NEW COUNTEREXAMPLES TO KNASTER'S CONJECTURE

AICKE HINRICHS AND CHRISTIAN RICHTER

Abstract. Given a continuous map $f : \mathbb{S}^{n-1} \to \mathbb{R}^m$ and $n-m+1$ points $p_1, \ldots, p_{n-m+1} \in \mathbb{S}^{n-1}$, does there exist a rotation $\varrho \in SO(n)$ such that $f(\varrho(p_1)) = \ldots = f(\varrho(p_{n-m+1}))$? We give a negative answer to this question for $m = 1$ if $n \in \{61, 63, 65\}$ or $n \geq 67$ and for $m = 2$ if $n \geq 5$.

1. Introduction and notation

In 1947 B. Knaster posed the following question (see [9]): *Given a continuous function f mapping the $(n-1)$-dimensional Euclidean sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$, $m \leq n-1$, and $k = n-m+1$ points $p_1, \ldots, p_k \in \mathbb{S}^{n-1}$, does there exist a rotation $\varrho \in SO(n)$ such that $f(\varrho(p_1)) = \ldots = f(\varrho(p_k))$?* Knaster's problem had been motivated by a theorem of H. Hopf (see [6]), that answers the above question in the affirmative for $k = 2$ thus generalizing the Borsuk-Ulam theorem on antipodal points of spheres (see [2]).

In 1955 E.E. Floyd proved Knaster's conjecture for $n = 3$, $m = 1$ (see [4]). All affirmative answers for further (n, m) do not cover the full generality of Knaster's question, but rest on restrictions on the geometry of the set $\{p_1, \ldots, p_k\}$ or on the nature of f (see e.g. [7, 11, 12, 13, 14, 15, 16, 17]). In particular, for the central case of real-valued functions f, i.e. $m = 1$, $k = n$, H. Yamabe and Z. Yujobô confirmed the conjecture if $\{p_1, \ldots, p_n\}$ is an orthonormal basis.

First counterexamples for $m \geq 3$ were found by V.V. Makeev and I.K. Babenko, S.A. Bogatyĭ in the 1980s (see [10, 1]). In 1998 W. Chen added counterexamples for the remaining dimensions n in the case $m \geq 3$ and gave a first one for $m = 2$, namely for $n = 4$ (see [3]). The recent paper [8] of B.S. Kashin and S.J. Szarek even provides counterexamples for $m = 1$, but only for large dimensions $n > 10^{12}$.

The aim of the present paper is to improve the case $m = 1$ by adding counterexamples for relatively small dimensions n, namely for $n \in \{61, 63, 65\}$ and $n \geq 67$, and to complete the case $m = 2$ by providing counterexamples for all $n \geq 5$. The result for $m = 2$ confirms a conjecture of Chen. Table 1 summarizes the current state of Knaster's problem.

As in [8], our methods give asymptotic lower estimates for the smallest possible dimension $n = n(m, r)$ such that for every continuous function f from $\mathbb{S}^{n-1}$ into $\mathbb{R}^m$ and r arbitrary points $p_1, \ldots, p_r \in \mathbb{S}^{n-1}$ there exists a rotation $\varrho \in SO(n)$ such

Date: July 1, 2003.

2000 *Mathematics Subject Classification.* Primary 55M20; Secondary 52A20, 54H25.

Key words and phrases. Knaster's conjecture, Borsuk-Ulam theorem, continuous functions on spheres, level sets of supremum norms.

Research of the first author was supported by DFG Grant HI 584/2-2.

Research of the second author was supported by DFG Grant RI 1087/2.

Table 1. Current state of Knaster's problem (general case)

	$k=2$	$k=3$	$k\geq 4$
$m=1$	true ([6])	true ([4])	open if $4\leq k\leq 60$ or $k\in\{62,64,66\}$, false for every other $k\geq 4$ ([8], Theorem 5)
$m=2$	true ([6])	false ([3])	false (Theorems 6 and 7)
$m\geq 3$	true ([6])	false ([3])	false ([10, 1, 3])

that $f(\varrho(p_j))$ is constant, $1\leq j\leq r$. Since these estimates are up to the absolute constants the same as the one obtained in [8], we do not state them explicitly .

We use the following notations. The cardinality of a set A is denoted by $|A|$. Open, half-open, and closed intervals with endpoints $\alpha,\beta\in\mathbb{R}$ are (α,β), $(\alpha,\beta]$, $[\alpha,\beta)$, and $[\alpha,\beta]$, respectively. Moreover, $\lceil\alpha\rceil$ is defined by $\lceil\alpha\rceil=\min\{l\in\mathbb{Z}: l\geq\alpha\}$. The i-th coordinate of a point $x\in\mathbb{R}^n$ is denoted by $x[i]$, the Euclidean norm of x by $\|x\|_2=\left(\sum_{i=1}^n x[i]^2\right)^{\frac{1}{2}}$. The Euclidean unit ball of $\mathbb{R}^n$ is $B_2^n=\{x\in\mathbb{R}^n : \|x\|_2\leq 1\}$, the unit sphere $\mathbb{S}^{n-1}=\{x\in\mathbb{R}^n:\|x\|_2=1\}$. The symbol $\operatorname{absconv}(M)$ stands for the convex hull of $M\cup(-M)$, M being a subset of $\mathbb{R}^n$.

