Counting spanning trees

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Spanning tree

A *spanning tree* T of a connected, undirected graph G is a tree composed of all the vertices and some (or perhaps all) of the edges of G. In other words, a spanning tree of G is a selection of edges of G that form a tree spanning every vertex. That is, every vertex lies in the tree, but no cycles (or loops) are allowed. On the other hand, every bridge of G must belong to T. A spanning tree of a connected graph G can also be defined as a maximal set of edges of G that contains no cycle, or as a minimal set of edges that connect all vertices.

Counting spanning trees

The number $t(G)$ of spanning trees of a connected graph is a well-studied invariant. In some cases, it is easy to calculate $t(G)$ directly. For example, if G is itself a tree, then $t(G) = 1$, while if G is the cycle graph C_n with n vertices, then $t(G) = n$. For any graph G, the number $t(G)$ can be calculated using Kirchhoff's matrix-tree theorem.

Here are some known results concerning counting spanning trees of graphs.

1. Complete graph K_n : $t(K_n) = n^{n-2}$ (Cayley's formula),
2. Complete bipartite graph $K_{n,m}$: $t(K_{n,m}) = m^{n-1}n^{m-1}$,
3. n-dimensional cube graph Q_n : $t(Q_n) = 2^{2^n-n-1}\prod\limits_{k=2}^{n} k^{\binom{n}{k}}$.

Kirchhoff Matrix-Tree Theorem

The celebrated Kirchhoff Matrix-Tree Theorem is the following statement.

Theorem (Kirchhoff (1847))

All cofactors of Laplacian matrix $L(G)$ are equal to $t(G)$.

More convenient form of this result were obtained by A. K. Kel'mans and V. M. Chelnokov.

Theorem (Kel'mans, Chelnokov (1974))

Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ denote the eigenvalues of the Laplace matrix $L(G)$ of a n point graph G. Then

$$t(G) = \frac{1}{n} \prod_{k=2}^{n} \lambda_k.$$

The Temperley's formula

One more convenient way to count spanning treses.

Theorem (Temperley, H. N. V. (1964))

The number of spanning trees of a n point graph G is given by the formula

$$t(G) = \det(L(G) + \frac{1}{n^2}J),$$

where J is $n \times n$ matrix all of whose elements are unity.

Now we will present an uniform proof for all the three previous theorems.

Proof of Kirchhoff, Kal'mans - Chelnokov and Temperley theorems

Proof.

Let $L = L(G)$ be the Laplacian matrix of G with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. The (i, j)-cofactor of a matrix M is by definition $(-1)^{i+j} \det M(i, j)$, where $M(i, j)$ is the matrix obtained from M by deleting row i and column j.
Let l_{xy} be the (x, y)-cofactor of L. Note that l_{xy} does not depend on an ordering of the vertices of G.
We set $N = t(G)$ and show that

$$N = l_{xy} = \det(L + \frac{1}{n^2}J) = \frac{1}{n}\lambda_2 \ldots \lambda_n \text{ for any } x, y \in V(G).$$

Spanning Trees

Let L^S, for $S \subset V(G)$, denote the matrix obtained from L by deleting the rows and columns indexed by S, so that $l_{xx} = \det L^{\{x\}}$. The equality $N = l_{xx}$ follows by induction on n, and for fixed $n > 1$ on the number of edges incident with x. Indeed, if $n = 1$ then $l_{xx} = 1$. Otherwise, if x has degree 0, then $l_{xx} = 0$ since $L^{\{x\}}$ has zero row sums. Now, if xy is an edge, then deleting this edge from G decreases l_{xx} by $\det L^{\{x,y\}}$, which by induction is the number of spanning trees of G with edge xy collapsing to a point, which is the number of spanning trees containing the edge xy. This shows $N = l_{xx}$. Since the sum of the columns of L is zero, so that one column is minus the sum of the other columns, we have $l_{xx} = l_{xy}$ for any x, y.

Now we consider the Laplacian polynomial
$\mu(G,t) = \det(tI - L) = t\prod_{i=2}^{n}(t-\lambda_i)$ for graph G. Then
$(-1)^{n-1}\lambda_2\ldots\lambda_n$ is the coefficient of t, that is, $\frac{d}{dt}\det(tI-L)|_{t=0}$.
We note that

$$\frac{d}{dt}\det(tI-L) = \sum_x \det(tI - L^{\{x\}}).$$

Putting $t=0$ we obtain $\lambda_2\ldots\lambda_n = \sum_x l_{xx} = nN$.
Finally, the eigenvalues of $L+\frac{1}{n^2}J$ are $\frac{1}{n}$ and $\lambda_2,\ldots,\lambda_n$, so

$$\det(L+\frac{1}{n^2}J) = \frac{1}{n}\lambda_2\ldots\lambda_n.$$

Spanning Trees

A generalization of the Matrix-Tree-Theorem was obtained by Kelmans (1967) who gave a combinatorial interpretation to all the coefficients of $\mu(G,x)$ in terms of the numbers of certain subforests of the graph. This result has been obtained even in greater generality (for weighted graphs) by Fiedler and Sedláček.

