Almost totally branched coverings between regular hypermaps

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What is a map?

Topological map

$\mathcal{M} = (\Gamma, S)$ is a 2-cell embedding of a connected graph Γ into a closed surface S.

Combinatorial map

$\mathcal{M} = (D; r_0, r_1)$ where

- D: set of darts,
- r_0, r_1, $r_1^2 = 1$: permutations on D,
- $\mathrm{Mon}(\mathcal{M}) = \langle r_0, r_1 \rangle$ is a transitive on D.

Algebraic representation of maps

- Each map $\mathcal{M}$ determines a finite transitive permutation representation of Grothendieck's cartographic group

$$\mathscr{C}_2^+ = \langle \rho_0, \rho_1, \rho_2 \mid \rho_1^2 = \rho_0\rho_1\rho_2 = 1 \rangle$$

given by

$$\theta : \mathscr{C}_2^+ \to \mathrm{Mon}(\mathcal{M}) \leq Sym(D), \quad \rho_i \mapsto r_i,$$

where $r_2 = (r_0 r_1)^{-1}$.

- Conversely, every finite transitive permutation representation of $\mathscr{C}_2^+$ determines a map.

What is a hypermap?

Definition (Algebraic hypermap)

A finite transitive permutation representation of the hypercartographic group

$$\mathscr{H}_2^+ = \langle \rho_0, \rho_1, \rho_2 \mid \rho_0\rho_1\rho_2 = 1 \rangle \cong \Delta(\infty, \infty, \infty),$$

given by

$$\theta : \mathscr{H}_2^+ \to \mathrm{Mon}(\mathcal{H}) \leq Sym(B), \quad \rho_i \mapsto r_i.$$

- B: set of brins,
- $\mathrm{Mon}(\mathcal{H})$: monodromy group,
- the stabilizer $H \leq \mathscr{H}_2^+$ of a brin: hypermap subgroup,
- (d_0, d_1, d_2): type, where $d_i = o(r_i)$.

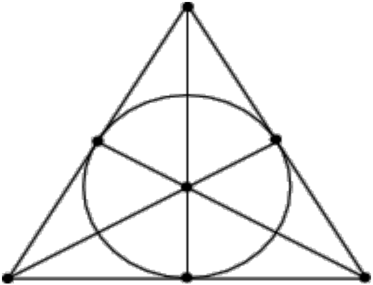
Topological hypermaps

Definition (Topological hypermap)

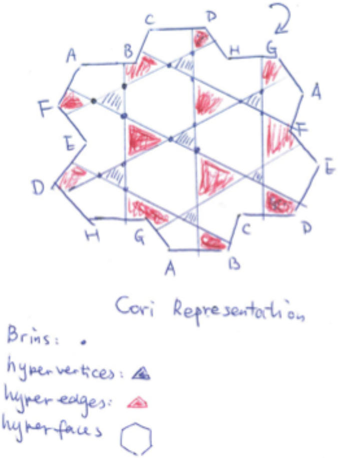
An embedding of a hypergraph into a closed (oriented) surface.

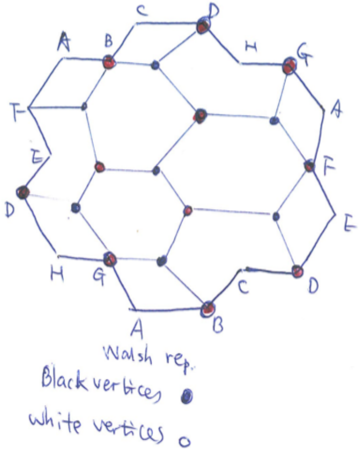
- Hypergraph: a set $B \neq \emptyset$ with two partitions V and E (hypervertices and hyperedges), and two parts are incident if they have non-empty intersection .
- Cori representation: V and E are identified with closed discs, $B = V \cap E$, and $S - (V \cup E)$ are hyperfaces.
- Walsh map $W(\mathcal{H})$: a 2-colored bipartite map corresponding to $\mathcal{H}$.

The Fano plane



Cori Rep. and its Walsh map $W(H)$





Hypermaps and Belyǐ functions

Theorem (Belyǐ, 1979)

A compact Riemann surface S is defined over the field $\bar{\mathbb{Q}}$ of algebraic numbers iff there is a meromorphic function $\beta : S \to \Sigma = \mathbb{C} \cup \{\infty\}$ with at most 3 critical values (these can be chosen as $\{0, 1, \infty\}$).