The counterexamples to be given for the case $m=1$ rest on the function $\|x\|_\infty=\max\{|x[1]|,\ldots,|x[n]|\}$ on $\mathbb{R}^n$. In the case $m=2$ we shall use the maps

$$f_{(l,n-l)}(x)=(f_1(x),f_2(x))=(\max_{1\leq i\leq l}|x[i]|,\max_{l+1\leq i\leq n}|x[i]|),$$

$1\leq l\leq n$. Finally we repeat a notation from [8]: Given a set $M\subseteq\mathbb{S}^{d-1}$ and a continuous function $f:\mathbb{S}^{n-1}\to\mathbb{R}^m$, a linear Euclidean isometry $\varrho:\mathbb{R}^d\to\mathbb{R}^n$ is called a *Knaster embedding* of M with respect to f if there exists a constant $c\in\mathbb{R}^m$ such that $f(\varrho(p))=c$ for all $p\in M$.

2. Local properties of supremum norms

Following the principal idea from [8] we present counterexamples based on two lemmas. Lemma 1, that generalizes Lemma 3 from [8], describes subsets of spheres whose Knaster embeddings ϱ necessarily have "large" constants c. In contrast with that Lemma 3, which plays the role of Lemma 4 from [8], characterizes sets that give rise to "small" constants. Suitable unions of sets of the first and of the second kind then do not allow any Knaster embedding and thus serve as counterexamples to Knaster's conjecture.

Lemma 1. *Let $M\subseteq\mathbb{S}^{d-1}$ and assume that $\delta>0$ is such that $\delta B_2^d\subseteq\operatorname{absconv}(M)$. Then any Knaster embedding ϱ of M into $\mathbb{R}^n$ w.r.t. $f=(f_1,f_2)=f_{(l,n-l)}$ with constant $c=(c_1,c_2)$ satisfies*

$$lc_1^2+(n-l)c_2^2\geq\delta^2 d.$$

Proof. Let $s=1,2$. Since f_s is convex and symmetric, $f_s(\varrho(p))=c_s$ for $p\in M$ implies that $f_s(x)\leq c_s$ for $x\in\operatorname{absconv}(\varrho(M))=\varrho(\operatorname{absconv}(M))$. Now the assumption $\delta B_2^d\subseteq\operatorname{absconv}(M)$ and the homogeneity of f_s imply that

$$f_s(x)\leq\tfrac{c_s}{\delta}\quad\text{for}\quad x\in\varrho\big(B_2^d\big).$$

Let $y_1, \ldots, y_d$ be an orthonormal basis of $\varrho(\mathbb{R}^d)$. Define $y, x_1, \ldots, x_n \in \mathbb{R}^n$ by

$$y[i] = \left(\sum_{j=1}^d y_j[i]^2\right)^{\frac{1}{2}} \quad \text{and} \quad x_i = \tfrac{1}{y[i]} \sum_{j=1}^d y_j[i] y_j.$$

Then $x_i[i] = y[i]$ implies $f_s(x_i) \geq y[i]$ for $s = 1$ if $i \leq l$ and for $s = 2$ if $i > l$, respectively. We obtain

$$d = \sum_{j=1}^d \|y_i\|_2^2 = \sum_{j=1}^d \sum_{i=1}^n y_j[i]^2 = \sum_{i=1}^n y[i]^2 \leq \sum_{i=1}^l f_1(x_i)^2 + \sum_{i=l+1}^n f_2(x_i)^2$$

and, since $\|x_i\|_2 = 1$, finally

$$d \leq l \tfrac{c_1^2}{\delta^2} + (n-l) \tfrac{c_2^2}{\delta^2}.$$

□

In the extremal case $l = n$ Lemma 1 yields the following.

Corollary 2. *Let $M \subseteq \mathbb{S}^{d-1}$ and let $\delta > 0$ be such that $\delta B_2^d \subseteq \operatorname{absconv}(M)$. Then any Knaster embedding of M into $\mathbb{R}^n$ w.r.t. $f = \|\cdot\|_\infty$ with constant c satisfies*

$$nc^2 \geq \delta^2 d.$$

Lemma 3. *Let $0 < \varepsilon < \sqrt{2}$ and let $p_1, \ldots, p_r \in \mathbb{S}^1$ be mutually distinct points such that $\|p_1 - p_j\|_2 \leq \varepsilon$, $1 < j \leq r$. Then any Knaster embedding ϱ of $\{p_1, \ldots, p_r\}$ into $\mathbb{R}^n$ w.r.t. $f = (f_1, f_2) = f_{(l,n-l)}$ with constant $c = (c_1, c_2)$ satisfies*

$$\left\lceil \tfrac{r}{2} \right\rceil (c_1^2 + c_2^2 - 4\varepsilon) \leq 1.$$

Proof. It suffices to show that

$$\left\lceil \tfrac{r}{2} \right\rceil (c_1^2 - 2\varepsilon) \leq \sum_{i=1}^l \varrho(p_1)[i]^2 \tag{1}$$

and, analogously, $\left\lceil \frac{r}{2} \right\rceil (c_2^2 - 2\varepsilon) \leq \sum_{i=l+1}^n \varrho(p_1)[i]^2$, because then the claim follows by

$$\left\lceil \tfrac{r}{2} \right\rceil (c_1^2 + c_2^2 - 4\varepsilon) \leq \sum_{i=1}^n \varrho(p_1)[i]^2 = \|\varrho(p_1)\|_2^2 = 1.$$

We can assume that $c_1 > 0$, since otherwise the estimate (1) is trivial. Let $\{q_1, q_2\}$ be an orthonormal basis of $\varrho(\mathbb{R}^2)$. Then $\varrho(\mathbb{S}^1) = \{q(\varphi) : 0 \leq \varphi < 2\pi\}$ where $q(\varphi) = \cos(\varphi) q_1 + \sin(\varphi) q_2$. There exist angles $\varphi_j \in [0, 2\pi)$, $1 \leq j \leq r$, such that $\varrho(p_j) = q(\varphi_j)$. Clearly, for every $1 \leq i \leq n$ there are $a_i, b_i \in \mathbb{R}$ such that

$$q(\varphi)[i] = a_i \cos(\varphi + b_i). \tag{2}$$

Let

$$A = \{i \in \{1, \ldots, l\} : |q(\varphi_j)[i]| = c_1 \text{ for some } j \in \{1, \ldots, r\}\}.$$

