Spanning Trees
Exercises 2. Solutions

Instructor: Mednykh I. A.

Sobolev Institute of Mathematics
Novosibirsk State University

Winter School in Harmonic Functions on Graphs and Combinatorial Designs

20 - 24 January, 2014

Exercises

Exercise 2.1.
Prove that the number of spanning trees for the path graph P_n is 1.

Exercise 2.2.
Prove that the number of spanning trees for the cyclic graph C_n is n.

Exercise 2.3.
Prove the Cayley formula for the number of spanning trees for the complete graph K_n : $t(K_n) = n^{n-2}$.

Exercise 2.4.
Prove that the number of spanning trees for the complete bipartite graph $K_{n,m}$ is given by the formula $t(K_{n,m}) = m^{n-1}n^{m-1}$.

Exercise 2.5. Denote by $G - e$ the graph obtained by removing edge e from the graph G. Let $G \backslash e$ be the graph obtained from graph G by contracting edge e. In other words, $G \backslash e$ is obtained by deleting edge e and identifying its ends. Prove the following formula

$$t(G) = t(G - e) + t(G \backslash e).$$

Exercise 2.6. Denote by $G_{s,e}$ the graph resulting from subdivision of an edge e of a graph G. Then

$$t(G_{s,e}) = t(G \backslash e) + 2t(G - e) = t(G) + t(G - e).$$

Exercise 2.7.

Find the number of spanning trees for the wheel graph $W_n = K_1 * C_n$.
Answer: If n is odd then $t(W_n) = \ell_k^2$, if n is even then $t(W_n) = 5f_k^2$, where ℓ_j is j-th Lukas number and f_k is k-th Fibonacci number.
Note:
$\ell_1 = 1,\ \ell_2 = 3,\ \ell_{k+2} = \ell_{k+1} + \ell_k,\ k \geq 1.$
$f_1 = 1,\ f_2 = 1,\ f_{k+2} = f_{k+1} + f_k,\ k \geq 1.$
$f_{2n} = \ell_n \cdot f_n$ and $\ell_n = f_{n-1} + f_{n+1}.$

Exercise 2.8.

Find the number of spanning trees for the fan graph $F_n = K_1 * P_n$.

Solution: By Exercise 1.7 and Kel'mans-Chelnokov theorem we obtain

$$t(F_n) = \prod_{k=1}^{n-1}(3 - 2\cos\frac{\pi k}{n}) = 2^{n-1}\prod_{k=1}^{n-1}(\frac{3}{2} - \cos\frac{\pi k}{n}).$$

$$= U_{n-1}(\frac{3}{2}) = \frac{1}{\sqrt{5}}\Big((\frac{3+\sqrt{5}}{2})^n - (\frac{3-\sqrt{5}}{2})^n\Big)$$

$$= \frac{1}{\sqrt{5}}\Big((\frac{1+\sqrt{5}}{2})^{2n} - (\frac{1-\sqrt{5}}{2})^{2n}\Big) = f_{2n}.$$

Exercise 2.9.

Find the number of spanning trees for the lattice graph $L_{m,n} = K_m \times K_n$.

Solution: The Laplace spectrums of the graphs K_m and K_n are

$\mu_0 = 0,\ \mu_i = m,\ i = 1, \dots, m-1$ and $\lambda_0 = 0,\ \lambda_j = n,\ j = 1, \dots, n-1.$

Then $t(K_m \times K_n) = \frac{1}{m\,n} \prod\limits_{\substack{i=0 \\ i+j>0}}^{m-1} \prod\limits_{j=0}^{n-1} (\mu_i + \lambda_j) =$

$$\frac{1}{m\,n} \prod_{j=1}^{n-1} \lambda_j \prod_{i=1}^{m-1} \mu_i \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} (\lambda_j + \mu_i) = m^{m-2} n^{n-2} (m+n)^{(m-1)(n-1)}.$$

Exercise 2.10.

Prove that the following result by Boesch and Prodinger

$$t(K_m \times C_n) = \frac{n}{m} 2^{m-1} (T_n(1 + \frac{m}{2}) - 1),$$

where $T_n(x) = \cos(n \arccos x)$ is the Chebyshev polynomial of the first kind.

