Matematika++ (topologie), 1. přednáška (8.3.2021)

Ida Kantor

https://kam.mff.cuni.cz/Matematika++/
https://kam.mff.cuni.cz/~dbulavka/teaching/ss2021/mathpp.html

Metric spaces, topological spaces

Topology … generalize the notion of metric spaces, capture the notion of continuity

Definition

A metric space is a pair (X,ρ) where X is a set and $\rho: X \times X \to \mathbb{R}$

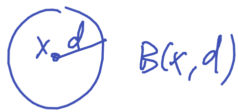
Satisfying $\forall a,b,c \in X$

(i) $\rho(a,b) \geq 0$ $\quad \rho(a,b)=0 \iff a=b$

(ii) $\rho(a,b)=\rho(b,a)$

(iii) $\rho(a,b) \leq \rho(a,c)+\rho(c,b)$

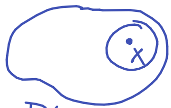
metric



Definition

$B(x,d) = \{y \in X ;\ \rho(x,y) \leq d\}$

Open sets in metric spaces:



A is open if $x \in A \Rightarrow \exists\, \varepsilon > 0$ $B(x,\varepsilon) \subseteq A$.

called topology

Definition

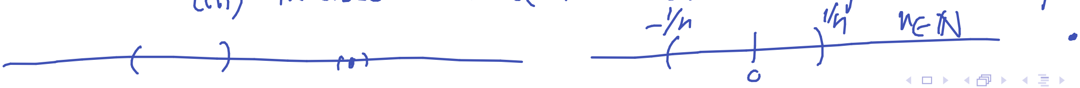
A *topological space* is a pair $(X,\mathcal{O})$, where X is a set and $\mathcal{O} \subseteq \mathcal{P}(X)$

Satisfying

(i) $\emptyset \in \mathcal{O}$, $X \in \mathcal{O}$

(ii) union of an arbitrary collection of open sets is also open

(iii) intersection of a finite collection of open sets is open



Homeomorphisms, subspaces

The notion of "sameness" is called "homeomorphism"

Definition

$(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are *homeomorphic* (writen as $X \cong Y$) if

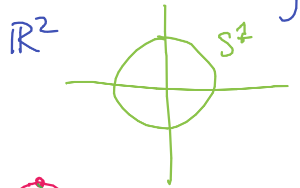
$\exists$ a bijection $f : X \to Y$ s.t. $A \in \mathcal{O}_X \iff f(A) \in \mathcal{O}_Y$

f ... called homeomorphism

Definition

$B^n = B(0,1) \subseteq \mathbb{R}^n$ (with euclidean metric) $= \{ y \in \mathbb{R}^n ; \|y\| \leq 1 \}$

$S^{n-1} = \{ y \in \mathbb{R}^n ; \|y\| = 1 \}$ sphere



Example

(i) boundary of □ $\cong S^1$

(ii) interval $(0,1) \cong \mathbb{R}$

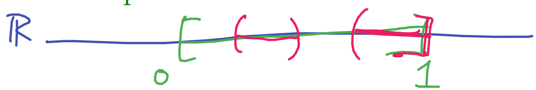
Definition If $(X, \mathcal{O})$ is a topo space, $Y \subseteq X$ we have a subspace topology $\mathcal{O}_Y$ on Y:

$$\mathcal{O}_Y = \{ U \cap Y ; U \in \mathcal{O}_X \}$$

Y is a subspace

Caution: different from e.g. graphs

Example



Closure, boundary

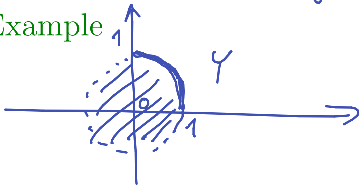
Definition

- F is *closed* if $X \setminus F$ is open

  U, V, W open F, G, H closed

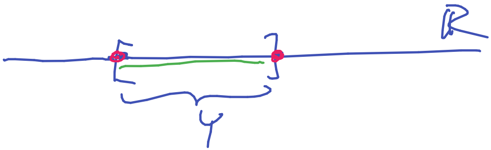
- *Closure* $\mathrm{cl}\, Y$ of Y is the intersection of all closed sets containing Y.
- *Boundary* $\partial Y = \mathrm{cl}(Y) \cap \mathrm{cl}(X \setminus Y)$
- *Interior* $\mathrm{int}\, Y = Y \setminus \partial Y$
- A set N is a *neighborhood* of a point x if $\exists$ open U s.t. $x \in U \subseteq N$.

