Mathematics++ (topology), lecture #4 (March 29, 2021)

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Homotopy: intuition and examples

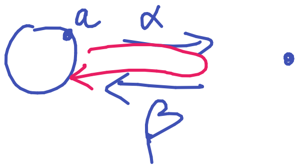
Beware:

- S^1 not homeomorphic to $\{a\}$
- maps $f, g : S^1 \longrightarrow \mathbb{R}^2$

  $f(x) = x$, $g(x) = a$

  are homotopic
- the two spaces are not homotopy eq.

  if yes:

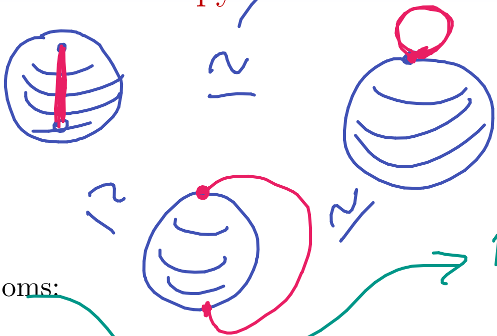


not possible

Homotopy: intuition and examples II

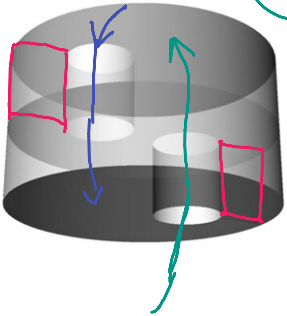
Exercise:

homot. eq.



House with two rooms:

Bing's house



homotopy equivalent to a single point

= contractible.

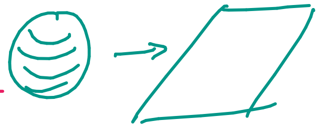
Definition $X \subseteq \mathbb{R}^m$ antipodally symmetric if $x \in X \Rightarrow -x \in X$

Let $X \subseteq \mathbb{R}^m$, $Y \subseteq \mathbb{R}^n$ antipodally sym.

ct $f : X \to Y$ is antipodal map if $f(-x) = -f(x)$ $\forall x \in X$.

Theorem (Borsuk—Ulam theorem, 3 versions)

(i) ct $f : S^n \to \mathbb{R}^n$ $\exists$ pt $x \in S^n$ s.t. $f(x) = f(-x)$.



(ii) $g : S^n \to \mathbb{R}^n$ antipodal $\Rightarrow \exists x \in S^n$ s.t. $g(x) = 0$

(iii) there is no antipodal mapping $S^n \to S^{n-1}$

Theorem (Lyusternik—Schnirelman)

$A_1, \dots, A_{n+1} \subseteq S^n$ be sets that cover S^n, each open or closed.

$\Rightarrow \exists i$ s.t. A_i contains a pair of antipodal points.

: $\exists$ covering with $n+2$ sets (closed), none has antip. pts.

Proof of Lyusternik—Schnirel'man

Theorem (Lyusternik—Schnirel'man)

Let $A_1, \dots, A_{n+1} \subseteq S^n$ be sets that cover S^n, each A_i either open or closed. Then some A_i contains a pair of antipodal points.

Proof. 1) All A_i closed. Define $f: S^n \longrightarrow \mathbb{R}^n$ by

$f(x) = (\mathrm{dist}(x, A_1), \mathrm{dist}(x, A_2), \dots, \mathrm{dist}(x, A_n))$ ... continuous

By (i) of B–U $\exists x \in S^n$ $f(x) = f(-x)$.

- if $\exists i$ $f(x)_i = 0$ ... $\mathrm{dist}(x, A_i) = 0 \Rightarrow x \in A_i$, $-x \in A_i$
- if $\forall i$ $f(x)_i \neq 0$ ... $x, -x$ don't belong to $A_1, \dots, A_n$ $\Rightarrow x, -x \in A_{n+1}$.

2) All A_i open: want $F_i \subseteq A_i$ closed, $F_1, \dots, F_{n+1}$ cover S^n.