For every $i \in A$ there is $j \in \{1, \ldots, r\}$ such that $|\varrho(p_j)[i]| = |q(\varphi_j)[i]| = c_1$. It follows from

$$\big| |\varrho(p_1)[i]| - c_1 \big| = \big| |\varrho(p_1)[i]| - |\varrho(p_j)[i]| \big| \leq \|\varrho(p_1) - \varrho(p_j)\|_2 \leq \varepsilon$$

that $|\varrho(p_1)[i]| \in [c_1 - \varepsilon, c_1 + \varepsilon]$ and $\varrho(p_1)[i]^2 \geq (\max\{c_1 - \varepsilon, 0\})^2$. Since $c_1 \leq 1$ implies $(\max\{c_1 - \varepsilon, 0\})^2 \geq c_1^2 - 2\varepsilon$, we conclude that

$$c_1^2 - 2\varepsilon \leq \varrho(p_1)[i]^2 \quad \text{for} \quad i \in A.$$

For every $j \in \{1, \ldots, r\}$ there exists $i \in A$ such that $|q(\varphi_j)[i]| = c_1$, because $\max\{|q(\varphi_j)[i]| : 1 \leq i \leq l\} = f_1(\varrho(p_j)) = c_1$. However, the representation (2) shows that a function $|q(\cdot)[i]|$ attains the value $c_1 > 0$ for at most four angles φ in the interval $[0, 2\pi)$ and, since $\{p_1, \ldots, p_r\}$ and so also $\{q(\varphi_1), \ldots, q(\varphi_r)\}$ does not contain a pair of antipodal points, for at most two angles from $\{\varphi_1, \dots, \varphi_r\}$. This yields $r \le 2|A|$ and $\lceil \frac{r}{2} \rceil \le |A|$. Now we obtain (1) by estimating

$$\lceil \tfrac{r}{2} \rceil (c_1^2 - 2\varepsilon) \le \textstyle\sum_{i \in A} \varrho(p_1)[i]^2 \le \sum_{i=1}^{l} \varrho(p_1)[i]^2.$$

□

An analogous proof yields the following for the case $m = 1$.

Corollary 4. *Let $0 < \varepsilon < \sqrt{2}$ and let $p_1, \dots, p_r \in \mathbb{S}^1$ be mutually distinct points such that $\|p_1 - p_j\|_2 \le \varepsilon$, $1 < j \le r$. Then any Knaster embedding of $\{p_1, \dots, p_r\}$ into $\mathbb{R}^n$ w.r.t. $f = \|\cdot\|_\infty$ with constant c satisfies*

$$\lceil \tfrac{r}{2} \rceil (c^2 - 2\varepsilon) \le 1.$$

Theorem 5. *Knaster's conjecture fails for $m = 1$ if $n \in \{61, 63, 65\}$ or $n \ge 67$.*

Proof. In the sphere $\mathbb{S}^2$ there exists a symmetric net $N = -N \subseteq \mathbb{S}^2$ of 22 points such that the spherical caps of angular radius $\alpha = 27.82$ degrees around the points of N cover $\mathbb{S}^2$ (see [5], the covering property of the net claimed on the web page has been confirmed by independent calculations of the authors).

We consider the function $f = \|\cdot\|_\infty$ on $\mathbb{S}^{n-1}$. For fixed $0 < \varepsilon < \sqrt{2}$ we choose $k = n$ points $p_1, \dots, p_n$ on spheres $\mathbb{S}^1 \subseteq \mathbb{S}^2 \subseteq \mathbb{S}^{n-1}$ as follows. We pick $p_1, \dots, p_{n-10} \in \mathbb{S}^1$ and $p_{n-9}, \dots, p_n \in \mathbb{S}^2$ such that $\|p_1 - p_j\|_2 \le \varepsilon$, $1 < j \le n - 10$, and $\{p_{n-10}, \dots, p_n\} \cup \{-p_{n-10}, \dots, -p_n\} = N$.

Now we assume that there is $\varrho \in SO(n)$ such that $f(\varrho(p_1)) = \ldots = f(\varrho(p_n)) = c$. We apply Corollary 2 to $M = \{p_{n-10}, \dots, p_n\}$. Since $(\cos\alpha) B_2^3 \subseteq \operatorname{absconv}(N) = \operatorname{absconv}(M)$, we obtain

$$nc^2 \ge 3\cos^2\alpha.$$

Application of Corollary 4 to $\{p_1, \dots, p_{n-10}\}$ yields

$$\lceil \tfrac{n-10}{2} \rceil (c^2 - 2\varepsilon) \le 1.$$

Consequently,

$$\varepsilon \ge \tfrac{1}{2} \Big(\tfrac{3\cos^2\alpha}{n} - \lceil \tfrac{n-10}{2} \rceil^{-1} \Big).$$

However, the right-hand side is strictly positive for $n \in \{61, 63, 65\}$ and $n \ge 67$. Thus we can obtain a contradiction by choosing ε sufficiently small. □

Theorem 6. *Knaster's conjecture fails for $m = 2$ if $n \ge 8$.*

Proof. We consider the function $f = f_{(\lceil \frac{n}{2} \rceil, n - \lceil \frac{n}{2} \rceil)}$. Let $0 < \varepsilon < \sqrt{2}$.

