Mathematics++ (topology), lecture #3 (March 22, 2021)

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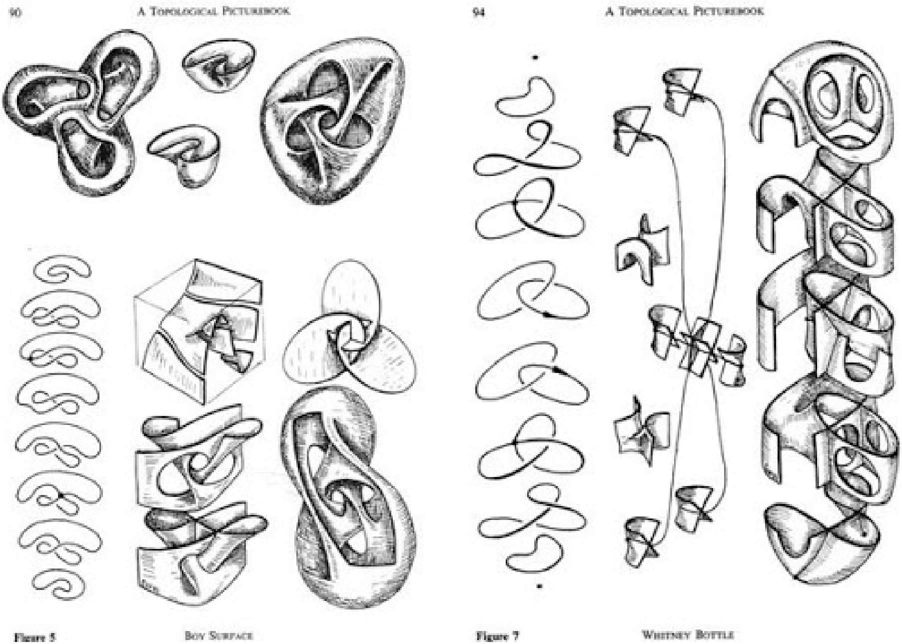


Compactness II

Last time:

Lemma: (iii) $f: X \to Y$ ct. $K \subseteq X$ cp $\Rightarrow f(K)$ cp.

(ii) X Hausdorff $K \subseteq X$ cp $\Rightarrow K$ is closed

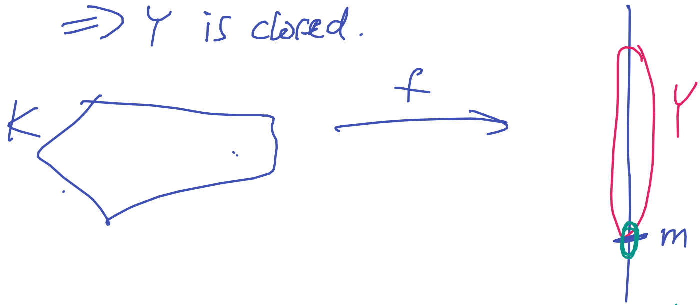
Theorem (Continuous real-valued function on compact set attains its minimum)

If K cp., $f: K \to \mathbb{R}$ is ct., then $\exists\, x_0 \in K$ s.t. $\inf_{x \in K} f(x) = f(x_0)$.

Proof.

Set $Y := f(K)$ Y is cp., $\mathbb{R}$ is Hausdorff

$\Rightarrow Y$ is closed.



Let $m = \inf Y$

if $m \notin Y$:

$\mathbb{R} \setminus Y$ open $\Rightarrow \exists\, \varepsilon > 0$

s.t. $(m-\varepsilon, m+\varepsilon) \subseteq \mathbb{R} \setminus Y$.

Contradiction with $m = \inf Y$.

$\boxtimes$

□

Products of spaces

Definition $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ topo spaces. their product has $X \times Y$ as ground set and basis $= \{U \times V ; U \in \mathcal{O}_X ; V \in \mathcal{O}_Y\}$

infinite products: Let $(X_i, \mathcal{O}_i)_{i \in I}$ might not even be countable

their product has groundset $\prod_{i \in I} X_i$ and basis $= \{\prod U_i ; U_i \in \mathcal{O}_i$ and moreover $U_i = X_i$ for all except a finite set $\}$ "product topology"

Remark

why not take $\{\prod U_i ; U_i \in \mathcal{O}_i\}$ as the basis?

