Spanning Trees
Exercises 2.

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Spanning tree

A *spanning tree* T of a connected, undirected graph G is a tree composed of all the vertices and some (or perhaps all) of the edges of G. In other words, a spanning tree of G is a selection of edges of G that form a tree spanning every vertex. That is, every vertex lies in the tree, but no cycles (or loops) are allowed. On the other hand, every bridge of G must belong to T. A spanning tree of a connected graph G can also be defined as a maximal set of edges of G that contains no cycle, or as a minimal set of edges that connect all vertices.

Counting spanning trees

The number $t(G)$ of spanning trees of a connected graph is a well-studied invariant. In some cases, it is easy to calculate $t(G)$ directly. For example, if G is itself a tree, then $t(G) = 1$, while if G is the cycle graph C_n with n vertices, then $t(G) = n$. For any graph G, the number $t(G)$ can be calculated using Kirchhoff's matrix-tree theorem.

Here are some known results concerning counting spanning trees of graphs.

1. Complete graph K_n : $t(K_n) = n^{n-2}$ (Cayley's formula),
2. Complete bipartite graph $K_{n,m}$: $t(K_{n,m}) = m^{n-1}n^{m-1}$,
3. n-dimensional cube graph Q_n : $t(Q_n) = 2^{2^n-n-1}\prod\limits_{k=2}^{n} k^{\binom{n}{k}}$.

Kirchhoff Matrix-Tree Theorem

The celebrated Kirchhoff Matrix-Tree Theorem is the following statement.

Theorem (Kirchhoff (1847))

All cofactors of Laplacian matrix $L(G)$ are equal to $t(G)$.

More convenient form of this result were obtained by A. K. Kel'mans and V. M. Chelnokov.

Theorem (Kel'mans, Chelnokov (1974))

Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ denote the eigenvalues of the Laplace matrix $L(X)$ of a n point graph X. Then

$$t(X) = \frac{1}{n} \prod_{k=2}^{n} \lambda_k.$$

Spanning Trees

A generalisation of the Matrix-Tree-Theorem was obtained by Kelmans (1967) who gave a combinatorial interpretation to all the coefficients of $\mu(X, x)$ in terms of the numbers of certain subforests of the graph. This result has been obtained even in greater generality (for weighted graphs) by Fiedler and Sedláček.

Theorem (Kel'mans (1967))

If

$$\mu(X, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^i c_i x^{n-i} + \ldots + (-1)^{n-1} c_{n-1} x$$

then

$$c_i = \sum_{S \subset V, |S| = n-i} t(X_S),$$

where $t(H)$ is the number of spanning trees of H, and X_S is obtained from X by identifying all vertices of S to a single one.

From the last theorem we can derive useful corollary.

Corollary

The degree of Laplacian polynomial $\mu(X, x)$ is equal to $n = |V(X)|$. Its coefficients c_1 and c_{n-1} are given by the formulas $c_1 = 2|E(X)|$ and $c_{n-1} = |V(X)| \cdot t(X)$.

Hence, the number of vertices $|V(X)|$, number of edges $|E(X)|$ and the number of spanning trees $t(X)$ are uniquely defined by the Laplacian polynomial.

Exercises

Exercise 2.1.
Prove that the number of spanning trees for the path graph P_n is 1.

Exercise 2.2.
Prove that the number of spanning trees for the cyclic graph C_n is n.

Exercise 2.3.
Prove the Cayley formula for the number of spanning trees for the complete graph K_n : $t(K_n) = n^{n-2}$.

Exercise 2.4.
Prove that the number of spanning trees for the complete bipartite graph $K_{n,m}$ is given by the formula $t(K_{n,m}) = m^{n-1}n^{m-1}$.