First let n be an even number. We choose the points $p_1, \dots, p_k$, $k = n - m + 1 = n - 1$, on a great circle $\mathbb{S}^1 \subseteq \mathbb{S}^{n-1}$ as follows. $p_1, \dots, p_{n-3}$ are selected such that $\|p_1 - p_j\|_2 \le \varepsilon$ for $1 < j \le n - 3$. The remaining two points p_{n-2}, p_{n-1} are chosen such that $\{p_{n-3}, p_{n-2}, p_{n-1}\} \cup \{-p_{n-3}, -p_{n-2}, -p_{n-1}\}$ form a regular hexagon.

Let us assume that there exists a rotation $\varrho \in SO(n)$ such that $f(\varrho(p_j)) = c = (c_1, c_2)$ is constant for $1 \le j \le n - 1$. The set $M = \{p_{n-3}, p_{n-2}, p_{n-1}\}$ satisfies $\frac{\sqrt{3}}{2} B_2^2 \subseteq \operatorname{absconv}(M)$, since $M \cup (-M)$ is a regular hexagon. Lemma 1 with $l = \lceil \frac{n}{2} \rceil = \frac{n}{2}$ yields

$$\tfrac{n}{2}(c_1^2 + c_2^2) \ge \tfrac{3}{2}.$$

By applying Lemma 3 to $p_1, \dots, p_{n-3}$ we obtain

$$\tfrac{n-2}{2}(c_1^2 + c_2^2 - 4\varepsilon) \le 1,$$

because $\lceil \frac{n-3}{2} \rceil = \frac{n-2}{2}$. Combining the two inequalities we arrive at $\varepsilon \geq \frac{n-6}{4n(n-2)} > 0$. Thus we obtain a contradiction if we choose the initial configuration such that $\varepsilon < \frac{n-6}{4n(n-2)}$.

Now let $n \geq 9$ be odd. We pick $p_1, \ldots, p_{n-4} \in \mathbb{S}^1$ such that $\|p_1 - p_j\|_2 \leq \varepsilon$ for $1 < j \leq n-4$ and $p_{n-3}, p_{n-2}, p_{n-1} \in \mathbb{S}^1$ such that $M \cup (-M)$ is a regular octagon where $M = \{p_{n-4}, \ldots, p_{n-1}\}$.

Again we suppose that there is a rotation $\varrho \in SO(n)$ such that $f(\varrho(p_j)) = c = (c_1, c_2)$ is constant for $1 \leq j \leq n-1$. In the present case we have $\delta B_2^2 \subseteq \operatorname{absconv}(M)$ with $\delta^2 = \frac{2+\sqrt{2}}{4}$. Lemma 1 with $l = \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ shows that $\frac{n+1}{2} c_1^2 + \frac{n-1}{2} c_2^2 \geq \frac{2+\sqrt{2}}{2}$ and thus

$$\tfrac{n+1}{2}(c_1^2 + c_2^2) \geq \tfrac{2+\sqrt{2}}{2}.$$

Application of Lemma 3 to $p_1, \ldots, p_{n-4}$ yields

$$\tfrac{n-3}{2}(c_1^2 + c_2^2 - 4\varepsilon) \leq 1,$$

since $\lceil \frac{n-4}{2} \rceil = \frac{n-3}{2}$. Now we obtain $\varepsilon \geq \frac{\sqrt{2}n-8-3\sqrt{2}}{4(n+1)(n-3)} > 0$, again a contradiction if ε is sufficiently small. $\square$

3. Another family of counterexamples for $m = 2$

In the case $m = 2$ we already have counterexamples for the dimensions $n = 4$ (see [3]) and $n \geq 8$ (Theorem 6). In the following we cover the gap between 4 and 8 by a class of counterexamples for all $n \geq 5$. Though this class rests on point configurations similar to that from Theorem 6, the arguments become slightly more technical.

Given $0 < \varepsilon < \frac{\pi}{2}$, an *$\varepsilon$-set* on the sphere $\mathbb{S}^{n-1}$ is meant to be a set of $r \geq 2$ points $p_1, \ldots, p_r$ on a great circle of $\mathbb{S}^{n-1}$, consecutively ordered following an orientation of the circle, such that the angular distance between p_1 and p_j is $\frac{\pi}{2}$ if $j = r$ and at most ε for $2 \leq j \leq r-1$.

Theorem 7. *Let $n = 4s+t$ with integers $s \geq 1$ and $t \in \{1,2,3,4\}$ and consider the function $f = (f_1, f_2) = f_{(\lceil \frac{n}{2} \rceil, n - \lceil \frac{n}{2} \rceil)}$ on $\mathbb{S}^{n-1}$. If $\{p_1, \ldots, p_r\} \subseteq \mathbb{S}^{n-1}$ is an ε-set such that*

$$r = \begin{cases} 2s+t+1 & \text{for } t \neq 4, \\ 2s+t & \text{for } t = 4 \end{cases} \qquad \text{and} \qquad \varepsilon \leq \tfrac{1}{16n^4},$$

then (f_1, f_2) is not constant on $\{p_1, \ldots, p_r\}$.

The following lemma is to be used in the proof of Theorem 7.

Lemma 8. *Let $\{p_1, \ldots, p_r\} \subseteq \mathbb{S}^{n-1}$ be an ε-set, $4 \leq r \leq n$, and consider the function $f(x) = \max\{|x[1]|, \ldots, |x[l]|\}$, $l \leq n$, on $\mathbb{S}^{n-1}$. If $f(p_j) = c$ is constant for $1 \leq j \leq r$, then the set*

$$A = \{i \in \{1, \ldots, l\} : |p_j[i]| = c \text{ for some } j \in \{1, \ldots, r-1\}\}$$

and the subset

$$B = \{i \in \{1, \ldots, l\} : |p_j[i]| = c \text{ for some } j \in \{2, \ldots, r-2\}\}$$

satisfy the following conditions.