Also a topology, box topology.

Spps $f : X \to \prod X_i$ is a function $f(x) = (f_1(x), f_2(x), \dots \quad)$

with product topo, if $\forall f_i$ is ct. $\Rightarrow f$ is ct. (Prove that!)

with box topo, not true:

let $X = \mathbb{R} = X_i \ \forall i$ $\quad f(x) = (x, x, \dots \quad)$

If $U_i = \left(-\frac{1}{2^i}, \frac{1}{2^i}\right)$, $U = \prod U_i$ $\quad f^{-1}(U) = \{0\}$ not open!, $\Rightarrow f$ not continuous!

Example

$\forall i \quad X_i =$ 2-pt discrete space, then $\prod_{i \in \mathbb{N}} X_i$ is homeom. to Cantor set.

Exercise

Product of Hausdorff spaces is Hausdorff.

Theorems of Tychonoff and de Bruijn—Erdős (about graph colorings)

Theorem (Tychonoff) Product of arbitrary collection of cp spaces is cp.

not necess. countable

Theorem (de Bruijn—Erdős): Let G be an infinite graph. If every finite subgraph of G is k-colorable, then G is k-colorable.

Proof. For every vertex $v \in V$, let X_v be a copy of the discrete topo space $\{1, 2, \dots, k\}$.

Let $X = \prod_{v \in V} X_v$. X_v are compact $\Rightarrow X$ is cp (by Tychonoff).

Assignment of colors to $V \longleftrightarrow$ a point in X.

For $e \in E$, let $F_e \subseteq X$ be those assignments where endpoints of e get different colors.

Want $\bigcap_{e \in E} F_e \neq \emptyset$. We know: if $E_0 \subseteq E$ finite, then $\bigcap_{e \in E_0} F_e \neq \emptyset$.

(since every finite subgraph k-colorable).

$\forall e$: assignments s.t. endpts of e get the same color:

$$e = \{u, v\} \qquad X \setminus F_e = \underbrace{\Big[[k] \times [k] \times \dots \times [k] \times \{1\} \times \dots \times \{1\} \times [k] \dots \Big] \cup \Big[[k] \dots \{2\} \dots \{2\} \dots \Big]}_{\text{one of the basis rectangles}} \cup \Big[\dots \{3\} \ \{3\} \dots \Big]$$

$\Rightarrow$ $\boxed{X \setminus F_e \text{ is open}}$

If $\bigcap_{e \in E} F_e = \emptyset \Rightarrow \{X \setminus F_e\}$ is open cover of X. $\Rightarrow \exists$ open subcover $\{X \setminus F_e;\ e \in E_0\}$

$\Rightarrow \bigcap_{e \in E_0} F_e = \emptyset$ a contradiction with

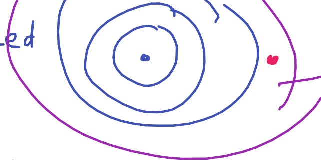
Compactness of subsets of $\mathbb{R}^n$

Theorem $A \subseteq \mathbb{R}^n$ is cp $\iff$ A is closed and bounded.

$\exists\, m \in \mathbb{R},\ m > 0$, $A \subseteq [-m, m]^n$

Proof. "$\Rightarrow$": $\mathbb{R}^n$ is Hausdorff $\Rightarrow$ A closed.

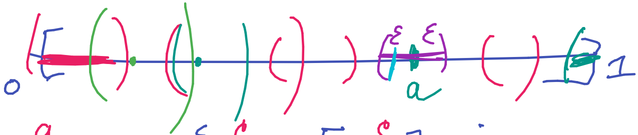
if not bounded



open balls $B(0, n)$ are a cover with no finite subcover.

"$\Leftarrow$": enough to show: $[-m, m]^n$ is compact.

actually enough: $[0, 1]$ is compact



Let $\mathcal{U}$ be open cover.

