Coverings of graphs and uniformisation theory

Lector: Alexander Mednykh

Sobolev Institute of Mathematics
Novosibirsk State University

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Graph coverings and covering groups

Let X and Y be connected graphs. A surjective morphism $\varphi : X \to Y$ is called a *(graph) covering* if for any vertex $x \in V(X)$ the restriction $\varphi|_{\mathrm{St}_X(x)} : \mathrm{St}_X(x) \to \mathrm{St}_Y(\varphi(x))$ is an isomorphism. The coverings $\varphi : X \to Y$ and $\varphi' : X' \to Y$ are said to be *equivalent* if there is an isomorphism $h : X \to X'$ such that $\varphi = \varphi' \circ h$.
A *covering group* of φ is defined as

$$\mathrm{Cov}(\varphi) = \{h \in \mathrm{Aut}(X) : \varphi = \varphi \circ h\}.$$

The covering φ is called *regular* if $\mathrm{Cov}(\varphi)$ act transitively on each fibre of φ and *irregular* otherwise. If $\varphi : X \to Y$ is a regular covering then $Y \cong X/\mathrm{Cov}(\varphi)$. A finite sheeted covering $\varphi : X \to Y$ is regular if and only if the order of covering group $|Cov(\varphi)|$ coincides with the number of sheets of the covering.

Graph coverings and voltage assignments

Permutation voltage assignments were introduced by J. L. Gross and T. W. Tucker. Let X be a finite connected graph, possibly including multiple edges or loops. It is *directed* if each edge (even a loop) is provided by the two possible directions. Let $D(X)$ be the set of the directed edges of X (also known as *darts, arcs* and so on in the literature). A *permutation voltage assignment* of X with voltages in the symmetric group $\mathbb{S}_n$ of degree n is a function $\phi : D(X) \to \mathbb{S}_n$ such that inverse edges have inverse assignments. The pair $(D(X), \phi)$ is called a permutation voltage graph.

Graph coverings and voltage assignments

The *(permutation) derived graph* X^{ϕ} derived from a permutation voltage assignment ϕ is defined as follows: $V(X^{\phi}) = V(X) \times \{1, \cdots, n\}$, and $((u, j), (v, k)) \in D(X^{\phi})$ if and only if $(u, v) \in D(X)$ and $k = \phi(u, v)(j)$. The natural projection $\pi : X^{\phi} \to X$ that is a function from $V(X^{\phi})$ onto $V(X)$ which erases the second coordinates gives a *graph covering*. J. L. Gross and T. W. Tucker showed that every covering of a given graph arises from some permutation voltage assignment in a symmetric group. Moreover, such a covering is connected if and only if $\phi(D(X))$ is a transitive subgroup in $\mathbb{S}_n$.

Regular coverings and ordinary voltage assignments

Ordinary voltage assignments were introduced by J. L. Gross. Let G be a finite group. Then a mapping $\omega : D(X) \to G$ is called an *ordinary voltage assignment* if $\omega(v,u) = \omega(u,v)^{-1}$ for each $(u,v) \in D(X)$. The *(ordinary) derived graph* X^{ω} derived from an ordinary voltage assignment ω is defined as follows: $V(X^{\omega}) = V(X) \times G$, and $((u,j),(v,k)) \in D(X^{\omega})$ if and only if $(u,v) \in D(X)$ and $k = \omega(u,v)j$. Consider the natural projection $\pi : X^{\omega} \to X$ that is a function from $V(X^{\omega})$ onto $V(X)$ which erases the second coordinates. Then the map $\pi : X^{\omega} \to X$ is a *G-covering* of X, that is a $|G|$-fold regular covering of X with the covering group G. Every regular covering of X can be obtained in such a way.