- **(a)** $|A| \geq \lceil \frac{r-1}{2} \rceil$ *and* $|p_1[i]| \geq (1 - \sqrt{2}\varepsilon)\, c$ *for all* $i \in A$.
- **(b)** $|B| \geq \lceil \frac{r-3}{2} \rceil$ *and* $|p_r[i]| \leq \sqrt{2}\varepsilon c$ *for all* $i \in B$.
- **(c)** *If* $|A| = \frac{r-1}{2}$ *then* $A = B$.

Proof. If $c = 0$ then $p_j[i] = 0$ for all $1 \le i \le n$, $1 \le j \le r$ and the assertions are trivial. In the following we assume that $c > 0$.

Since p_1 and p_r are perpendicular, the great circle containing $\{p_1, \ldots, p_r\}$ is the set $\{p(\varphi) : 0 \le \varphi < 2\pi\}$ where $p(\varphi) = \cos(\varphi)p_1 + \sin(\varphi)p_2$. Clearly, $p_j = p(\varphi_j)$, $1 \le j \le r$, where $0 = \varphi_1 < \varphi_2 < \ldots < \varphi_{r-1} \le \varepsilon < \varphi_r = \frac{\pi}{2}$.

For every $1 \le i \le l$ the function $|p(\cdot)[i]|$ is of the form

$$|p(\varphi)[i]| = |a_i \cos(\varphi + b_i)|. \tag{3}$$

$|p(\cdot)[i]|$ is a π-periodic $|a_i|$-Lipschitz function. If it is not constant, $|p(\cdot)[i]|$ attains its maximum $|a_i|$ for exactly one argument η_i in the interval $[0, \pi)$ and its minimum 0 for the corresponding angle $\eta_i - \frac{\pi}{2}$ or $\eta_i + \frac{\pi}{2}$ in $[0, \pi)$. Since $|p(0)[i]| = |p_1[i]| \le c$ and $\left|p\left(\frac{\pi}{2}\right)[i]\right| = |p_r[i]| \le c$, one obtains $|a_i| \le \sqrt{2}c$ and

$$\big||p(\varphi)[i]| - |p(\eta)[i]|\big| \le \sqrt{2}c|\varphi - \eta| \tag{4}$$

for arbitrary angles φ, η.

For proving $|A| \ge \left\lceil \frac{r-1}{2} \right\rceil$ we first note that, according to $f(p_1) = \ldots = f(p_{r-1}) = c$, for every $j \in \{1, \ldots, r-1\}$ there exists $i \in A$ such that $|p(\varphi_j)[i]| = |p_j[i]| = c$. However, the representation (3) shows that a function $|p(\cdot)[i]|$ attains the value c at most two times in the interval $[0, \pi)$. Hence $2\,|A| \ge r - 1$. This yields $|A| \ge \left\lceil \frac{r-1}{2} \right\rceil$.

In the same way one obtains $|B| \ge \left\lceil \frac{r-3}{2} \right\rceil$.

For the proof of the second part of (a) let $i \in A$ be fixed. We find $j \in \{1, \ldots, r-1\}$ such that $|p_j[i]| = c$. By (4), we obtain

$$|p_1[i]| \ge |p_j[i]| - \big||p_j[i]| - |p_1[i]|\big| = c - \big||p(\varphi_j)[i]| - |p(0)[i]|\big| \ge c - \sqrt{2}c\varphi_j \ge (1 - \sqrt{2}\varepsilon)c,$$

which is our claim.

Now we fix $i \in B$ for verifying the second part of (b). We choose $j \in \{2, \ldots, r-2\}$ such that $|p_j[i]| = c$. Since $|p_{j-1}[i]| \le f(p_{j-1}) = c$ and $|p_{j+1}[i]| \le f(p_{j+1}) = c$, the function $|p(\cdot)[i]|$ attains its maximum for an angle $\eta_i \in (\varphi_{j-1}, \varphi_{j+1})$. In particular, $0 \le \eta_i \le \varepsilon$. Then $\left|p\left(\eta_i + \frac{\pi}{2}\right)[i]\right| = 0$ and, by (4),

$$|p_r[i]| = \left|\, \left|p\left(\tfrac{\pi}{2}\right)[i]\right| - \left|p\left(\eta_i + \tfrac{\pi}{2}\right)[i]\right| \,\right| \le \sqrt{2}c\eta_i \le \sqrt{2}c\varepsilon, \tag{5}$$

as asserted.

For proving (c) we suppose that $|A| = \frac{r-1}{2}$. Let us assume that $A \ne B$. Then there exists $i_0 \in A \backslash B$, that is, $|p(\varphi_1)[i_0]| = c$ or $|p(\varphi_{r-1})[i_0]| = c$, but $|p(\varphi_j)[i_0]| \ne c$ for $2 \le j \le r-2$. Since $2\,|A| = r-1$, the above argument showing that $|A| \ge \left\lceil \frac{r-1}{2} \right\rceil$ now implies that, for every $i \in A$, the function $|p(\cdot)[i]|$ necessarily attains the value c for two of the angles $\varphi_1, \ldots, \varphi_{r-1}$. For $i = i_0$ this yields $|p(\varphi_1)[i_0]| = |p(\varphi_{r-1})[i_0]| = c$. The representation (3) then yields $|p(\varphi_2)[i_0]| > c$ or $|p(\varphi_r)[i_0]| > c$, because $0 = \varphi_1 < \varphi_2 < \varphi_{r-1} < \varphi_r = \frac{\pi}{2}$. However, $|p(\varphi_2)[i_0]| \le f(p_2) = c$ and $|p(\varphi_r)[i_0]| \le f(p_r) = c$. This contradiction proves $A = B$. □

Proof of Theorem 7. We assume that $(f_1, f_2)(p_j) = (c_1, c_2)$ is constant for $1 \le j \le r$. Then

$$\tfrac{1}{\sqrt{n}} \le \max\{c_1, c_2\} \le 1, \tag{6}$$

for $\max\{c_1, c_2\} = \|p_j\|_\infty$ and $\|p_j\|_2 = 1$.