Exercise 2.5. Denote by $G - e$ the graph obtained by removing edge e from the graph G. Let $G \backslash e$ be the graph obtained from graph G by contracting edge e. In other words, $G \backslash e$ is obtained by deleting edge e and identifying its ends. Prove the following formula

$$t(G) = t(G - e) + t(G \backslash e).$$

Exercise 2.6. Denote by $G_{s,e}$ the graph resulting from subdivision of an edge e of a graph G. Then

$$t(G_{s,e}) = t(G \backslash e) + 2t(G - e) = t(G) + t(G - e).$$

Exercise 2.7.
Find the number of spanning trees for the wheel graph $W_n = K_1 * C_n$.
Answer: If n is odd then $t(W_n) = \ell_k^2$, if n is even then $t(W_n) = 5f_n^2$, where ℓ_j is j-th Lukas number and f_k is k-th Fibonacci number.
Note:
$\ell_1 = 1,\ \ell_2 = 3,\ \ell_{k+2} = \ell_{k+1} + \ell_k,\ k \geq 1.$
$f_1 = 1,\ f_2 = 1,\ f_{k+2} = f_{k+1} + f_k,\ k \geq 1.$
$f_{2n} = \ell_n \cdot f_n$ and $\ell_n = f_{n-1} + f_{n+1}.$

Exercise 2.8.
Find the number of spanning trees for the fan graph $F_n = K_1 * P_n$.
Answer: $t(F_n) = f_{2n}.$

Exercise 2.9.
Find the number of spanning trees for the lattice graph $L_{m,n} = K_m \times K_n$.
Answer: $t(K_m \times K_n) = m^{m-2} n^{n-2} (m+n)^{(m-1)(n-1)}$.

Exercise 2.10.
Prove that the following result by Boesch and Prodinger

$$t(K_m \times C_n) = \frac{n}{m} 2^{m-1} (T_n(1 + \frac{m}{2}) - 1)^{m-1},$$

where $T_n(x) = \cos(n \arccos x)$ is the Chebyshev polynomial of the first kind.

Exercise 2.11.

Prove that the number of spanning trees for the prism $P_2 \times C_n$ is given by the formula

$$t(P_2 \times C_n) = \frac{n}{2}((2+\sqrt{3})^n + (2-\sqrt{3})^n - 2).$$

Exercise 2.12.

Prove that the number of spanning trees for the Moebius ladder graph M_n is given by the formula

$$t(M_n) = \frac{n}{2}((2+\sqrt{3})^n + (2-\sqrt{3})^n + 2).$$

Addendum

Chebyshev polynomials

The Chebyshev polynomial of the first kind is defined by the formula

$$T_n(x) = \cos(n \arccos x).$$

Equivalently,

$$T_n(x) = \frac{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n}{2}.$$

Also, $T_n(x)$ satisfies the recursive relation

$$T_0(x) = 1,\ T_1(x) = x,\ T_n(x) = 2x \cdot T_{n-1}(x) - T_{n-2}(x),\ n \geq 2.$$

Chebyshev polynomials

The Chebyshev polynomial of the second kind is defined by the formula

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}.$$

Equivalently,

$$U_n(x) = \frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}.$$

Also, $U_n(x)$ satisfies the recursive relation

$$U_0(x) = 1,\ U_1(x) = 2x,\ U_n(x) = 2x \cdot U_{n-1}(x) - U_{n-2}(x),\ n \geq 2.$$

We have $U_n(\cos\frac{k\pi}{n+1}) = 0, k = 1, 2, \ldots, n$. Hence

$$U_n(x) = 2^n \prod_{k=1}^{n} (x - \cos\frac{k\pi}{n+1}).$$

Chebyshev polynomials

Since

$$U_n(x) = (-1)^n U_n(-x) = 2^n \prod_{k=1}^{n} (x + \cos \frac{k\pi}{n+1})$$

we obtain

$$U_n^2(x) = \prod_{k=1}^{n} (4x^2 - 4\cos^2 \frac{k\pi}{n+1}).$$

Polynomials $T_n(x)$ and $U_{n-1}(x)$ are related by the following identity

$$T_n^2(x) + (x^2 - 1)U_{n-1}^2(x) = 1.$$

Chebyshev polynomials

Consider $n \times n$ matrix

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2x & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2x & -1 & \dots & 0 & 0 \\ 0 & 0 & -1 & 2x & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2x & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2x \end{pmatrix}.$$

Then $\det A_n(x) = U_n(x)$.