\ &= \sum_{x \in \varphi^{-1}(y)} |\{\tilde{e} \in E(G) |\, x \in \tilde{e},\ \varphi(\tilde{e}) = e''\}| \\ &= |\{\tilde{e} \in E(G) |\, \varphi(\tilde{e}) = e''\}|. \end{aligned} \tag{1}$$

Now suppose e', e'' are arbitrary edges of G'. Since G is connected, the result follows by applying (1) to each pair of consecutive edges in any path connecting e' and e''.

According to the next result, the degree of a harmonic morphism $\varphi : G \to G'$ is just the number of pre-images under φ of any vertex of G', counting multiplicities.

Lemma 2.

For any vertex $y \in G$, we have

$$\deg(\varphi) = \sum_{x \in V(G),\, \varphi(x) = y} m_\varphi(x).$$

Proof. Choose an edge $e' \in E(G')$ with $y \in e'$. Then

$$\sum_{x \in \varphi^{-1}(y)} m_\varphi(x) = \sum_{x \in \varphi^{-1}(y)} \sum_{e \in \varphi^{-1}(e'),\, x \in e} 1 = |\varphi^{-1}(e')| = \deg(\varphi).$$

As with morphisms of Riemann surfaces, a harmonic morphism of graphs must be either constant or surjective.

Lemma 3.

Let $\varphi : G \to G'$ be a harmonic morphism. Then $\deg(\varphi) = 0$ if and only if φ is constant, and $\deg(\varphi) > 0$ if and only if φ is surjective.

Proof. If φ is constant, then clearly $\deg(\varphi) = 0$. Moreover, it follows from Lemmas 1 and 2 that φ is surjective if and only if $\deg(\varphi) > 0$. So it remains only to be shown that if $\deg(\varphi) = 0$, then φ is constant. For this, suppose we have $\varphi(x) = y$. Since $m_\varphi(x) = 0$, it follows that $\varphi(e) = y$ for every edge e with $x \in e$. Thus $\varphi(x') = y$ for every neighbor x' of x. As G is connected, it follows that every vertex and every edge of G is mapped under φ to y.

Riemann-Hurwitz formula for graphs

The following version of Riemann-Hurwitz formula for harmonic maps was established by M. Baker and S. Norine. We define genus of graph G as $g = |E(G)| - |V(G)| + 1$, that is as cyclomatic number of G.

Theorem (M. Baker, S. Norine, 2009)

Let G be a graph of genus g and G' be a graph of genus g'. Consider a surjective harmonic map $\varphi : G \to G'$. Then we have

$$g - 1 = deg(\varphi)(g' - 1) + \sum_{x \in V(G)} (m_\varphi(x) - 1) + N_{ver},$$

where $V(G)$ is the set of vertices of G, $m_\varphi(x)$ is the horizontal multiplicity of φ at x, and N_{ver} is the number of vertical edges of φ.

Riemann-Hurwitz formula for graphs

The following statement immediately follows from the Riemann-Hurwitz formula.

Theorem (Schreier formula)

Let $\varphi : G \to G'$ be a graph covering. Suppose that G and G' are graphs of genera g and g' respectively. Then we have

$$g - 1 = deg(\varphi)(g' - 1).$$

Harmonic Maps and Graphs of Groups

From now on we restrict ourself by harmonic maps without vertical edges. Them we employ the Bass-Serre theory of graphs of groups to prove uniformisation theorems for this class of maps.

Following H. Bass we define a *graph of groups* to be a pairs $\mathbb{A} = (A, \mathcal{A})$, where A is a connected graph, and $\mathcal{A} = \{A_a\}_{a \in A}$ assigns group A_a to each vertex $a \in A$.

Let $\mathbb{A} = (A, \mathcal{A})$ and $\mathbb{A}' = (A', \mathcal{A}')$ be graphs of groups. By a *covering of graph of groups*

$$\mathbb{F} = (\varphi, \Phi) : \mathbb{A} \to \mathbb{A}'$$

we mean

(i) a harmonic morphism $\varphi : A \to A'$;

(ii) a set Φ of injective homomorphisms

$$\varphi_a : \mathcal{A}_a \to \mathcal{A}'_{\varphi(a)} \ (a \in A) \text{ such that } m_\varphi(a)|\mathcal{A}_a| = |\mathcal{A}'_{\varphi(a)}|,$$

where $m_\varphi(a)$ is the multiplicity of φ at the point a.

The *fundamental group* of a graph of group $\mathbb{A} = (A, \mathcal{A})$, denoted $\pi_1(\mathcal{A})$, is defined as the free product

$$(*_{a \in A} \mathcal{A}_a) * \pi_1(A),$$

where $\pi_1(A) = \pi_1(A, a)$ denotes the fundamental group of the graph A.

To every graph of groups $\mathbb{A}$ one can associate a *Bass-Serre universal covering tree* $\widetilde{\mathbb{A}}$, which is a tree with $\pi_1(\widetilde{\mathbb{A}}) = \langle 1 \rangle$ that comes equipped with a natural group action of the fundamental group $\pi_1(\mathbb{A})$ without edge-inversions. Moreover, the quotient graph $\widetilde{\mathbb{A}}/\pi_1(\mathbb{A})$ is isomorphic to $\mathbb{A}$.

Bass-Serre uniformization theorem

Theorem (H. Bass, J.-P. Serre)

Let $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$ be a graph of group covering. Then $\mathbb{X}$ and $\mathbb{Y}$ share the same universal covering tree $\widetilde{\mathbb{Y}}$. Moreover, the groups $H = \pi_1(\mathbb{X})$ and $\Gamma = \pi_1(\mathbb{Y})$ are acting on $\widetilde{\mathbb{Y}}$ in such a way that $\mathbb{X} \cong \widetilde{\mathbb{Y}}/H$, $\mathbb{Y} \cong \widetilde{\mathbb{Y}}/\Gamma$ and the covering

$$\mathbb{F} : \mathbb{X} = \widetilde{\mathbb{Y}}/H \to \mathbb{Y} = \widetilde{\mathbb{Y}}/\Gamma$$

is induced by the group inclusion $H < \Gamma$.