Math++ Problems

Problem set 2 – Fourier analysis

hints after **15. 4. 2020**, solutions due **22. 4. 2020**

Definition: We say that $y \in \mathbb{Z}_p$ is a *quadratic residue* in $\mathbb{Z}_p$, if there is $x \in \mathbb{Z}_p$ such that $y = x^2$.

1. Prove that characters are eigenvectors for the convolution operator (with respect to a fixed function). That is, given a finite Abelian group G, a map $f \in \mathbb{C}^G$ and a character χ show that $f * \chi = \lambda \cdot \chi$. What is the value of the eigenvalue λ? **[1]**

2. Let G be a finite Abelian group and H a subgroup of G. Let $f \in \mathbb{C}^G$ and $a \in G$. Prove that
$$\frac{1}{|H|} \sum_{x \in H} f(x+a) = \sum_{y \in H^\perp} \widehat{f}(y)\chi_y(a).$$
Let us recall that $H^\perp = \{a \in G \colon \chi_a(x) = 1, \forall x \in H\}$. **[2]**

3. A function $f\colon \{0,1\}^n \to \{-1,1\}$ is *non-increasing*, if $f(x) \leq f(y)$ for every $x, y \in \{0,1\}^n$ such that $x_i \geq y_i$ for every i.
   (a) Show that if $f\colon \{0,1\}^n \to \{-1,1\}$ is non-increasing, then $\mathrm{Inf}_i(f) = \widehat{f}(e_i)$. Here we identify $\{0,1\}^n$ with $\mathbb{Z}_2^n$ and e_i is the ith vector of the standard basis of $\mathbb{Z}_2^n$. The influence $\mathrm{Inf}_i(f)$ is defined similarly as in the previous problem set: $\mathrm{Inf}_i(f) := \Pr[f(x) \neq f(x_1, \dots, x_{i-1}, 1 - x_i, x_{i+1}, \dots, x_n)]$ where the addition is considered in $\mathbb{Z}_2^n$. **[2]**
   (b) Show that if n is odd, then $f(x) = \mathrm{sgn}(n/2 - \sum_i x_i)$ maximizes the total influence among all non-increasing functions of n variables from $\{0,1\}^n$ to $\{-1,1\}^n$. By the total influence, we mean the value $\mathrm{Inf}(f) = \sum_{i=1}^n \mathrm{Inf}_i(f)$. **[2]**

4. Let p be a prime number and $r \in \mathbb{Z}_p$. Let us define $\mathrm{Gau}(r) := \sum_{x \in \mathbb{Z}_p} e(rx^2/p)$ (the Gauss summation). Prove that:
   (a) $\mathrm{Gau}(rs^2) = \mathrm{Gau}(r)$ for $s \in \mathbb{Z}_p \setminus \{0\}$; **[1]**
   (b) if -1 is not a quadratic residue in $\mathbb{Z}_p$, then $\mathrm{Gau}(-r) = -\mathrm{Gau}(r)$; **[2]**
   (c) $\mathrm{Gau}(1)^2 = \pm p$ for an odd prime number p. **[2]**

5. Let $f\colon \mathbb{R} \to \mathbb{C}$ be a 1-periodic function with continuous derivative.
   (a) Show that $\widehat{f'}(n) = 2\pi i n \cdot \widehat{f}(n)$. (A formula for $\widehat{f}(n)$ and integration per partes may be helpful.) **[1]**
   (b) Show that if $g, h\colon \mathbb{R} \to \mathbb{C}$ are arbitrary two continuous 1-periodic functions satisfying $\widehat{g}(n) = \widehat{h}(n)$ for every $n \in \mathbb{T}$, then $g = h$. (Fejér's theorem may be helpful.) **[1]**
   (c) Find all 1-periodic functions $g\colon \mathbb{R} \to \mathbb{C}$, with continuous second derivative which satisfy
$$g''(x) + 2g'(x) + g(x) = f(x).$$
[1]