We put $C_1 = \{1, \dots, \lceil \frac{n}{2} \rceil\}$, $C_2 = \{\lceil \frac{n}{2} \rceil + 1, \dots, n\}$ and

$$\begin{aligned} A_q &= \{i \in C_q : |p_j[i]| = c_q \text{ for some } j \in \{1, \dots, r-1\}\}, \\ B_q &= \{i \in C_q : |p_j[i]| = c_q \text{ for some } j \in \{2, \dots, r-2\}\} \end{aligned}$$

for $q = 1, 2$.

Let $i' \in C_1 \setminus A_1$. We estimate $1 = \|p_1\|_2^2 = \sum_{i=1}^n p_1[i]^2 \geq p_1[i']^2 + \sum_{i \in A_1 \cup A_2} p_1[i]^2$. Lemma 8 (a) yields $p_1[i]^2 \geq (1 - \sqrt{2}\varepsilon)^2 c_q^2 \geq (1 - 2\sqrt{2}\varepsilon) c_q^2$ for $i \in A_q$. Hence

$$1 \geq p_1[i']^2 + |A_1| c_1^2 + |A_2| c_2^2 - 2\sqrt{2}\varepsilon(|A_1| c_1^2 + |A_2| c_2^2).$$

By (6), we obtain $2\sqrt{2}\varepsilon(|A_1| c_1^2 + |A_2| c_2^2) \leq \frac{3}{16n^4}(|A_1| + |A_2|) \leq \frac{3}{16n^3}$ and

$$1 \geq p_1[i']^2 + |A_1| c_1^2 + |A_2| c_2^2 - \tfrac{3}{16n^3} \quad \text{for} \quad i' \in C_1 \setminus A_1. \tag{7}$$

In particular

$$1 \geq |A_1| c_1^2 + |A_2| c_2^2 - \tfrac{3}{16n^3}, \tag{8}$$

even if $A_1 = C_1$.

Now let $i' \in C_1 \setminus B_1$. The coordinates of p_r satisfy $|p_r[i]| \leq f_q(p_r) = c_q$ if $i \in C_q$ and $|p_r[i]| \leq \sqrt{2}\varepsilon c_q$ for $i \in B_q$ by Lemma 8 (b). Thus

$$1 = \|p_r\|_2^2 \leq 2\varepsilon^2(|B_1| c_1^2 + |B_2| c_2^2) + p_r[i']^2 + (|C_1| - |B_1| - 1) c_1^2 + (|C_2| - |B_2|) c_2^2.$$

Estimate (6) yields $2\varepsilon^2(|B_1| c_1^2 + |B_2| c_2^2) \leq \varepsilon(|B_1| + |B_2|) \leq \frac{1}{16n^3}$ and

$$1 \leq p_r[i']^2 + \big(\lceil \tfrac{n}{2} \rceil - |B_1| - 1\big) c_1^2 + \big(n - \lceil \tfrac{n}{2} \rceil - |B_2|\big) c_2^2 + \tfrac{1}{16n^3} \quad \text{for} \quad i' \in C_1 \setminus B_1. \tag{9}$$

Since $|p_r[i']| \leq c_1$, we have in particular

$$1 \leq \big(\lceil \tfrac{n}{2} \rceil - |B_1|\big) c_1^2 + \big(n - \lceil \tfrac{n}{2} \rceil - |B_2|\big) c_2^2 + \tfrac{1}{16n^3}. \tag{10}$$

If $B_1 = C_1$ formula (10) can be directly deduced in analogy with (9).

Combining (8) and (10) we arrive at

$$\big(|A_1| + |B_1| - \lceil \tfrac{n}{2} \rceil\big) c_1^2 + \big(|A_2| + |B_2| - n + \lceil \tfrac{n}{2} \rceil\big) c_2^2 \leq \tfrac{1}{4n^3}. \tag{11}$$

Case 1: $t = 1$. The definition of n and r and Lemma 8 (a) and (b) yield

$$\lceil \tfrac{n}{2} \rceil = 2s + 1, \quad n - \lceil \tfrac{n}{2} \rceil = 2s, \quad |A_q| \geq \lceil \tfrac{r-1}{2} \rceil = s + 1, \quad |B_q| \geq \lceil \tfrac{r-3}{2} \rceil = s \tag{12}$$

for $q = 1, 2$.

Case 1.1: $|A_1| + |B_1| > 2s + 1$. Then $|A_1| + |B_1| - \lceil \frac{n}{2} \rceil \geq 1$, $|A_2| + |B_2| - n + \lceil \frac{n}{2} \rceil \geq 1$ and (11) yields $c_1^2 + c_2^2 \leq \frac{1}{4n^3}$, a contradiction with (6).

