Laplacian for Graph
Exercises 1. Solutions

Instructor: Mednykh I. A.

Sobolev Institute of Mathematics
Novosibirsk State University

Winter School in Harmonic Functions on Graphs and Combinatorial Designs

20 - 24 January, 2014

Exercise 1.1.

Find Laplacian spectrum of the complete graph on n vertices K_n.

Solution: We want to show that $\mu(K_n, x) = x(x-n)^{n-1}$. To solve this problem we use induction by number of vertices n. For $n = 1$, K_1 is a singular vertex. Its Laplacian matrix $L(K_n) = \{0\}$. Hence $\mu(K_1, x) = x$. Hence the statement is true for $n = 1$. Suppose that for given n the equality $\mu(K_n, x) = x(x-n)^{n-1}$ is already proved. It is easy to see that K_{n+1} is a join of K_n and K_1. By Kel'mans theorem we get

$$\mu(K_{n+1}, x) = \frac{x(x-n-1)}{(x-1)(x-n)}\mu(K_1, x-n)\mu(K_n, x-1) =$$

$$= \frac{x(x-n-1)}{(x-1)(x-n)}(x-n)(x-1)(x-n-1)^{n-1} = x(x-n-1)^n.$$

Hence, the Laplacian spectrum of K_n is $\{0^1,\ n^{n-1}\}$.

Exercise 1.2.

Find Laplacian spectrum of the complete bipartite graph $K_{n,m}$.

Solution: Let us note that $K_{n,m}$ is a join of X_m and X_n, where X_k is a disjoint union of k vertices. We have

$L(X_k) = D(X_k) - A(X_k) = O_k - O_k = O_k$, where O_k is $k \times k$ zero matrix. Hence, $\mu(X_k, x) = x^k$. By Kel'mans theorem we obtain

$$\mu(K_{n,m}, x) = \frac{x(x-m-n)}{(x-n)(x-m)}\mu(X_n, x-m)\mu(X_m, x-n) =$$

$$= \frac{x(x-m-n)}{(x-n)(x-m)}(x-m)^n(x-n)^m = x(x-m-n)(x-n)^{m-1}(x-m)^{n-1}.$$

Hence, the Laplacian spectrum of $K_{n,m}$ is $\{0^1,\ n^{m-1},\ m^{n-1}, (m+n)^1\}$.

Exercise 1.3.

Find Laplacian spectrum of the cycle graph C_n.

Solution: The Laplacian matrix $L(C_n)$ is the circulant matrix with entities

$$v_0 = 2,\ v_1 = -1,\ v_2 = \ldots = v_{n-2} = 0,\ v_{n-1} = -1.$$

Then by properties of circulant matrices its eigenvalues are

$$\lambda_k = v_0 + v_1\varepsilon^k + \ldots + v_{n-1}\varepsilon^{(n-1)k},\ k = 0,\ \ldots,\ n-1,$$

where $\varepsilon = e^{\frac{2\pi i}{n}}$ is the n-th primitive root of the unity. Hence,

$$\lambda_k = 2 - \varepsilon^k - \varepsilon^{(n-1)k}.$$

Since

$$e^{\frac{2\pi i}{n}k} + e^{\frac{2\pi i}{n}(n-1)k} = 2\left(\frac{e^{\frac{2\pi i}{n}k} + e^{-\frac{2\pi i}{n}k}}{2}\right) = 2\cos\frac{2\pi k}{n},$$

we have $\lambda_k = 2 - 2\cos\frac{2\pi k}{n},\ k = 0,\ \ldots,\ n-1.$

Exercise 1.4.
Find Laplacian spectrum of the path graph P_n.
Solution: The Laplacian matrix for path graph P_n has the form

$$L_n = L(P_n) = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix}.$$

Then its characteristic matrix is given by

$$L_n - \lambda I_n = \begin{pmatrix} 1-\lambda & -1 & 0 & \dots & 0 & 0 \\ -1 & 2-\lambda & 1 & \dots & 0 & 0 \\ 0 & -1 & \lambda-2 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2-\lambda & -1 \\ 0 & 0 & 0 & \dots & -1 & 1-\lambda \end{pmatrix}.$$