Apply compactness: For $\forall x \in S^n$, pick $i(x)$ s.t. $x \in A_{i(x)}$

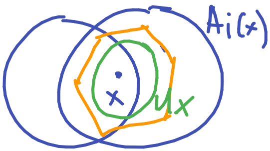
$\exists U_x$ open s.t. $\mathrm{cl}\, U_x \subseteq A_i$

... this is open cover of S^n

... pick finite subcover $U_{x_1}, \dots, U_{x_m}$

$\forall i$ let $F_i = \bigcup \mathrm{cl}(U_{x_j})$

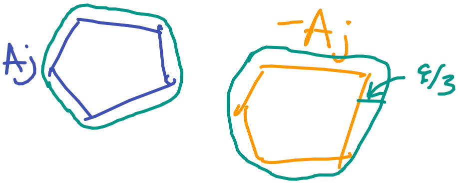
↳ union over those x_j for which we picked A_i



Application: χ of Kneser graphs

3) $A_1, \dots, A_k$ open, $A_{k+1}, \dots, A_{n+1}$ closed.

If no antipodal pair: if A_j closed, has no antipodal pair:



$\Rightarrow \exists\, \varepsilon > 0 \quad \operatorname{dist}(A_i, -A_i) > \varepsilon$

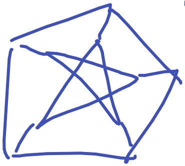
Let $A_i' = \{x \in S^n : \operatorname{dist}(x, A_i) < \frac{\varepsilon}{3}\}$

$\hookrightarrow$ open, still no antipodal pair

... we are back in case 2) (for open sets) $\blacksquare$

Def: $K(n,k)$ Kneser graph $\quad V = \binom{[n]}{k} \quad \{A,B\} \in E \iff A \cap B = \emptyset$.

EX. $K(5,2)$

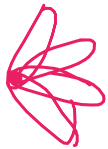


Thm (Lovász–Kneser): $\chi(K(n,k)) \geq n - 2k + 2$, $n \geq 2k$

Actually $\chi \dots =$

$n \leq \chi(G) \cdot \alpha(G)$

$\Rightarrow \chi(G) \geq \frac{n}{\alpha(G)}$



: k small, $n \to \infty$ ... χ is large

$w(K(n,k)) = \lfloor \frac{n}{k} \rfloor$

But here $\alpha = \binom{n-1}{k-1}$

$\Rightarrow$

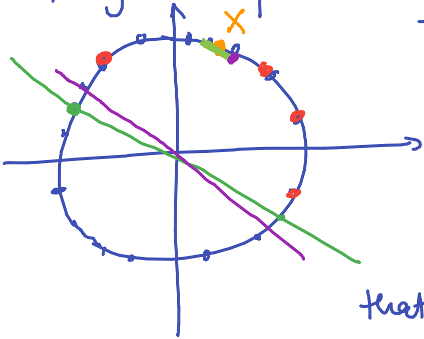
$$\chi(G) \geq \frac{n}{\binom{n-1}{k-1}} = \frac{n}{\left(\frac{ne}{k}\right)^k} = \text{const.}\ \frac{1}{n^{k-1}} \ll n-2k+2$$

Application: χ of Kneser graphs

Theorem (Lovász—Kneser)

If $n \geq 2k$, then $\chi(K(n,k)) \geq n-2k+2$.

Proof. Let $d = n-2k+1$. Place n pts on S^d (in $\mathbb{R}^{d+1}$) in general position ... no $d+1$ are on a common hyperplane passing through 0.



This is possible (e.g. by arguments from measure theory)

Suppose we can color $K(n,k)$ with $n-2k+1$ colors.

Take such coloring, define sets A_i: $A_i \subseteq S^d$. Let $x \in S^d$. The hyperplane with normal vector x cuts S^d in two open half-spheres. If $\exists$ k-tuple of color i in the half-sphere that contains x, then let $x \in A_i$.