Let $a := \sup\{c : [0, c]$ is covered by finite # of elts. in $\mathcal{U}\}$

if $a < 1$: a is covered by open set U_a in $\mathcal{U}$ $\Rightarrow \exists\, \varepsilon > 0$ s.t. $(a-\varepsilon, a+\varepsilon) \subseteq U_a$ ... find some $a - \varepsilon < b < a$ s.t. $[0, b]$ is covered by finite subcollection $\mathcal{U}_0$. ... add U_a ... $\mathcal{U}_0 \cup \{U_a\}$ ... $\lightning$ with $a = \sup$.

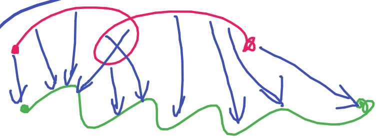
Sup = max ... i.e. $1 \in$

if not, find ε ....

$\blacksquare$

Homotopy: homotopic maps

In general, it is hard to decide whether two spaces are homeomorphic (actually, undecidable).
We take a coarser equivalence
... homotopy eq.
Still undecidable, but can use many tools

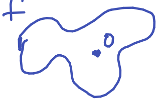


$f \sim g$

Definition $f,g : X \to Y$ are ct. maps. They are homotopic if $\exists$ ct $H : X \times [0,1] \to Y$ ... homotopy between f and g
s.t. $H(\cdot, 0) = f$ $\quad H(\cdot, 1) = g$

Example 1) $f,g : S^1 \to \mathbb{R}^2$

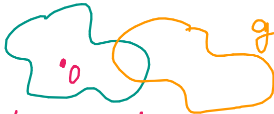
f



$g(x) = 0$

2) $f,g : S^1 \to \mathbb{R}^2 \setminus \{0\}$

f $\quad g$



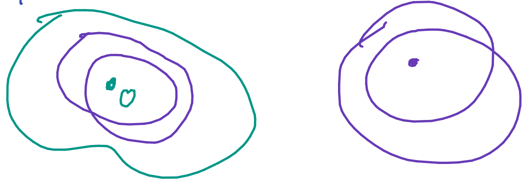
not homotopic

Exercise: Prove that every two maps $X \to B^n$ are homotopic.

: Given X, Y being homotopic is eq. relation

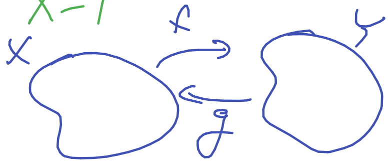
$[X, Y]$ ... set of classes.

maps $S^1 \to \mathbb{R}^2 \setminus \{0\}$ in bijection with $\mathbb{Z}$



Homotopy: homotopy equivalence

Definition X and Y are homotopy equivalent $X \simeq Y$
if $\exists$ ct $f: X \to Y$ and $g: Y \to X$ s.t.
$g \circ f : X \to X$ is homotopic to id_X
and $f \circ g : Y \to Y$ is homotopic to id_Y.



Example

1)

$X = \{ (x,y) ; 1 \le \sqrt{x^2+y^2} \le 2 \}$ $\quad = \|x\|$

$Y = \{ (x,y) ; \sqrt{x^2+y^2} = 1 \}$

$f: X \to Y \qquad f(x) = \frac{x}{\|x\|}$

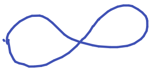
$g: Y \to X \qquad g(y) = y$

$\to g \circ f : X \to X$

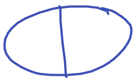
$H(x,t) = (1-t) \cdot x + t \cdot \frac{x}{\|x\|}$

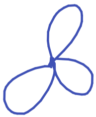
$f \circ g : Y \to Y$ is identity

2)

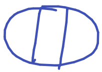


$\simeq$





$\simeq$



3) X, Y homeomorphic $\Rightarrow$ X, Y homotopy eq.

Homotopy: deformation retraction

To visualize, we use def. retraction:

Y is a subspace of X.

Definition A deformation retraction is a ct. map

$R: X \times [0,1] \to X$ s.t.

- $R(\cdot, 0) = id_X$
- $R(x, 1) \in Y \quad \forall x \in X$
- $\forall y \in Y \quad \forall t \in [0,1] \quad R(y,t) = y$.

If $Y \subseteq X$ and $\exists$ a def. retraction $X \to Y \Rightarrow Y$ is a $\boxed{\text{def. retract}}$ of X.

Fact $X \simeq Y \iff \exists Z$ s.t. both X and Y are def. retracts of Z.

Example