Short way to construct coverings

Let X be a graph of genus g. Choose a spanning tree T in X and g directed edges $e_1, e_2, \ldots, e_g$ from the compliment $X \setminus T$.
An arbitrary *reduced permutation assignment* $\psi : D(X) \to \mathbb{S}_n$ is uniquely determined by the following conditions:

(i) $\psi(e_i) = \xi_i$, where $\xi_i \in \mathbb{S}_n$ for $i = 1, 2, \ldots, g$ and $\psi(e) = 1$, for any edge e which is in T;

(ii) $\xi_1, \xi_2, \ldots, \xi_g$ generate a transitive subgroup in $\mathbb{S}_n$.

Then the permutation derived graph gives a required covering.
All connected n-fold coverings can be obtained in such a way. Two tuples $(\xi_1, \xi_2, \ldots, \xi_g)$ and $(\xi'_1, \xi'_2, \ldots, \xi'_g)$ give equivalent coverings if and only if there exists $h \in \mathbb{S}_n$ such that $\xi'_i = h\,\xi_i\,h^{-1}$ for all $i = 1, 2, \ldots, g$.

Monodromy group and covering group

The tranitive group $\mathrm{Mon}(\psi) = \langle \xi_1, \xi_2, \ldots, \xi_g \rangle$ is called the *monodromy group* of the covering ψ. Is has the following properties.

(i) Covering ψ is regular if and only if the group $\mathrm{Mon}(\psi)$ is regular, that is acts without fixed point on the set $\{1, 2, \ldots, n\}$;

(ii) In the case of regular covering $\mathrm{Cov}(\psi) \cong \mathrm{Mon}(\psi)$;

(iii) In the case of irregular covering one can use an isomorphism $\mathrm{Cov}(\psi) \cong C_{\mathbb{S}_n}(\mathrm{Mon}(\psi))$.

Coverings and transitive homomorphisms

Let $\Gamma = \pi_1(X, x)$ be the fundamental group of a graph X at vertex x. It is well known that there is a one-to-one correspondence between the classes of equivalent n-fold coverings of X and the equivalence classes of transitive homomorphisms from Γ to the symmetric group $\mathbb{S}_n$ on n symbols. Recall that a homomorphism to $\mathbb{S}_n$ is called *transitive* if its image is a transitive subgroup in $\mathbb{S}_n$. Two homomorphisms, $\theta, \theta' : \Gamma \to \mathbb{S}_n$ are said to be *equivalent* if there exists $h \in \mathbb{S}_n$ such that $\theta' = h\,\theta\,h^{-1}$. [A. Hatcher, Algebraic Topology, Cambridge Univ. Press, Cambridge, 2002, p. 68].

Coverings and transitive homomorphisms

Let X be a graph of genus g. Then Γ is a free group of rank g. Suppose that Γ is freely generated by the elements $x_1, x_2, \dots, x_g$. Then an arbitrary transitive homomorphism $\theta : \Gamma \to \mathbb{S}_n$ is uniquely determined by the following conditions:

(i) $\theta(x_i) = \xi_i$, where $\xi_i \in \mathbb{S}_n$ for $i = 1, 2, \dots, g$.

(ii) $\xi_1, \xi_2, \dots, \xi_g$ generate a transitive subgroup in $\mathbb{S}_n$.

Two homomorphisms defined by tuples $(\xi_1, \xi_2, \dots, \xi_g)$ and $(\xi'_1, \xi'_2, \dots, \xi'_g)$ are equivalent if and only if exists $h \in \mathbb{S}_n$ such that $\xi'_i = h\,\xi_i\,h^{-1}$ for all $i = 1, 2, \dots, g$.

Coverings and the fundamental group

If $\varphi : X \to Y$ is a covering and $\varphi(x) = y$ then there is a natural imbedding of the fundamental groups $\varphi_* : \pi_1(X, x) \to \pi_1(Y, y)$ induced by φ. Moreover, the index of subgroup $\varphi_* \pi_1(X, x)$ in $\pi_1(Y, y)$ coincides with the number of sheets of the covering. The covering φ is regular if and only if $\varphi_* \pi_1(X, x)$ is a normal subgroup in $\pi_1(Y, y)$. In the latter case, $Cov(\varphi)$ is canonically isomorphic to the factor-group $\pi_1(Y, y)/\varphi_* \pi_1(X, x)$. The coverings $\varphi : X \to Y$ and $\varphi' : X' \to Y$ are equivalent if and only if the corresponding subgroups $\varphi_* \pi_1(X, x)$ and $\varphi'_* \pi_1(X', x')$ are conjugate in $\pi_1(Y, y)$.