Case 1.2: $|A_1| + |B_1| \leq 2s + 1$. Then, by (12),

$$|A_1| = s + 1 \quad \text{and} \quad |B_1| = s. \tag{13}$$

Now (12) yields $|A_1| + |B_1| - \lceil \frac{n}{2} \rceil = 0$, $|A_2| + |B_2| - n + \lceil \frac{n}{2} \rceil \geq 1$, and, by (11), $c_2^2 \leq \frac{1}{4n^3}$. Then (6) gives $\frac{1}{\sqrt{n}} \leq c_1$. Thus

$$\tfrac{1}{n} \leq c_1^2 \quad \text{and} \quad c_2^2 \leq \tfrac{1}{2n} c_1^2. \tag{14}$$

In the present case $C_1 \setminus A_1 \neq \emptyset$, because $|C_1| = \lceil \frac{n}{2} \rceil = 2s + 1 > s + 1 = |A_1|$. For $i' \in C_1 \setminus A_1$ inequalities (7) and (10) show that

$$p_1[i']^2 \leq \big(\lceil \tfrac{n}{2} \rceil - |A_1| - |B_1|\big) c_1^2 + \big(n - \lceil \tfrac{n}{2} \rceil - |A_2| - |B_2|\big) c_2^2 + \tfrac{1}{4n^3}.$$

By (12), we have $\lceil \frac{n}{2} \rceil - |A_1| - |B_1| \le 0$ and $n - \lceil \frac{n}{2} \rceil - |A_2| - |B_2| < 0$. Therefore, with (14),

$$p_1[i']^2 \le \tfrac{1}{4n^3} \le \tfrac{1}{4n^2} c_1^2 \quad \text{for} \quad i' \in C_1 \setminus A_1. \tag{15}$$

If $i' \in C_1 \setminus B_1$ estimates (8) and (9) yield

$$p_r[i']^2 \ge \big(|A_1| + |B_1| + 1 - \lceil \tfrac{n}{2} \rceil\big) c_1^2 + \big(|A_2| + |B_2| - n + \lceil \tfrac{n}{2} \rceil\big) c_2^2 - \tfrac{1}{4n^3}.$$

By (12), $|A_1| + |B_1| + 1 - \lceil \frac{n}{2} \rceil \ge 1$ and $|A_2| + |B_2| - n + \lceil \frac{n}{2} \rceil > 0$. Thus, with (14),

$$p_r[i']^2 \ge c_1^2 - \tfrac{1}{4n^3} \ge c_1^2 - \tfrac{1}{4n^2} c_1^2 \ge \tfrac{1}{2} c_1^2 \quad \text{for} \quad i' \in C_1 \setminus B_1. \tag{16}$$

According to (13) there exists a unique i_0 such that $A_1 \setminus B_1 = \{i_0\}$. We use this for an estimate of the scalar product of the perpendicular vectors p_1 and p_r.

$$0 = |\langle p_1, p_r \rangle| \ge |p_1[i_0] p_r[i_0]| - \textstyle\sum_{i \in \{1,\dots,n\} \setminus \{i_0\}} |p_1[i] p_r[i]|. \tag{17}$$

We have

$$|p_1[i]| \begin{cases} \ge (1 - \sqrt{2}\varepsilon) c_1 \ge \frac{1}{\sqrt{2}} c_1 & \text{for } i = i_0 \in A_1 \setminus B_1 = \{i_0\} \quad \text{(Lemma 8 (a))}, \\ \le c_1 & \text{for } i \in B_1, \\ \le \frac{1}{2n} c_1 & \text{for } i \in C_1 \setminus A_1 \quad \text{(by (15))}, \\ \le c_2 \le \frac{1}{\sqrt{2n}} c_1 & \text{for } i \in C_2 \quad \text{(by (14))} \end{cases}$$

and

$$|p_r[i]| \begin{cases} \ge \frac{1}{\sqrt{2}} c_1 & \text{for } i = i_0 \in A_1 \setminus B_1 = \{i_0\} \quad \text{(by (16))}, \\ \le \sqrt{2}\varepsilon c_1 \le \frac{1}{2n} c_1 & \text{for } i \in B_1 \quad \text{(Lemma 8 (b))}, \\ \le c_1 & \text{for } i \in C_1 \setminus A_1, \\ \le c_2 \le \frac{1}{\sqrt{2n}} c_1 & \text{for } i \in C_2 \quad \text{(by (14))}. \end{cases}$$

Therefore (17) can be continued to

$$0 \ge \tfrac{1}{2} c_1^2 - (n-1) \tfrac{1}{2n} c_1^2 = \tfrac{1}{2n} c_1^2 > 0.$$

This contradiction completes the consideration of Case 1.

Case 2: $t = 2$. In this case

$$\lceil \tfrac{n}{2} \rceil = n - \lceil \tfrac{n}{2} \rceil = 2s + 1, \quad |A_q| \ge \lceil \tfrac{r-1}{2} \rceil = s + 1, \quad |B_q| \ge \lceil \tfrac{r-3}{2} \rceil = s. \tag{18}$$

This yields in particular $|A_2| + |B_2| - n + \lceil \frac{n}{2} \rceil \ge 0$. Hence (11) implies that

$$\big(|A_1| + |B_1| - \lceil \tfrac{n}{2} \rceil\big) c_1^2 \le \tfrac{1}{4n^3}. \tag{19}$$

Since the roles of f_1 and f_2 can be exchanged by a permutation of the coordinates, we can assume that $c_1 = \max\{c_1, c_2\}$ without loss of generality. Therefore

$$c_1^2 \ge \tfrac{1}{n} \tag{20}$$

by (6).

Case 2.1: $|A_1| > s+1$. Then, by (18), $|A_1| + |B_1| - \lceil \frac{n}{2} \rceil \ge (s+2) + s - (2s+1) = 1$ and (19) gives $c_1^2 \le \frac{1}{4n^3}$ in contradiction with (20).

