Harmonic Maps
Exercises 4.

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Definitions and basic properties

Let G, G' be graphs. A function $\varphi : V(G) \cup E(G) \to V(G') \cup E(G')$ is said to be a *morphism* from G to G' if $\varphi(V(G)) \subseteq V(G')$, and for every edge $e \in E(G)$ with endpoints x and y, either $\varphi(e) \in E(G')$ and $\varphi(x), \varphi(y)$ are the endpoints of $\varphi(e)$, or $\varphi(e) \in V(G')$ and $\varphi(e) = \varphi(x) = \varphi(y)$. We write $\varphi : G \to G'$ for brevity. If $\varphi(E(G)) \subseteq E(G')$ then we say that φ is a *homomorphism.* A bijective homomorphism is called an *isomorphism*, and an isomorphism $\varphi : G \to G$ is called an *automorphism.*

Basic definition:

A morphism $\varphi : G \to G'$ is said to be *harmonic* if, for all $x \in V(G)$, $y \in V(G')$ such that $y = \varphi(x)$, the quantity $|e \in E(G) : x \in e, \varphi(e) = e'|$ is the same for all edges $e' \in E(G')$ such that $y \in e'$.

Harmonic Maps

Let $\varphi : G \to G'$ be a morphism and let $x \in V(G)$. Define the *vertical multiplicity* of φ at x by

$$v_\varphi(x) = |e \in E(G) : x \in e, \varphi(e) = \varphi(x)|.$$

This is simply the number of *vertical edges* incident to x, where an edge e is called *vertical* if $\varphi(e) \in V(G')$ (and is called *horizontal* otherwise).
If φ is harmonic and $|V(G')| > 1$, we define the *horizontal multiplicity* of φ at x by

$$m_\varphi(x) = |e \in E(G) : x \in e, \varphi(e) = e'|$$

for any edge $e' \in E(G)$ such that $\varphi(x) \in e'$. By the definition of a harmonic morphism, $m_\varphi(x)$ is independent of the choice of e'.

Harmonic Maps

Define the degree of a harmonic morphism $\varphi : G \to G'$ by the formula

$$\deg(\varphi) := |e \in E(G) : \varphi(e) = e'|$$

for any edge $e' \in E(G')$. By virtue of the following lemma $\deg(\varphi)$ does not depend on the choice of e' (and therefore is well defined):

Lemma 1.

The quantity $|e \in E(G) : \varphi(e) = e'|$ is independent of the choice of $e' \in E(G')$.

Harmonic Maps

According to the next result, the degree of a harmonic morphism $\varphi : G \to G'$ is just the number of pre-images under φ of any vertex of G', counting multiplicities:

Lemma 2.

For any vertex $y \in G$, we have

$$\deg(\varphi) = \sum_{x \in V(G),\, \varphi(x)=y} m_\varphi(x).$$

As with morphisms of Riemann surfaces, a harmonic morphism of graphs must be either constant or surjective.

Lemma 3.

Let $\varphi : G \to G'$ be a harmonic morphism. Then $\deg(\varphi) = 0$ if and only if φ is constant, and $\deg(\varphi) > 0$ if and only if φ is surjective.

Harmonic Maps

In recent papers harmonic maps are called also as *quasi-covering, branched coverings of graphs*. Another, not so popular names, are *wrapped quasi-coverings* and *horizontally conformal* maps. Harmonic maps are generalisation of graph coverings. The simplest examples are given by the following list.

1. Any covering of graphs is a harmonic map.
2. A natural projection of the wheel graph W_6 onto the wheel graph W_2 is a harmonic map.

Harmonic Maps

We say that a group G acts on X if G is a subgroup of $\mathrm{Aut}(X)$.
A group G acts harmonically if G acts fixed point free on the set of directed edges $D(X)$ of a graph X.
In the latter case, the group G acts pure harmonically if G has no invertible edges on X.

Scott Corry (2012) made the following useful observation.

If a group G acts pure harmonically on a graph X then the canonical projection $X \to X/G$ is a harmonic map.

Exercises

Exercise 4.1.
Let $\varphi : G \to G'$ be a harmonic morphism of graphs. Prove the following formula

$$deg(x) = deg(\varphi(x))m_{\varphi}(x) + v_{\varphi}(x),$$

where x is any vertex of the graph G.

Exercise 4.2.
Let cyclic group $\mathbb{Z}_n$ acts on the wheel graph W_{nk} by rotation. Show that the factor graph $W_{nk}/\mathbb{Z}_n$ is isomorphic to W_k and the respective canonical projection $\pi : W_{nk} \to W_k = W_{nk}/\mathbb{Z}_n$ is a harmonic map.

Exercise 4.3.
Show that “zig-zag” map of the path graph P_4 onto the path graph P_2 is a harmonic map.

Exercise 4.4.
Construct a harmonic map of tree onto a tree with one branch point of order n.

Exercise 4.5.
Let group G acts purely harmonically on a graph X. Then the factor map $X \to X/G$ is harmonic map.

Exercise 4.6.
Construct a $\mathbb{Z}_6$-regular harmonic map of a complete bipartite graph $K_{2,3}$ onto a segment P_2.

Exercise 4.7.
Let a finite group G acts on a graph X fixing only one edge e. Replace e by $|G|$ parallel edges to get graph X'. Show that there is a harmonic action of G on X'.

Exercise 4.8.

Construct a 3-fold uniform harmonic map that is irregular.

Exercise 4.9.

Show that every genus 2 bridgeless graph G is hyperelliptic. That is there exists an involution τ acting on G harmonically such that the quotient graph $G/\langle\tau\rangle$ is a tree.