- The continuation around the critical values 0, 1 and ∞ of a Belyǐ function determines three permutations r_0, r_1 and r_2 on the sheets. These generate a transitive permutation group satisfying $r_0 r_1 r_2 = 1$, and hence, we obtain a hypermap.
- Given a hypermap on a close oriented Riemann surface S, one can construct a Belyǐ function on S (see Jones, Singerman, 1996).

Grothendick's theory of dessins d'enfants

- The Absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ has a natural action on the Riemann surfaces defining Belyĭ functions.
- It induces a (faithful) action on the category of hypermaps and their coverings.

Dessins d'Enfants (Children's Drawing)

A combinatorial approach to the study of the absolute Galois group. In other words, one can do Galois theory by drawing pictures.

Regular hypermaps

Observe that $\mathrm{Aut}(\mathcal{H}) = C_{Sym(B)}(\mathrm{Mon}(\mathcal{H}))$ and $\mathrm{Mon}(\mathcal{H})$ is transitive on B, we have $\mathrm{Aut}(\mathcal{H})$ is semiregular on B.

Definition (Regular hypermap)

$\mathrm{Aut}(\mathcal{H})$ is regular on B.

- $\mathcal{H}$ is regular

iff $\mathrm{Mon}(\mathcal{H})$ is regular,

iff $H \trianglelefteq \mathscr{H}_2^+$,

in which case

$$\mathrm{Mon}(\mathcal{H}) \cong \mathscr{H}_2^+/H \cong \mathrm{Aut}(\mathcal{H}).$$

Regular coverings

Identify a regular map $\mathcal{H}$ with a quadruple (G, x, y, z)

- $G = \mathrm{Aut}(\mathcal{H})$,
- $x \in G$ stabilizes a black vertex (in Walsh's rep),
- $y \in G$ stabilizes an incident white vertex, and $z = (xy)^{-1}$,
- $\mathrm{Mon}(\mathcal{H})$ and $\mathrm{Aut}(\mathcal{H})$ are identified with the left and right regular representation of G.

Coverings between regular hypermaps

Each covering $p\colon \mathcal{H}_1 \to \mathcal{H}_2$ between regular hypermaps is a regular covering, and $x_1 \mapsto x_2, y_1 \mapsto y_2, z_1 \mapsto z_2$ extends to an epimorphism $p : G_1 \to G_2$ such that $G_1/CT(p) \cong G_2$.

Motivation

Problem

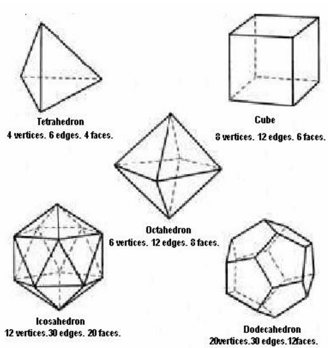
Construct and classify regular hypermaps which cover the platonic maps with cyclic covering transformation groups,

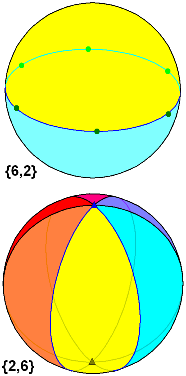
where platonic maps are defined to be regular maps (G, x, y, z) of type $(d_0, 2, d_2)$,

$$G = \langle x, y \mid x^{d_0} = y^2 = z^{d_2} = xyz = 1 \rangle,$$

where $1/d_0 + 1/d_2 > 1/2$.

Platonic maps





History

Before 21th Century certain extensions of the Platonic groups were studied in the context of polyhedral groups, by Miller (1907), Threlfall (1932), Shephard (1952), Coxeter (1940-1962), Sherk (1959).

Jones and Surowski(2000) classification of cyclic regular coverings of the Platonic maps, branched exclusively over the vertices, edges, or face-centres.

Širáň (2001) self-dual cyclic regular map coverings of the tetrahedral map.

H., Nedela, Wang(2013) complete classification of cyclic regular map coverings of the platonic maps.

Convention

Let $\mathcal{H}_1 = (G_1, x_1, y_1, z_1) \xrightarrow{p} \mathcal{H}_2 = (G_2, x_2, y_2, z_2)$ be a covering between two regular hypermaps, where $\mathcal{H}_2$ has type (d_0, d_1, d_2). Denote

$$A = \langle x_1^{d_0} \rangle, B = \langle y_1^{d_1} \rangle, C = \langle z_1^{d_2} \rangle$$

Observation

$A, B, C \leq CT(p)$ and hence $ABC \subseteq CT(p)$.