Let $V_n = \det(L_n - \lambda I_n)$. Then $\det V_n$ is equal

$$\begin{vmatrix} 1-\lambda & -1 & \dots & 0 \\ -1 & 2-\lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1-\lambda \end{vmatrix} =$$

$$\begin{vmatrix} 2-\lambda & -1 & \dots & 0 \\ -1 & 2-\lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 0 & \dots & 0 \\ -1 & 2-\lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1-\lambda \end{vmatrix} =$$

$$\underbrace{\begin{vmatrix} 2-\lambda & -1 & \dots & 0 \\ -1 & 2-\lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1-\lambda \end{vmatrix}}_{n\times n} - \underbrace{\begin{vmatrix} 2-\lambda & -1 & \dots & 0 \\ -1 & 2-\lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1-\lambda \end{vmatrix}}_{n-1\times n-1} =$$

$$D_n - D_{n-1}.$$

In a similar way $D_n = U_n - U_{n-1}$, where

$$U_n = \underbrace{\begin{vmatrix} 2-\lambda & -1 & \dots & 0 \\ -1 & 2-\lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 2-\lambda \end{vmatrix}}_{n\times n} = U_n(\frac{2-\lambda}{2}).$$

Here $U_n(x) = \dfrac{\sin(n+1)\arccos x}{\sin\arccos x}$ is a Chebyshev polynomial of the second kind. Since $U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0$, we obtain

$$V_n = D_n - D_{n-1} = U_n(x) - 2U_{n-1}(x) + U_{n-2}(x)$$

$$= (2x-2)U_{n-1}(x) = -\lambda U_{n-1}(\frac{\lambda-2}{2}),$$

where $x = \frac{\lambda-2}{2}$. The equation

$$\lambda U_{n-1}(\frac{\lambda-2}{2}) = 0$$

has the following solutions $\lambda_k = 2 - 2\cos(\frac{\pi k}{n}),\ k = 0, \ldots, n-1.$

Exercise 1.5.
Show that Laplacian polynomial of the path graph P_n has the following form

$$\mu(P_n, x) = x\, U_{n-1}(\frac{x-2}{2}),$$

where $U_{n-1}(x) = \dfrac{\sin(n \arccos x)}{\sin(\arccos x)}$ is the Chebyshev polynomial of the second kind.

Solution:
Follows from the previous exercise.

Exercise 1.6.
Find Laplacian spectrum of the wheel graph $W_n = K_1 * C_n$.
Answer: $\{0,\ n+1,\ 3-2\cos\frac{2\pi k}{n},\ k=1,\ \ldots,\ n-1\}$

Exercise 1.7.
Find Laplacian spectrum of the fan graph $F_n = K_1 * P_n$.
Answer: $\{0,\ n+1,\ 3-2\cos\frac{\pi k}{n},\ k=1,\ \ldots,\ n-1\}$

Exercise 1.8.
Show that the Laplacian polynomial of the fan graph $F_n = K_1 * P_n$ is given by the formula

$$\mu(F_n,\ x) = x(x-n-1)U_{n-1}(\frac{x-3}{2})$$

where $U_{n-1}(x)$ is the Chebyshev polynomial of the second kind.

Exercise 1.9.
Find Laplacian spectrum of the cylinder graph $P_m \times C_n$.
Solution:
The spectrum of P_m is $\lambda_j = 4\sin^2(\frac{\pi j}{2m})$, $j = 0, \ldots, m-1$ and the spectrum of C_n is $\mu_k = 4\sin^2(\frac{2\pi k}{2n})$, $k = 0, \ldots, n-1$. As a result we have the following spectrum for $P_m \times C_n$.

$$\ell_{j,k} = 4\sin^2(\frac{\pi j}{2m}) + 4\sin^2(\frac{2\pi k}{2n}),\ j = 0, \ldots, m-1,\ k = 0, \ldots, n-1.$$

Exercise 1.10.
Find Laplacian spectrum of the Moebius ladder graph M_n. Moebius ladder graph is a cycle graph C_{2n} with additional edges, connecting opposite vertices in cycle.
Solution:
We note that the Laplacian matrix for M_n is circulant $circ\{v_0 \ldots, v_{2n-1}\}$, where $v_0 = 3,\ v_1 = -1,\ v_2 = \ldots = v_{n-1} = 0,$
$v_n = -1,\ v_{n+1} = \ldots = v_{2n-2} = 0,\ v_{2n-1} = -1.$ Let $\varepsilon = e^{\frac{2\pi i}{2n}}$ be the $2n$-th primitive root of unity.
Then $L(M_n)$ has the following spectrum

$$\lambda_k = \sum_{j=0}^{2n-1} \varepsilon^{kj} v_j = 3 + (-1)^{k+1} - 2\cos\frac{\pi k}{n},\ k = 0, \ldots, 2n-1.$$