Example



$\mathrm{cl}(Y) = B^2$ $\mathrm{int}\, Y = \{y;\ \|y\| < 1\}$

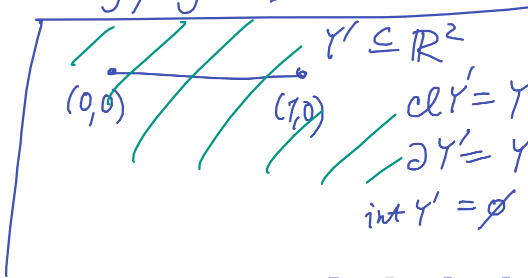
$\partial Y = S^1$



$Y = \mathrm{cl}\, Y = Y$

$\partial Y = \{0, 1\}$

$\mathrm{int}\, Y = (0, 1)$



$Y' \subseteq \mathbb{R}^2$

$\mathrm{cl}\, Y' = Y'$

$\partial Y' = Y'$

$\mathrm{int}\, Y' = \emptyset$

Bases

actually, both versions are used

Definition

- A ~~*base*~~ basis of $\mathcal{O}$ is $\mathcal{B} \subseteq \mathcal{O}$ s.t. every $U \in \mathcal{O}$ is a union of members of $\mathcal{B}$
- A ~~*subbase*~~ subbasis of $\mathcal{O}$ is $\mathcal{S}$ s.t. finite intersections of members of $\mathcal{S}$ form a basis.

Example

$\mathbb{R}$

Basis: all open intervals

another Basis all intervals with rational endpoints

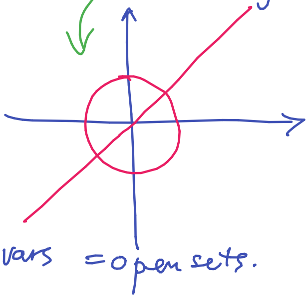
Examples of topological spaces

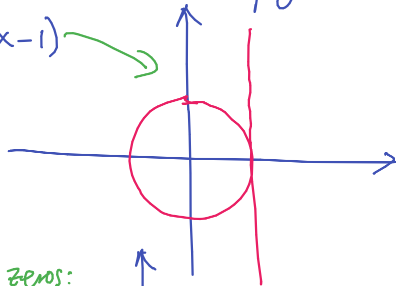
- *discrete topology* X any set ... $\mathcal{O}$ all subsets
  if $X = \mathbb{Z}$, this is the subspace topology of $\mathbb{R}$ (with standard topo)
- *indiscrete topology* = trivial topo X any set $\mathcal{O} = \{\emptyset, X\}$
- *topology of finite complements*
  ... all $X \setminus B$ where B is finite set. and $\emptyset$
  (Similarly topo of countable components for X uncountable)
- *algebraic variety* in $\mathbb{R}^n$ set of common zeros of a set of n-variable polynomials,

$x^3 + x^2z^2 - y^2 = 0$

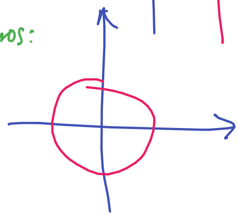
Ex. $n=2$ $f_1 = (x^2+y^2-1)\cdot(x-1)$

$f_2 = (x^2+y^2-1)\cdot(x-y)$





common zeros:



Zariski topology on $\mathbb{R}^n$

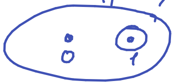
... complements of alg. vars = open sets.

Note: $n=1$

same as topology of finite complements

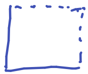
Examples of topological spaces II

- *two-point space* = Sierpiński space $X = \{1, 2\}$ open sets are $\emptyset, \{1\}, \{1,2\}$

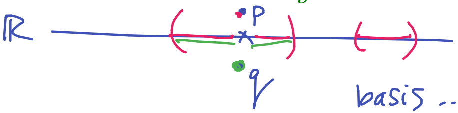


  Q: what are the closures of $\{0\}$, $\{1\}$?