Totally branched covering

The covering $p : \mathcal{H}_1 \to \mathcal{H}_2$ is called minimal if $CT(p) = ABC$.

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- totally branched at hyperfaces if $CT(p) = C$;
- totally branched if it is (simultaneously) totally branched at hypervertices, hyperedges and hyperfaces, that is, $CT(p) = A = B = C$.

Almost totally branched covering

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- almost totally branched if it is (simultaneously) almost totally branched at hypervertices, hyperedges and hyperfaces, that is, $A, B, C \trianglelefteq G_1$.

Observations

- Each cyclic regular hypermap covering of the platonic map is an almost totally branched covering.

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- The covering transformation group of an almost totally branched covering is a product of three cyclic groups.
- If an almost totally branched covering is smooth at one of the objects (hypervertices/hyperedges/hyperfaces), then $CT(p)$ is a metacyclic group, which can be abelian or nonabelian.

Classification in the category of maps: abelian case

Theorem (H., Nedela, Wang, 2013)

Up to isomorphism, the abelian almost totally branched map-covers over the platonic maps (i.e smooth at edges), are in 1-1 correspondence with 6-tuples $(q_F, q_V, h_0, e_F, e_V, e)$ *satisfying*

$$e_V \equiv 1 \pmod{h_0}, \qquad e_F \equiv 1 \pmod{h_0},$$

$$e_V^{\mathrm{hcf}\,(d_F,2)} \equiv 1 \pmod{h_0 q_V}, \qquad e_F^{\mathrm{hcf}\,(d_V,2)} \equiv 1 \pmod{h_0 q_F},$$

$$\Sigma_{i=0}^{l_V-1} e_V^i \equiv 0 \pmod{q_V}, \qquad \Sigma_{i=0}^{l_F-1} e_F^i \equiv 0 \pmod{q_F},$$

$$e\Sigma_{i=0}^{l_F-1} e_F^i / q_F + \Sigma_{i=0}^{l_V-1} e_V^i / q_V \equiv 0 \pmod{h_0}, \quad e \in \mathbb{Z}_{h_0}^*.$$

In particular, the cyclic regular map-coverings are such maps with an extra condition $\gcd(q_F, q_V) = 1$.

Classification in the category of maps: nonabelian case

Theorem (H., Jones, Nedela, Wang, 2013)

- *In the family of platonic maps, only the tetrahedral map, the icosahedral map and the dodecahedral map admit a nonabelian almost totally branched map-covering;*
- *The isomorphism classes of nonabelian almost totally branched map-coverings over an admissible platonic map are in 1-1 correspondence with the 4-tuples* (q_F, q_V, h_0, e) *of positive integers such that*
  - q_F, q_V *and* h_0 *are all even,*
  - $q_F | l_F$ *and* $q_V | l_V$,
  - $l_F e/q_F + l_V/q_V \equiv 0 \pmod{h_0}$ *where* $e \in \mathbb{Z}^*_{h_0}$.

  where l_F *and* l_V *are the numbers of faces and vertices of the platonic map.*

Partial classification in the category of hypermaps

Theorem

The isomorphism classes of abelian almost totally branched hypermap-coverings over the platonic maps, smooth at faces, are in 1-1 correspondence with the solutions $(q_0, q_1, h, e_0, e_1, e)$ *of the system of congruences*

$$e_0 \equiv 1 \pmod{h}, \qquad e_1 \equiv 1 \pmod{h},$$
$$e_0^{\gcd(2,d_2)} \equiv 1 \pmod{q_0 h}, \qquad e_1^{\gcd(d_0,d_2)} \equiv 1 \pmod{q_1 h},$$
$$\Sigma_{i=0}^{l_0-1} e_0^i \equiv 0 \pmod{q_0}, \qquad \Sigma_{i=0}^{l_1-1} e_1^i \equiv 0 \pmod{q_1}$$
$$e\Sigma_{i=0}^{l_0-1} e_0^i/q_0 + \Sigma_{i=0}^{l_1-1} e_1^i/q_1 \equiv 0 \pmod{h}, \qquad e \in \mathbb{Z}_h^*.$$

In particular, such a covering is cyclic iff $\gcd(q_0, q_1) = 1$.

Open problems

1. Complete the classification of cyclic regular hypermap coverings of the platonic maps.
2. Investigation of almost totally branched coverings.

The end

Thank you very much!