Solution:

$$t(K_m \times C_n) = \frac{1}{mn} \prod_{\substack{i=0 \\ i+j>0}}^{m-1} \prod_{j=0}^{n-1} (\mu_i + \lambda_j) = \frac{1}{mn} \prod_{j=1}^{n-1} \lambda_j \prod_{i=1}^{m-1} \mu_i \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} (\lambda_j + \mu_i)$$

$$= t(C_n) t(K_m) \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} (m + 2 - 2\cos(2\pi j/n))$$

$$= n\, m^{m-2} (\prod_{j=1}^{n-1} (m + 4 - 4\cos^2(\pi j/n)))^{m-1} = n\, m^{m-2} \left[U_{n-1}^2 (\sqrt{\tfrac{m+4}{4}}) \right]^{m-1}.$$

From elementary identities $\sin^2(u) = \frac{1-\cos(2u)}{2}$, $\cos(2u) = 2\cos^2(u) - 1$ and basic definitions of the Chebyshev polynomials one can derive the following relations

$$U_{n-1}^2(x) = \frac{1}{2(1-x^2)}(1 - T_{2n}(x))$$

$$= \frac{1}{2(1-x^2)}(1 - T_n(T_2(x))) = \frac{1}{2(1-x^2)}(1 - T_n(2x^2 - 1)).$$

Putting $x = \sqrt{\frac{m+4}{4}}$ we get the formula by Boesch and Prodinger

$$t(K_m \times C_n) = \frac{n}{m} 2^{m-1} (T_n(1 + \frac{m}{2}) - 1)^{m-1}.$$

Exercise 2.11.
Prove that the number of spanning trees for the prism $P_2 \times C_n$ is given by the formula

$$t(P_2 \times C_n) = \frac{n}{2}((2+\sqrt{3})^n + (2-\sqrt{3})^n - 2).$$

Solution: Since $P_2 = K_2$, we put $m = 2$ in the solution of Exercise 2.10 to obtain

$$t(P_2 \times C_n) = n(T_n(2) - 1) = \frac{n}{2}((2+\sqrt{3})^n + (2-\sqrt{3})^n - 2).$$

Exercise 2.12.
Prove that the number of spanning trees for the Moebius ladder graph M_n is given by the formula

$$t(M_n) = \frac{n}{2}((2+\sqrt{3})^n + (2-\sqrt{3})^n + 2).$$

Solution: Let us note that the Laplacian matrix for M_n is circulant $circ\{v_0 \ldots, v_{2n-1}\}$, where $v_0 = 3,\ v_1 = -1,\ v_2 = \ldots = v_{n-1} = 0$, $v_n = -1,\ v_{n+1} = \ldots = v_{2n-2} = 0,\ v_{2n-1} = -1$. Let $\varepsilon = e^{\frac{2\pi i}{2n}}$ be the $2n$-th primitive root of unity.
Then $L(M_n)$ has the following spectrum

$$\lambda_k = \sum_{j=0}^{2n-1} \varepsilon^{kj} v_j = 3 + (-1)^{k+1} - 2\cos\frac{\pi k}{n},\ k = 0, \ldots, 2n-1.$$

We have

$$t(M_n) = \frac{1}{2n} \prod_{k=1}^{2n-1} (3 + (-1)^{k+1} - 2\cos\frac{\pi k}{n})$$

$$= \frac{1}{2n} \prod_{j=1}^{n-1} (4 - 2\cos\frac{(2j-1)\pi}{n}) \prod_{j=1}^{n-1} (2 - 2\cos\frac{2j\pi}{n}).$$

We note that

$$\prod_{j=1}^{n-1} (2 - 2\cos\frac{2j\pi}{n}) = \prod_{j=1}^{n-1} (4 - 4\cos^2\frac{j\pi}{n}) = U_{n-1}^2(1) = n^2.$$

Remark:

$$U_{n-1}(1) = \frac{\sin(n \arccos 1)}{\sin(\arccos 1)} = \lim_{u\to 0} \frac{\sin(n\, u)}{\sin(u)} = n.$$

Now we simplify the first product

$$\prod_{j=1}^{n-1}(4-2\cos\frac{(2j-1)\pi}{n})=\prod_{j=1}^{2n-1}(4-2\cos\frac{2j\pi}{2n})\Big/\prod_{j=1}^{n-1}(4-2\cos\frac{2j\pi}{n}).$$

We get

$$\prod_{j=1}^{n-1}(4-2\cos\frac{2j\pi}{n})=\prod_{j=1}^{n-1}(6-4\cos^2\frac{j\pi}{n})=U_{n-1}^2(\sqrt{\frac{3}{2}}).$$

From the properties of Chebyshev polynomias we have

$$U_{n-1}^2(x)=\frac{1}{2(1-x^2)}(1-T_{2n}(x))=\frac{1}{2(1-x^2)}(1-T_n(2x^2-1)).$$

In particular, for $x=\sqrt{3/2}$ we obtain $U_{n-1}^2(\sqrt{\frac{3}{2}})=T_n(2)-1$. Simillary,

$$U_{2n-1}^2(\sqrt{\frac{3}{2}})=T_{2n}(2)-1.$$

By making use of the identity $\dfrac{\sin 2n\,u}{\sin n\,u} = 2\cos n\,u$, we have

$$t(M_n) = \frac{1}{2n} \cdot \frac{T_{2n}(2) - 1}{T_n(2) - 1} \cdot n^2 = n(T_n(2) + 1).$$

Equivalently,

$$t(M_n) = \frac{n}{2}((2+\sqrt{3})^n + (2-\sqrt{3})^n + 2).$$