Laplacian for Graphs
Exercises 1.

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Winter School in Harmonic Functions on Graphs and Combinatorial Designs

20 - 24 January, 2014

Laplacian matrix. Laplacian spectrum

The graphs under consideration are supposed to be unoriented and finite. They may have loops, multiple edges and to be disconnected.
Let a_{uv} be the number of edges between two given vertices u and v of G.
The matrix $A = A(G) = [a_{uv}]_{u,v \in V(G)}$, is called the *adjacency matrix* of the graph G.
Let $d(v)$ denote the degree of $v \in V(G)$, $d(v) = \sum_u a_{uv}$, and let $D = D(G)$ be the diagonal matrix indexed by $V(G)$ and with $d_{vv} = d(v)$.
The matrix $L = L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G.
It should be noted that loops have no influence on $L(G)$. The matrix $L(G)$ is sometimes called the *Kirchhoff matrix* of G.

Laplacian polynomial and Laplacian spectrum

We denote by $\mu(G, x)$ the characteristic polynomial of $L(G)$. We will call it the *Laplacian polynomial.* Its roots will be called the *Laplacian eigenvalues* (or sometimes just eigenvalues) of G. They will be denoted by $\lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_n(G)$, $(n = |V(G)|)$, always enumerated in increasing order and repeated according to their multiplicity.

We note that λ_1 is always equal to 0.
Graph G is connected if and only if $\lambda_2 > 0$.

If G consists of k components then

$$\lambda_1(G) = \lambda_2(G) = \ldots = \lambda_k(G) = 0 \text{ and } \lambda_{k+1}(G) > 0.$$

Preliminary results

The following theorems help a lot when dealing with computation of Laplacian polynomials of various graphs.

Theorem (Eigenvalues of circulant matrix)

Let $v = (v_0,\, v_1,\, \ldots,\, v_{n-1})$ be a row vector in $\mathbb{C}^n$, and $V = circ\{v\}$. If ε is primitive $n-th$ root of unity, then

$$\det\ V = \det \begin{pmatrix} v_0 & v_1 & \ldots & v_{n-2} & v_{n-1} \\ v_{n-1} & v_0 & \ldots & v_{n-3} & v_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_2 & v_3 & \ldots & v_0 & v_1 \\ v_1 & v_2 & \ldots & v_{n-1} & v_0 \end{pmatrix} = \prod_{l=0}^{n-1}\Big(\sum_{j=0}^{n-1} \varepsilon^{jl} v_j\Big).$$

Corollary

Eigenvalues of circulant matrix V is given by the formulae

$$\lambda_l = \sum_{j=0}^{n-1} \varepsilon^{jl} v_j,\ \ l = 0,\, \ldots,\, n-1.$$

Laplacian for Graphs.

Theorem (Kel'mans)

Let $X_1 * X_2$ denote the join of X_1 and X_2, i.e. the graph obtained from the disjoint union of X_1 and X_2 by adding all possible edges $uv,\ u \in V(X_1),\ v \in V(X_2)$. Then

$$\mu(X_1 * X_2, x) = \frac{x(x - n_1 - n_2)}{(x - n_1)(x - n_2)} \mu(X_1, x - n_2)\mu(X_2, x - n_1).$$

where n_1 and n_2 are orders of X_1 and X_2, respectively and $\mu(X, x)$ is the characteristic polynomial of the Laplacian matrix of X.

Theorem (M. Fiedler (1973))

The Laplacian eigenvalues of the Cartesian product $X_1 \times X_2$ of graphs X_1 and X_2 are equal to all the possible sums of eigenvalues of the two factors:

$$\lambda_i(X_1) + \lambda_j(X_2),\ i = 1, \ldots, |V(X_1)|,\ j = 1, \ldots, |V(X_2)|.$$

Using this theorem we can easily determine the spectrum of "lattice" graphs. The $m \times n$ lattice graph is just the Cartesian product of paths, $P_m \times P_n$. Below we will show that the spectrum of path-graph P_k is

$$\ell_i^{(k)} = 4\sin^2 \frac{\pi i}{2k},\ i = 0, 1, \ldots, k-1.$$

So $P_m \times P_n$ has eigenvalues

$$\lambda_{i,j} = \ell_i^{(m)} + \ell_j^{(n)} = 4\sin^2 \frac{\pi i}{2m} + 4\sin^2 \frac{\pi j}{2n},\ i = 0, 1, \ldots, m-1, j = 0, 1, \ldots, n-1.$$

Exercises

Exercise 1.1.
Find Laplacian spectrum of the complete graph on n vertices K_n.
Answer: $\{0^1,\ n^{n-1}\}$.

Exercise 1.2.
Find Laplacian spectrum of the complete bipartite graph $K_{n,m}$.
Answer: $\{0^1,\ n^{m-1},\ m^{n-1}, (m+n)^1\}$.

Exercise 1.3.
Find Laplacian spectrum of the cycle graph C_n.
Answer: $\{2 - 2\cos(\frac{2\pi k}{n}),\ k = 0,\ \ldots,\ n-1\}$.

Exercise 1.4.
Find Laplacian spectrum of the path graph P_n.
Answer: $\{2 - 2\cos(\frac{\pi k}{n}),\ k = 0,\ \ldots,\ n-1\}$.

Exercise 1.5.
Show that Laplacian polynomial of the path graph P_n has the following form

$$\mu(P_n, x) = x\, U_{n-1}(\frac{x-2}{2}),$$

where $U_{n-1}(x) = \dfrac{\sin(n \arccos x)}{\sin(\arccos x)}$ is the Chebyshev polynomial of the second kind.

Exercise 1.6.
Find Laplacian spectrum of the wheel graph $W_n = K_1 * C_n$.
Answer: $\{0,\ n+1,\ 3-2\cos\frac{2\pi k}{n},\ k=1,\ \ldots,\ n-1\}$.

Exercise 1.7.
Find Laplacian spectrum of the fan graph $F_n = K_1 * P_n$.
Answer: $\{0,\ n+1,\ 3-2\cos\frac{\pi k}{n},\ k=1,\ \ldots,\ n-1\}$.

Exercise 1.8.
Show that the Laplacian polynomial of the fan graph $F_n = K_1 * P_n$ is given by the formula

$$\mu(F_n,\ x) = x(x-n-1)U_{n-1}(\frac{x-3}{2}),$$

where $U_{n-1}(x)$ is the Chebyshev polynomial of the second kind.

Exercise 1.9.
Find Laplacian spectrum of the cylinder graph $P_m \times C_n$.

Exercise 1.10.
Find Laplacian spectrum of the Moebius ladder graph M_n. Moebius ladder graph is a cycle graph C_{2n} with additional edges, connecting opposite vertices in cycle.