Math++ Problems

Problem set 4 – Polynomials

hints after **27. 5. 2020**, solutions due **30. 6. 2020**

Definition: Let $\mathbb{K}$ be a field and let $f_1, \dots, f_k \in \mathbb{K}[x_1, \dots, x_n]$. We define the variety of $f_1, \dots, f_k$ as the set

$$V(f_1, \dots, f_k) := \bigcap_{i=1}^{k} \{(x_1, \dots, x_n) \in \mathbb{K}^n \colon f_i(x_1, \dots, x_n) = 0\}\,.$$

Definition: Let $f = \sum_{i=0}^{k} f_i x^i$ a $g = \sum_{j=0}^{l} g_j x^j$ be polynomials of a single variable over a field $\mathbb{K}$. Then the *resultant* $\mathrm{Res}(f, g, x)$ is the determinant of the Sylvester matrix $(F_l | G_k)$, where $F_l \in \mathbb{C}^{(l+k)\times l}$ and $(F_l)_{r,c} = f_{r-c}$, the indexing of the rows and the columns starts from 0, and $f_i = 0$ if $i \notin [0, k]$. We define G_k analogously. Here is an example of the Sylvester matrix for $k = 2$ and $l = 3$:

$$\begin{pmatrix} f_0 & 0 & 0 & g_0 & 0 \\ f_1 & f_0 & 0 & g_1 & g_0 \\ f_2 & f_1 & f_0 & g_2 & g_1 \\ 0 & f_2 & f_1 & g_3 & g_2 \\ 0 & 0 & f_2 & 0 & g_3 \end{pmatrix}$$

1. Is it true that a polynomial of degree at most d over a ring has at most d roots? Prove or disprove. **[1]**

2. Let $\mathbb{K}$ be a field and $S \subset \mathbb{T}$ be a finite set. For any n and $d \le |S|$ find a polynomial of n variables of degree d, such that the zero set of this polynomial contains contains exactly $d|S|^{n-1}$ points in S^n. **[1]**

3. Prove that the sets $\mathbb{Z} \subseteq \mathbb{R}$ and $[0,1]^2 \subseteq \mathbb{R}^2$ are not algebraic varieties (over $\mathbb{R}$). **[1]**

4. Let $\mathbb{F}$ be a finite field.

   (a) Let $f \in \mathbb{F}[x_1, \dots, x_n]$ be a polynomial, which contains only monomials $x_1^{i_1} \cdots x_n^{i_n}$ such that $\min(i_1, \dots, i_n) < |\mathbb{F}| - 1$. Show that $\sum_{x \in \mathbb{F}^n} f(x) = 0$. **[2]**

   (b) Assume that $\mathrm{char}(\mathbb{F}) = p$ where p is a prime number. Let $f_1, \dots, f_k \in \mathbb{F}[x_1, \dots, x_n]$ be nonzero polynomials such that $\deg(f_1) + \dots + \deg(f_k) < n$. Prove that the size of $V(f_1, \dots, f_k)$ is divisible by p. **[2]**

   (c) Let $n \ge 1$ be an integer and $f \in \mathbb{F}[x]$ be polynomial of degree n. Show that either $f(\mathbb{F}) = \mathbb{F}$, or $|f(\mathbb{F})| \le |\mathbb{F}| - \frac{|\mathbb{F}|-1}{n}$. **[3]**

5. Let $f, g \in \mathbb{R}[x, y]$ be irreducible polynomials. Is it true that $V(f, g)$ is irreducible as well? **[1]**

6. Prove the weak Nullstellensatz for $n = 1$. More precisely, let $\mathbb{K}$ be an algebraically closed field a let $I \subseteq \mathbb{K}[x]$ ba an ideal such that $V(I) = \emptyset$. Show that $I = \mathbb{K}[x]$. **[2]**

7. Let $\mathbb{K}$ be a field. Show that if polynomials $f, g \in \mathbb{K}[x]$ have no common non-constant factor, then there are polynomials $u, v \in \mathbb{K}[x]$, such that $uf + vg = 1$. Hint: Use Euclid's algorithm. **[2]**

8. Let $\mathbb{K}$ be a field. Show that for every $f, g \in \mathbb{K}[x]$, if $\mathrm{Res}(f, g, x) = 0$, then the polynomials f, g contain a common non-constant factor. **[2]**