- *Sorgenfrey line*
  $\mathbb{R}$ with topology whose basis are all half-open intervals $[a,b)$
  Sorgenfrey plane → $\mathbb{R}^2$ with basis of rectangles $[a,b)\times[c,d)$



- *line with two origins*



  $X = (\mathbb{R}\setminus\{0\}) \cup \{p, q\}$
  basis ... all open intervals $\subseteq \mathbb{R}$ not containing 0
  all all sets $(a,0) \cup \{p\} \cup (0,b)$ for $a<0<b$
  — " — $(a,0) \cup \{q\} \cup (0,b)$ — " —

Note $X\setminus\{p\}$ is homeomorphic to $\mathbb{R}$ (with standard topo)

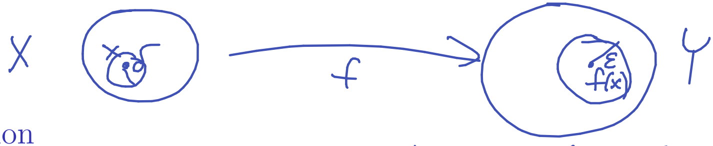
Exercise None of the examples except discrete topo are metrizable

- Prove ↓ for line w/ two origins, for the indiscrete space
- Prove that discrete topo is metrizable
- check that all these are topo spaces

Continuous functions

In metric spaces: $f : X \to Y$ is continuous if

$$\forall x \in X \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad f(B(x,\delta)) \subseteq B(f(x), \varepsilon)$$



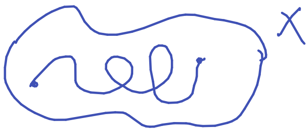
Definition

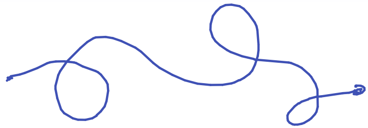
A *continuous mapping* of $(X, \mathcal{O}_X)$ into $(Y, \mathcal{O}_Y)$ is mapping $f : X \to Y$ s.t. $U \in \mathcal{O}_Y \Rightarrow f^{-1}(U) \in \mathcal{O}_X$.

Note $f^{-1}(U) := \{ x \in X \, ; \, f(x) \in U \}$

Exercise Check that if X, Y are metric spaces, the two definitions are equivalent.

Connected and path-connected spaces





Definition

- X is *connected* if it can't be written as a union of two disjoint open sets
- X is *path-connected* if every two points $x, y \in X$ are connected by a path (in X).

where a path from x to y is a continuous $f : [0,1] \to X$ s.t. $f(0) = x$ $f(1) = y$.

Example

Subspace of $\mathbb{R}$
disconnected!
(union of open sets $[0,1]$ and $[2,3]$)
also not path-connected

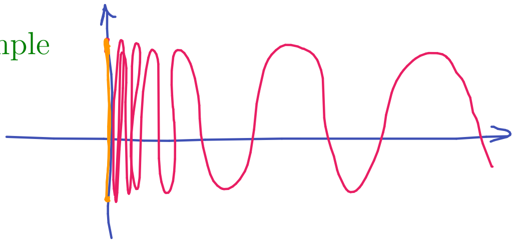
Remark

X path-connected $\Rightarrow$ X connected

reverse not true $\not\Leftarrow$

Strange examples

Example



topologist sine curve

$\subseteq \mathbb{R}^2$ consisting of line segment from $(0,-1)$ to $(0,1)$ and graph of $x \mapsto \sin \frac{1}{x}$

connected, not path-connected $\Big\}$ try to prove!

Definition

connected components of X are inclusion-maximal subsets that are connected (as subspaces of X)

Example Cantor set

C_0

$C_1 = \frac{1}{3} C_0 \cup \left(\frac{2}{3} + \frac{1}{3} C_0\right)$

$C_i = \frac{1}{3} C_{i-1} \cup \left(\frac{2}{3} + \frac{1}{3} C_{i-1}\right)$

$C = \bigcap_{i=1}^{\infty} C_i$

Many nice properties:

in bijection with $[0,1]$

connected components: single points

Remark

Example