Theorem (Kel'mans (1967))

If

$$\mu(G,x) = x^n - c_1 x^{n-1} + \ldots + (-1)^i c_i x^{n-i} + \ldots + (-1)^{n-1} c_{n-1} x$$

then

$$c_i = \sum_{S \subset V, |S| = n-i} t(G_S),$$

where $t(H)$ is the number of spanning trees of H, and G_S is obtained from G by identifying all vertices of S to a single one.

From the last theorem we can derive useful corollary.

Corollary

The degree of Laplacian polynomial $\mu(G,x)$ is equal to $n = |V(G)|$. Its coefficients c_1 and c_{n-1} are given by the formulas $c_1 = 2|E(G)|$ and $c_{n-1} = |V(G)| \cdot t(G)$.

Hence, the number of vertices $|V(G)|$, number of edges $|E(G)|$ and the number of spanning trees $t(G)$ are uniquely defined by the Laplacian polynomial.

Some recursive formulas for t(G)

Now we give a few recursive formulas for the number of spanning trees employed in graph theory by W. Feussner (1904) and J. W. Moon (1970).

Denote by $G - e$ the graph obtained by removing edge e from the graph G. Let $G \backslash e$ be the graph obtained from graph G by contracting edge e. In other words, $G \backslash e$ is obtained by deleting edge e and identifying its ends. Then the following formula takes a place.

$$t(G) = t(G - e) + t(G \backslash e).$$

Proof. We note that the set of spanning trees of a given graph G decomposed in two disjoint sets. First set consist of tree containing selected edge $e \in E(G)$ and second set consist of trees that do not contain e. The number of spanning trees that contains e is exactly $t(G \backslash e)$ because each of them corresponds to a spanning tree of $G \backslash e$. The number of spanning tree that do not contain e is $t(G - e)$, since each of them is also a spanning tree of $G - e$ and vice versa.

Denote by $G_{s,e}$ the graph resulting from subdivision of an edge e of a graph G. Then

$$t(G_{s,e}) = t(G \backslash e) + 2t(G - e) = t(G) + t(G - e).$$

Proof. Again, the number of spanning trees of X that contains e is $t(G \backslash e)$. All of them also the spanning trees of $t(G_{s,e})$. If a spanning tree of G do not contains e then it can be extended to be a spanning tree of $t(G_{s,e})$ in two different ways. Hence, $t(G_{s,e}) = t(G \backslash e) + 2t(G - e)$. By the previous statement $t(G \backslash e) + 2t(G - e) = t(G) + t(G - e)$.

Let $G_{p,e}$ denotes the results of adding an edge in parallel an edge e of a graph G. Then

$$t(G_{p,e}) = t(G) + t(G\backslash e).$$

Proof. Distinguishing spanning trees that contain an edge e and that are not we have

$$t(G_{p,e}) = t(G - e) + 2t(G\backslash e) = t(G) + t(G\backslash e).$$

Spanning Trees

Let G_1 and G_2 are the graphs with exactly one vertex in common. Then

$$t(G_1 \cup G_2) = t(G_1) \cdot t(G_2),$$

where $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Proof. Let T be a spanning tree of $G_1 \cup G_2$. Then $T_1 = T \cap G_1$ and $T_2 = T \cap G_2$ are spanning trees for G_1 and G_2 respectively. Moreover $G_1 \cap G_2 = \{v\}$, where v is the common vertex of G_1 and G_2. Conversely, let T_1 and T_2 are respective spanning trees for G_1 and G_2. Then $T_1 \cap T_2 = \{v\}$ and $T = T_1 \cup T_2$ is a spanning tree of $G_1 \cup G_2$.

Chebyshev polynomials

The Chebyshev polynimial of the first kind is defined by the formula

$$T_n(x) = \cos(n \arccos x).$$

Eqivalently,

$$T_n(x) = \frac{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n}{2}.$$

Also, $T_n(x)$ satisfies the recursive relation

$$T_0(x) = 1,\ T_1(x) = x,\ T_n(x) = 2x \cdot T_{n-1}(x) - T_{n-2}(x),\ n \geq 2.$$

Chebyshev polynomials

The Chebyshev polynimial of the second kind is defined by the formula

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}.$$

Eqivalently,

$$U_n(x) = \frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}.$$

Also, $U_n(x)$ satisfies the recursive relation

$$U_0(x) = 1,\ U_1(x) = 2x,\ U_n(x) = 2x \cdot U_{n-1}(x) - U_{n-2}(x),\ n \geq 2.$$

We have $U_n(\cos\frac{k\pi}{n+1}) = 0, k = 1, 2, \ldots, n$. Hence

$$U_n(x) = 2^n \prod_{k=1}^{n} (x - \cos\frac{k\pi}{n+1}).$$

Chebyshev polynomials

Since

$$U_n(x) = (-1)^n U_n(-x) = 2^n \prod_{k=1}^{n} (x + \cos \frac{k\pi}{n+1})$$

we obtain

$$U_n^2(x) = \prod_{k=1}^{n} (4x^2 - 4\cos^2 \frac{k\pi}{n+1}).$$

Polynomials $T_n(x)$ and $U_{n-1}(x)$ are related by the following identity

$$T_n^2(x) + (x^2 - 1)U_{n-1}^2(x) = 1.$$

Chebyshev polynomials

Consider $n \times n$ matrix

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2x & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2x & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2x & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2x & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2x \end{pmatrix}.$$

Then $\det A_n(x) = U_n(x)$.