Case 2.2: $|A_1| \le s + 1$. This yields necessarily $|A_1| = s + 1 = \frac{r-1}{2}$. Now Lemma 8 (c) shows that $A_1 = B_1$. Therefore $|A_1| + |B_1| - \lceil \frac{n}{2} \rceil = 2|A_1| - \lceil \frac{n}{2} \rceil = 1$ and, by (19), $c_1^2 \le \frac{1}{4n^3}$. This contradiction with (20) finishes Case 2.

Case 3: $t = 3$. Then

$$\left\lceil \tfrac{n}{2} \right\rceil = 2s+2,\ n - \left\lceil \tfrac{n}{2} \right\rceil = 2s+1,\ |A_q| \geq \left\lceil \tfrac{r-1}{2} \right\rceil = s+2,\ |B_q| \geq \left\lceil \tfrac{r-3}{2} \right\rceil = s+1.$$

Accordingly, inequality (11) yields $c_1^2 + 2c_2^2 \leq \frac{1}{4n^3}$, a contradiction with (6).

Case 4: $t = 4$. In this case

$$\left\lceil \tfrac{n}{2} \right\rceil = n - \left\lceil \tfrac{n}{2} \right\rceil = 2s+2, \quad |A_q| \geq \left\lceil \tfrac{r-1}{2} \right\rceil = s+2, \quad |B_q| \geq \left\lceil \tfrac{r-3}{2} \right\rceil = s+1.$$

Now (11) gives $c_1^2 + c_2^2 \leq \frac{1}{4n^3}$, again a contradiction with (6). □

Acknowledgment

The authors thank Stanisław Szarek for drawing their attention to the problem and for placing a preprint of [8] at their disposal.

References

[1] I.K. Babenko, S.A. Bogatyĭ: *Mapping a sphere into Euclidean space*, Math. Notes **46** (1989), no. 3-4, 683–686 (1990) (Russian original in Mat. Zametki **46** (1989), no. 3, 3–8).
[2] K. Borsuk: *Drei Sätze über die n-dimensionale euklidische Sphäre* (German) [Three theorems on the n-dimensional Euclidean sphere], Fund. Math. **20** (1933), 177–190.
[3] W. Chen: *Counterexamples to Knaster's conjecture*, Topology **37** (1998), 401–405.
[4] E.E. Floyd: *Real-valued mappings of spheres*, Proc. Amer. Math. Soc. **6** (1955), 957–959.
[5] R.H. Hardin, N.J.A. Sloane, W.D. Smith: *Spherical codes*, in preparation (for coverings of $\mathbb{S}^2$ see *Spherical coverings. A library of putatively optimal coverings of the sphere with n equal caps* at http://www.research.att.com/~njas/coverings/index.html, for the particular code of 22 points on $\mathbb{S}^2$ see http://www.research.att.com/~njas/coverings/dim3/cover.3.22.txt).
[6] H. Hopf: *Eine Verallgemeinerung bekannter Abbildungs- und Überdeckungssätze* (German) [A generalization of well-known mapping and covering theorems], Portugaliae Math. **4** (1944), 129–139.
[7] S. Kakutani: *A proof that there exists a circumscribing cube around any bounded closed convex set in $\mathbb{R}^3$*, Ann. of Math. (2) **43** (1942), 739–741.
[8] B.S. Kashin, S.J. Szarek: *The Knaster problem and the geometry of high-dimensional cubes*, C. R. Acad. Sci. Paris, Ser. I **336** (2003), 931–936.
[9] B. Knaster: *Problème 4* (French) [Problem 4], Colloq. Math. **1** (1947), 30–31.
[10] V.V. Makeev: *Some properties of continuous mappings of spheres and problems in combinatorial geometry* (Russian), Geometric questions in the theory of functions and sets (Russian), 75–85, Kalinin. Gos. Univ., Kalinin, 1986.
[11] V.V. Makeev: *Knaster's problem on continuous maps of a sphere into Euclidean space*, J. Soviet Math. **52** (1990), no. 1, 2854–2860 (Russian original in Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **167** (1988), Issled. Topol. 6, 169–178).
[12] V.V. Makeev: *The Knaster problem and almost spherical sections*, Math. USSR-Sb. **66** (1990), no. 2, 431–438 (Russian original in Mat. Sb. **180** (1989), no. 3, 424–431).
[13] V.V. Makeev: *Solution of the Knaster problem for polynomials of second degree on a two-dimensional sphere*, Math. Notes 53 (1993), no. 1-2, 106–107 (Russian original in Mat. Zametki **53** (1993), no. 1, 147–148).
[14] A.Yu. Volovikov: *A Bourgin-Yang-type theorem for $\mathbb{Z}_p^n$-action*, Russian Acad. Sci. Sb. Math. **76** (1993), no. 2, 361–387 (Russian original in Mat. Sb. **183** (1992), no. 7, 115–144).
[15] A.Yu. Volovikov: *On a property of functions on a sphere*, Math. Notes **70** (2001), no. 5-6, 616–627 (Russian original in Mat. Zametki **70** (2001), no. 5, 679–690).
[16] H. Yamabe, Z. Yujobô: *On the continuous function defined on a sphere*, Osaka Math. J. **2** (1950), 19–22.
[17] C.T. Yang: *On maps from spheres to Euclidean spaces*, Amer. J. Math. **79** (1957), 725–732.

Mathematisches Institut, Friedrich-Schiller-Universität Jena, D-07740 Jena, Germany

E-mail address: nah@rz.uni-jena.de

Equipe d'Analyse, Université Paris VI, Case 186, 4, Place Jussieu, 75252 Paris Cedex 05, France

E-mail address: richterc@minet.uni-jena.de