Coverings and the fundamental group

Let Y be a graph with fundamental group $\Gamma = \pi_1(Y, y)$ and $H < \Gamma$ be an arbitrary subgroup of Γ. Then there exists a covering $\varphi : X \to Y$ is a covering with $\varphi(x) = y$ such that $H \cong \pi_1(X, x)$.

Universal covering

The covering $\tilde{Y}$ corresponding to the trivial subgroup $H = \{e\} < \pi_1(Y, y)$ is called the *universal covering.* It has the following properties.

(a) The unversal covering $\tilde{Y}$ exists for any connected graph Y and is uniquely determined up to equivalency.

(b) The universal covering of a graph is a tree.

(c) Let $\varphi : X \to Y$ be a covering. Then there exists a covering $\psi : \tilde{Y} \to X$ such that the composition $\varphi \circ \psi : \tilde{Y} \to Y$ is the universal covering of Y.

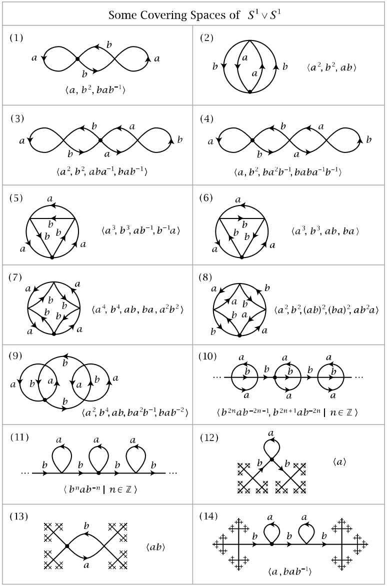
(d) The group $\Gamma = \pi_1(Y, y)$ acts freely on $\tilde{Y}$ in such a way that the factor graph $\tilde{Y}/\Gamma$ is isomorphic to Y.

Coverings and uniformisation theory

If $\varphi : X \to Y$ is a covering and $\varphi(x) = y$ and is $H = \varphi_\star : \pi_1(X, x) \to \Gamma = \pi_1(Y, y)$ the natural imbedding of the fundamental groups induced by φ. Denote by $\tilde{Y}$ the universal covering of Y. Then there is a free action of groups Γ and H on $\tilde{Y}$ such that $\tilde{Y}/\Gamma$ is isomorphic to Y, $\tilde{Y}/H$ is isomorphic to X and the covering

$$\varphi : X = \tilde{Y}/H \to Y = \tilde{Y}/\Gamma$$

is induced by the group inclusion $H < \Gamma$.


Some Covering Spaces of $S^1 \vee S^1$
(1) $\langle a, b^2, bab^{-1} \rangle$
(2) $\langle a^2, b^2, ab \rangle$
(3) $\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$
(4) $\langle a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} \rangle$
(5) $\langle a^3, b^3, ab^{-1}, b^{-1}a \rangle$
(6) $\langle a^3, b^3, ab, ba \rangle$
(7) $\langle a^4, b^4, ab, ba, a^2b^2 \rangle$
(8) $\langle a^2, b^2, (ab)^2, (ba)^2, ab^2a \rangle$
(9) $\langle a^2, b^4, ab, ba^2b^{-1}, bab^{-2} \rangle$
(10) $\langle b^{2n}ab^{-2n-1}, b^{2n+1}ab^{-2n} \mid n \in \mathbb{Z} \rangle$
(11) $\langle b^nab^{-n} \mid n \in \mathbb{Z} \rangle$
(12) $\langle a \rangle$
(13) $\langle ab \rangle$
(14) $\langle a, bab^{-1} \rangle$