1. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Show that the following statements are equivalent.

   (i) For all $x \in \mathbb{R}^n$ we have $x^T Ax \geq 0$.

   (ii) All eigenvalues of A are non-negative.

   (iii) There exists a matrix $U \in \mathbb{R}^{n\times n}$ such that $A = U^T U$.

2. Show that the following statement is not true:

   If a semidefinite program has a finite optimum value, then there exists a solution which attains it.

   It is enough to show a counterexample.

   Note. This does not mean that our algorithms are useless. In our applications, we always solve semidefinite programs approximately within arbitrary precision.

3. Another reason why semidefinite programs may not be able solvable in polynomial time is that the optimum solution is superpolynomial in the number of variables.

   In particular, consider the following problem:

   $$\min\{x_n : x_1 \geq 2, \text{ and } x_i \geq x_{i-1}^2 \text{ for } i = 2, \ldots n\}.$$

   Clearly, $x_i = 2^{2^{i-1}}$ for $i = 1, \ldots, n$, which requires an exponential number of bits to be written.

   Is this a semidefinite program?

   Note. Thankfully, the optimum solution of our algorithms are always polynomial.

4. Suppose that we want to approximately solve a reasonable SDP. Reasonable means that the situation in Tasks 2 and 3 (among others) do not happen.

   We will use the ellipsoid method. In order to do so, we need to show that we can solve the following task.

   Let $P \in \mathbb{R}^{n\times n}$. Show that we can in polynomial time determine, whether P satisfies our semidefinite program, or produce separating hyperplane between P and all feasible solutions.

   Hint. Feel free to assume that we can compute all eigenvalues of a linear operator in polynomial time. See Cholesky factorization for more details.