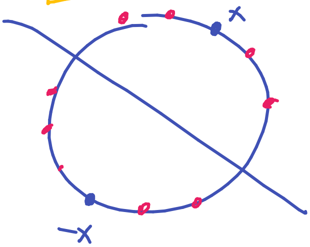
Note: $\forall x$, $x \in A_i \Rightarrow \exists \varepsilon > 0$ s.t. points ε-close to x are in A_i
$\Rightarrow A_i$ open.

Let $A_{d+1} = S^d \setminus \bigcup^d A_i$. This is closed.

By L–S $\exists i \in [d+1]$ $\exists x \in S^d$ ... $x, -x \in A_i$

Application: χ of Kneser graphs

$\boxed{\text{If } i \le d:}$



open "northern hemisphere" wrt x has k-tuple of color i

$-x \in A_i$ ... "southern — " — " — "

... endpoints of an edge got the same color $\not\downarrow$.

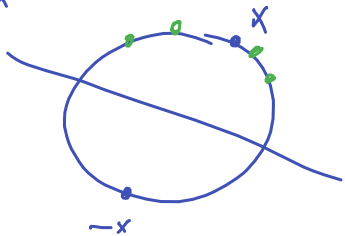
$\boxed{\text{If } i = d+1}$ $\Rightarrow$ open northern hemisphere wrt x has $\le k-1$ pts.

south. — " — as well

$\Rightarrow$ equator has $\ge n-(2k-2) = n-2k+2$ pts

$= d+1$

$\Rightarrow \exists$ hyperplane w/ $d+1$ pts and origin.



$\square$

More operations: disjoint union

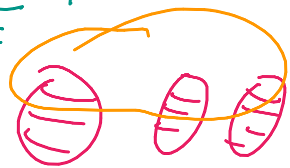
Last time: products

we mentioned: we want projections π_i to be ct.

where $\pi_i : (x_j)_{j\in I} \longmapsto x_i$. But too many open sets can cause trouble (see example from last time)

Remark product topo = the coarsest topo such that projections ct.

Definition $(X_i)_{i\in I}$ topo spaces .. disjoint union $\coprod_{i\in I} X_i$



on set $\bigcup_{i\in I} X_i \times \{i\}$.

U open iff $\forall i$ $X_i \cap U$ is open.

Remark

Now we have inclusion maps $\iota_i : X_i \longrightarrow \coprod_{i\in I} X_i$

this topo is finest s.t. all inclusion maps are ct.

More operations: quotient

shrinking subset to a point, gluing spaces together....

Definition X is topo, $\approx$ eq. relation on X

Quotient space $X/\approx$ ... ground set = equivalence classes of $\approx$.

$q : X \to X/\approx$ maps x to its eq. class = quotient map

$U \subseteq X/\approx$ is open iff $q^{-1}(U)$ is open.

Notation: If $A \subseteq X$ we write X/A for $X/\approx$

where $\approx$' ... classes A, $\forall x \in X \setminus A$ ... class $\{x\}$

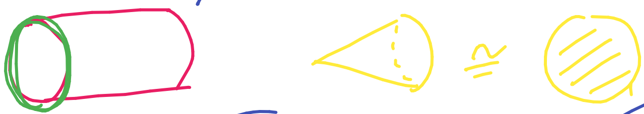
Similarly $X/(A_i)_{i\in I}$ for a collection of disjoint subspaces

Example

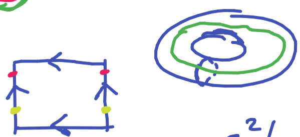
1) $[0,1]/\{0,1\}$ ...

.. actually homeomorphic to S^1

2) $S^n \times [0,1] / S^n \times \{0\} \cong B^{n+1}$

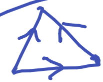


3)



$\rightsquigarrow [0,1]^2/\approx$ $\cong$ $S^1 \times S^1$

4)



Remark

Remark: $\mathbb{R}^2/B^2$ (closed) ... nice

$\mathbb{R}^2/\operatorname{int} B^2$ ... not even Hausdorff.

More operations: join

Definition

Special cases:

Cone:

Suspension: