Graph Covering
Exercises 3.

Instructor: Mednykh I. A.

Sobolev Institute of Mathematics
Novosibirsk State University

Winter School in Harmonic Functions on Graphs and Combinatorial Designs

20 - 24 January, 2014

Graph coverings and covering groups

Let X and Y be connected graphs. A surjective morphism $\varphi : X \to Y$ is called a *(graph) covering* if for any vertex $x \in V(X)$ the restriction $\varphi|_{\mathrm{St}_X(x)} : \mathrm{St}_X(x) \to \mathrm{St}_Y(\varphi(x))$ is an isomorphism. A *covering group* of φ is defined as

$$\mathrm{Cov}(\varphi) = \{h \in \mathrm{Aut}(X) : \varphi = \varphi \circ h\}.$$

The covering φ is called *regular* if $\mathrm{Cov}(\varphi)$ act transitively on each fibre of φ and *irregular* otherwise. If $\varphi : X \to Y$ is a regular covering then $Y \cong X/\mathrm{Cov}(\varphi)$. A finite sheeted covering $\varphi : X \to Y$ is regular if and only if the order of covering group $|Cov(\varphi)|$ coincides with the number of sheets of the covering.

Graph coverings and voltage assignments

Permutation voltage assignments were introduced by J. L. Gross and T. W. Tucker. Let X be a finite connected graph, possibly including multiple edges or loops. It is *directed* if each edge (even a loop) is provided by the two possible directions. Let $D(X)$ be the set of the directed edges of X (also known as *darts, arcs* and so on in the literature). A *permutation voltage assignment* of X with voltages in the symmetric group $\mathbb{S}_n$ of degree n is a function $\phi : D(X) \to \mathbb{S}_n$ such that inverse edges have inverse assignments. The pair $(D(X), \phi)$ is called a permutation voltage graph.

Graph coverings and voltage assignments

The *(permutation) derived graph* X^{ϕ} derived from a permutation voltage assignment ϕ is defined as follows: $V(X^{\phi}) = V(X) \times \{1, \cdots, n\}$, and $((u,j), (v,k)) \in D(X^{\phi})$ if and only if $(u,v) \in D(X)$ and $k = \phi(u,v)(j)$. The natural projection $\pi : X^{\phi} \to X$ that is a function from $V(X^{\phi})$ onto $V(X)$ which erases the second coordinates gives a *graph covering*.
J. L. Gross and T. W. Tucker showed that every covering of a given graph arises from some permutation voltage assignment in a symmetric group. Moreover, such a covering is connected if and only if $\phi(D(X))$ is a transitive subgroup in $\mathbb{S}_n$.

Regular coverings and ordinary voltage assignments

Ordinary voltage assignments were introduced by J. L. Gross. Let G be a finite group. Then a mapping $\omega : D(X) \to G$ is called an *ordinary voltage assignment* if $\omega(v, u) = \omega(u, v)^{-1}$ for each $(u, v) \in D(X)$. The *(ordinary) derived graph* X^ω derived from an ordinary voltage assignment ω is defined as follows: $V(X^\omega) = V(X) \times G$, and $((u, j), (v, k)) \in D(X^\omega)$ if and only if $(u, v) \in D(X)$ and $k = \omega(u, v)j$. Consider the natural projection $\pi : X^\omega \to X$ that is a function from $V(X^\omega)$ onto $V(X)$ which erases the second coordinates. Then the map $\pi : X^\omega \to X$ is a *G-covering* of X, that is a $|G|$-fold regular covering of X with the covering group G. Every regular covering of X can be obtained in such a way.

Short way to construct coverings

Let X be a graph of genus g. Choose a spanning tree T in X and g directed edges $e_1, e_2, \ldots, e_g$ from the compliment $X \setminus T$.
An arbitrary *reduced permutation assignment* $\psi : D(X) \to \mathbb{S}_n$ is uniquely determined by the following conditions:

(i) $\psi(e_i) = \xi_i$, where $\xi_i \in \mathbb{S}_n$ for $i = 1, 2, \ldots, g$ and $\psi(e) = 1$, for any edge e which is in T;

(ii) $\xi_1, \xi_2, \ldots, \xi_g$ generate a transitive subgroup in $\mathbb{S}_n$.

Then the permutation derived graph gives a required covering.
All connected n-fold coverings can be obtained in such a way. Two tuples $(\xi_1, \xi_2, \ldots, \xi_g)$ and $(\xi'_1, \xi'_2, \ldots, \xi'_g)$ give equivalent coverings if and only if there exists $h \in \mathbb{S}_n$ such that $\xi'_i = h\,\xi_i\,h^{-1}$ for all $i = 1, 2, \ldots, g$.

Exercises

Exercise 3.1.
Draw all 2-fold coverings of the figure-eight graph. Show that all of them are regular.

Exercise 3.2.
Draw all 3-fold coverings of the figure-eight graph. How many of them are regular?

Exercise 3.3.
Let X be a connected graph and X is not a tree. Show that G has infinitely many non-equivalent coverings.

Exercise 3.4.
Construct the universal covering tree for the following graphs:

1° Cyclic graph C_n,

2° The figure eight graph.

Exercise 3.5.
Show that two cyclic graphs C_m and C_n share a finite sheeted covering.

Exercise 3.6.
Describe all coverings of a cyclic graph C_n.

Exercise 3.7.
Let Y be a bipartite graph and $\varphi : X \to Y$ is a graph covering. Show that X is also a bipartite graph.

Exercise 3.8.
Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be regular graph coverings. Is it true that $\psi \circ \varphi : X \to Z$ is also regular graph covering?