Math++ Problems

Problem set 1 – Harmonic analysis

hints after **18. 3. 2020**, solutions due **25. 3. 2020**

Definition: Given a map $f\colon \{0,1\}^n \to \{0,1\}$ the *influence of the ith variable* is defined as

$$\mathrm{Inf}_i(f) := \Pr[f(x) \neq f(x_1, \ldots, x_{i-1}, 1 - x_i, x_{i+1}, \ldots, x_n)].$$

Definition: Let G be an Abelian group and $S \subseteq G$ be a symmetric set (ie. $a \in S \Rightarrow -a \in S$). The *Cayley graph* $\mathrm{Cay}(G,S)$ is a graph where G is the vertex set of $\mathrm{Cay}(G,S)$ and two vertices a, b are connected with an edge iff $b - a \in S$.

Definition: Given two functions $f, g\colon G \to \mathbb{C}$ their *convolution* is defined as

$$(f * g)(z) := \mathop{\mathbb{E}}_{x \in G} f(x) g(z - x).$$

Definition: Given $f\colon G \to \mathbb{C}$, the *support* $\mathrm{Supp}(f)$ of f is defined as $\{x \in G\colon f(x) \neq 0\}$.

1. Compute the influences of all variables for the following maps $f\colon \{0,1\}^n \to \{0,1\}$:
   (a) $f(x) = x_1$, **[0.5]**
   (b) $f(x) = \sum_i x_i \bmod 2$, **[0.5]**
   (c) $f(x)$ is the "majority vote". That is, $f(x)$ is the value which is more frequent among x_i. In case of tie, we set $f(x) = 0$. **[1]**

2. Find a map $f\colon \{0,1\}^n \to \{0,1\}$ which attains values 0 and 1 with the same frequency, while the influence of each variable is at most $\mathcal{O}(\log(n)/n)$. **[3]**

3. Let G be a finite Abelian group, χ a character of G and $S \subseteq G$ be a symmetric subset of G with $0 \notin S$. Let M be the rescaled incidence matrix of $\mathrm{Cay}(G,S)$ so that $M_{ij} = 1/|S|$ if $j - i \in S$ and $M_{ij} = 0$ otherwise.
   (a) Consider a vector $x \in \mathbb{C}^G$ such that $x_a = \chi(a)$. Prove that x is an eigenvector of $\mathrm{Cay}(G,S)$ (that is of the matrix M). **[1]**
   (b) Compute the eigenvalues of the cycle on n vertices. **[1]**
   (c) Compute the eigenvalues of the d-dimensinal hypercube H_d, i.e., $V(H_d) = \{0,1\}^d$ and (a,b) is an edge if a and b differ in exactly one coordinate. **[2]**

4. Find the matrix for the linear map corresponding to the Fourier transform over $\mathbb{Z}_n$. That is, find a matrix M_n such that for every $f\colon \mathbb{Z}_n \to \mathbb{C}$ we get

   $$(\widehat{f}(\chi_0), \ldots, \widehat{f}(\chi_{n-1}))^T = M_n (f(0), \ldots, f(n-1))^T.$$

   Compute $\det(M_n)$ and verify that the Fourier transform is a bijection. **[2]**

5. Let G be a finite Abelian group and $f, g\colon G \to \mathbb{C}$. Prove the following assertions:

   (a) $\mathrm{Supp}(f * g) \subseteq \mathrm{Supp}(f) + \mathrm{Supp}(g)$, **[1]**

   (b) $\|f * g\|_\infty \leq \|f\|_p \cdot ||g||_q$, where $1/p + 1/q = 1$, **[2]**

   (c) $\widehat{f \cdot g}(\chi) = \sum_{\zeta \in \widehat{G}} \widehat{f}(\chi - \zeta)\widehat{g}(\zeta)$. **[1]**