Locally constrained homomorphisms on graphs of bounded treewidth and bounded degree

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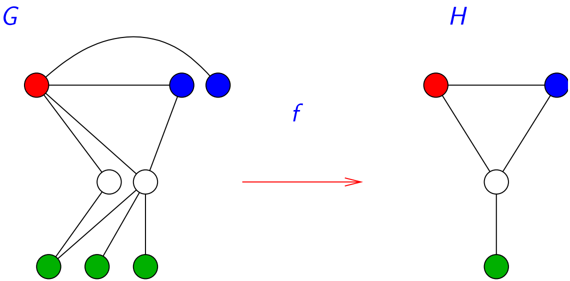
Graph homomorphisms

A mapping $f : V_G \to V_H$ is a *graph homomorphism* if

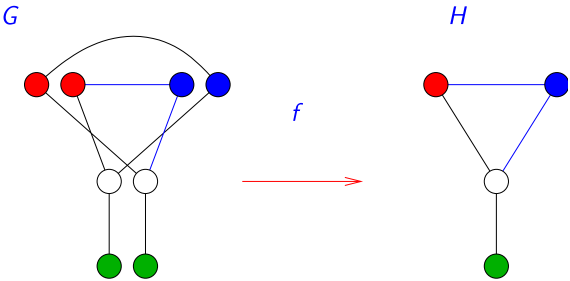
$$(u, v) \in E_G \Rightarrow (f(u), f(v)) \in E_H$$



Locally bijective homomorphisms

A homomorphism $f : V_G \to V_H$ is *locally bijective* if

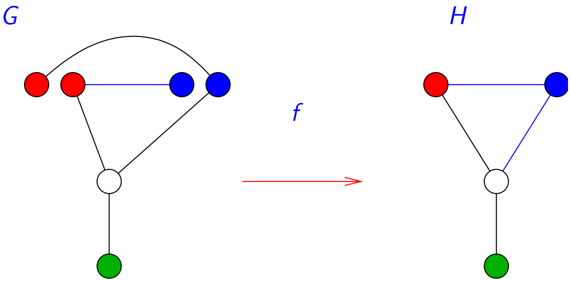
f acts bijectively between $N(u)$ and $N(f(u))$ for all $u \in V_G$



Locally injective homomorphisms

A homomorphism $f : V_G \to V_H$ is *locally injective* if

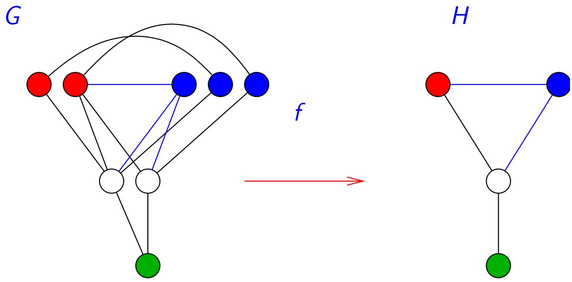
f acts injectively between $N(u)$ and $N(f(u))$ for all $u \in V_G$



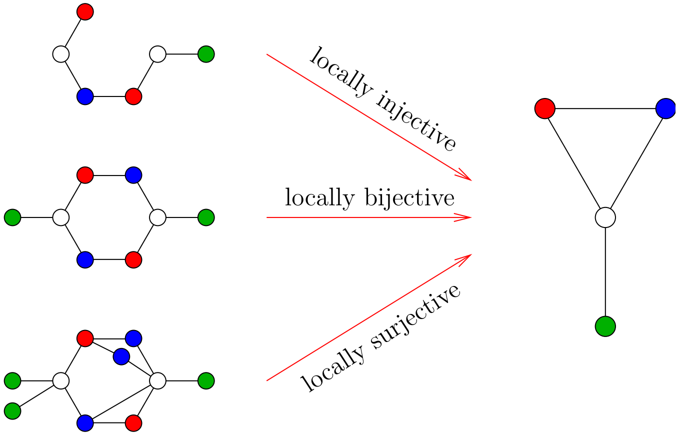
Locally surjective homomorphisms

A homomorphism $f : V_G \rightarrow V_H$ is *locally surjective* if

f acts surjectively between $N(u)$ and $N(f(u))$ for all $u \in V_G$



Summary



Decision problems

Instance: Graphs G and H.

Problem:	Query: Does G allow:
HOM	— a homomorphism to H?
LBHOM	— a locally bijective homomorphism to H?
LIHOM	— a locally injective homomorphism to H?
LSHOM	— a locally surjective homomorphism to H?

Theorem [Hell, Nešetřil, 1990]
HOM is polynomial-time solvable if H is bipartite, and it is NP-complete otherwise.

Bounding the maximum degree

Theorem [Kratochvíl, Křivánek, 1988]
LBHom is NP-complete on input pairs (G, K_4),

... G must be cubic in this case

Theorem [Kratochvíl, Proskurowski, Telle 1997, F. 2000]
LBHom is NP-complete on input pairs (G, H),
where H is any k-regular graph with $k \geq 3$.

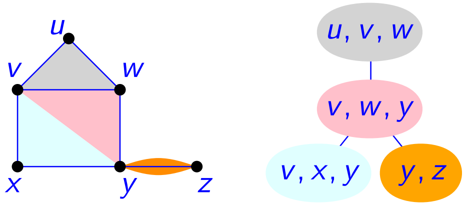
Corollary
LBHom, LIHom and LSHom are NP-complete on input pairs (G, H), where G has maximum degree $k \geq 3$.

Treewidth and pathwidth

A *tree decomposition* of a graph G is a tree T,
whose nodes are subsets of V_G satisfying:

- each edge of G is a subset of some node of T,
- each vertex has connected appearance in the nodes of T.

The *width* of T is the maximum size of its nodes $+1$.
The *treewidth* of G is the minimum possible width of its tree decomposition (*pathwidth* when T is a path).



$$tw(G) = \min\{\omega(H) : G \subseteq H, H \text{ is chordal}\} + 1$$

Bounding the treewidth

Theorem

(i) LBHom is NP-complete on input pairs (G, H), where G has pathwidth at most 5 and H has pathwidth at most 3,

(ii) LSHom is NP-complete on input pairs (G, H), where G has pathwidth at most 4 and H has pathwidth at most 3,

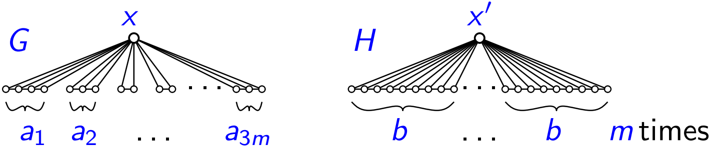
(iii) LIHom is NP-complete on input pairs (G, H), where G has pathwidth at most 2 and H has pathwidth at most 2.

Proof of statement (iii)

Reduce the strongly NP-complete problem 3-PARTITION:

Instance: A multiset $A = \{a_1, a_2, \ldots, a_{3m}\}$ and an integer b s.t. $\sum A = mb$, and $\forall a_i : \frac{b}{4} < a_i < \frac{b}{2}$.

Query: Does A have a *3-partition*, i.e. a partition into m disjoint triplets $A_1, \ldots, A_m$, s.t. $\sum A_i = b$ for each A_i?



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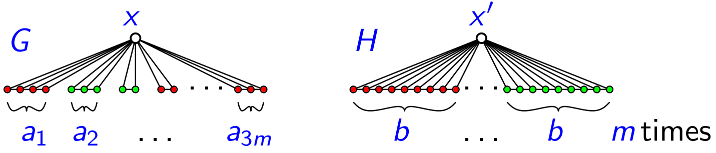

— (A, b) has a 3-partition if and only if $G \xrightarrow{l} H$.
— G and H have pathwidth 2.

What if we bound the treewidth and the maximum degree?

Bounding the treewidth and the maximum degree

Theorem

LBHom, LIHom and LSHom are polynomially solvable when G has bounded treewidth and G or H has bounded maximum degree.

Proof Idea: Use dynamic programming.

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Alternative proof for LBHom and LIHom:

Locally bijective and injective homomorphisms can be expressed as homomorphisms between relational structures.

> **Theorem** [Dalmau, Kolaitis, Vardi, 2002]
> *The existence of a homomorphism between two relational structures $\mathcal{A}$ and $\mathcal{B}$ can be tested in polynomial time if the treewidth of the Gaifman graph $G_{\mathcal{A}}$ is bounded by a constant.*

Here: $G_{\mathcal{A}} \simeq G^2$, which is the graph arising from G by adding an edge between any two vertices at distance 2.
One can show that $\text{tw}(G^2) \leq \Delta(G)(\text{tw}(G) + 1) - 1$.

Open problems

Recall our Theorem:

(i) LBHom is NP-complete on input pairs (G, H), where G has pathwidth at most 5 and H has pathwidth at most 3

(ii) LSHom is NP-complete on input pairs (G, H), where G has pathwidth at most 4 and H has pathwidth at most 3

(iii) LIHom is NP-complete on input pairs (G, H), where G has pathwidth at most 2 and H has pathwidth at most 2.

Can we reduce the bounds on the pathwidth of G for LBHom and LSHom?

Recall our Theorem:
LBHom, LIHom and LSHom are polynomially solvable when G has bounded treewidth and G or H has bounded maximum degree.

The running time for LSHom is

$$O\left(|V_G|\left(|V_H|^{\mathrm{tw}(G)+1}2^{\Delta(H)(\mathrm{tw}(G)+1)}\right)^2(\mathrm{tw}(G)+1)\Delta(H)\right).$$

Note that $G \xrightarrow{S} H$ implies that $\Delta(G) \geq \Delta(H)$.

Are LBHom, LSHom *and* LIHom *fixed-parameter tractable when parameterized by* $\mathrm{tw}(G)+\Delta(G)$, *that is, can they be solved in time*

$$f(\mathrm{tw}(G), \Delta(G)) \cdot (|V_G| + |V_H|)^{O(1)}$$

for some function f *that does not depend on the sizes of* G *and* H*?*

Specific classes od the guest graph G

Guest graph	LBHom	LIHom	LSHom
Chordal	GI-complete [3]	NP-complete	NP-complete [3]
Interval	Polynomial [3]	NP-complete	*open*
Proper Interval	Polynomial	NP-complete	Polynomial [3]
Complete	Polynomial	NP-complete [3]	Polynomial
Tree	Polynomial [2]	Polynomial [1]	Polynomial [2]

[1] [Chaplick, F., van 't Hof, Paulusma, Tesař, 2013]
[2] [F., Paulusma, 2008]
[3] [Heggernes, van 't Hof, Paulusma, 2010]