A Quick Summary

Chapter 1

- Topological space. Subspace. Continuous map (preimage of an open set is open). Homeomorphism (bijective, continuous, inverse continuous). Closed set, closure, boundary, interior.
- Compact space (in $\mathbb{R}^d$: closed and bounded). A continuous function attains its minimum on a compact space. A continuous mapping on a compact metric space is uniformly continuous.
- **Homotopy** (of maps; $f \sim g$). **Homotopy equivalence** (of spaces; $X \simeq Y$). Deformation retract (Y can be shrunk continuously to X, "within Y" and keeping X fixed). $X \simeq Y$ iff X and Y are both deformation retracts of some Z.
- Simplex. **Geometric simplicial complex. Abstract simplicial complex** K (a hereditary system of finite sets, includes $\emptyset$). Geometric realization, polyhedron (unique up to homeomorphism). Associates a topological space with any finite hereditary set system. Further notions: $V(\mathsf{K})$, $\dim \mathsf{K}$, $\mathrm{supp}(\boldsymbol{x})$ (the support of a point).
- Triangulation ($X \cong \|\mathsf{K}\|$). Every d-dimensional K can be realized in $\mathbb{R}^{2d+1}$ (use the moment curve).
- **Simplicial map** (vertex $\mapsto$ vertex, preserves simplices). A combinatorial counterpart of a continuous map. Yields a continuous map by affine extension on each simplex. Isomorphism.
- Order complex $\Delta(P, \preceq)$ of a poset (simplices = chains). **Barycentric subdivision** $\mathrm{sd}(\mathsf{K}) := \Delta(\mathsf{K}, \subseteq)$. Geometric view (add barycenters as new vertices); $\|\mathrm{sd}(\mathsf{K})\| \cong \|\mathsf{K}\|$.

Chapter 2

- **Borsuk–Ulam theorem:**
  - For every $f\colon S^n \to \mathbb{R}^n$ there is $\boldsymbol{x}$ with $f(\boldsymbol{x}) = f(-\boldsymbol{x})$.
  - Every antipodal $f\colon S^n \to \mathbb{R}^n$ has a zero.
  - There is no antipodal $f\colon S^n \to S^{n-1}$.
  - There is no $f\colon B^n \to S^{n-1}$ antipodal on the boundary.

- ∘ For every cover of S^n by $n+1$ sets, each of them either open or closed, one of the sets contains a pair of antipodal points (Lyusternik–Shnirel'man theorem).

  Many proofs known.
- Tucker's lemma (a discrete version of Borsuk–Ulam): If T is a triangulation of B^n antipodally symmetric on the boundary and $\lambda: V(\mathsf{T}) \to \{\pm 1, \ldots, \pm n\}$ is a labeling antipodal on the bounary, then there is a complementary edge (labels $+i$ and $-i$). Reformulation: There is no simplicial map of T into $\diamond^{n-1}$ (boundary of the crosspolytope) antipodal on the boundary.

Chapter 3

- Mass distribution μ in $\mathbb{R}^d$ ($\mu(\mathbb{R}^d) < \infty$, open sets measurable, hyperplanes have measure 0). **Ham sandwich theorem:** Every d mass distributions in $\mathbb{R}^d$ can be simultaneously bisected by a hyperplane.
- Every d finite point sets in $\mathbb{R}^d$ can be simultaneously bisected by a hyperplane (A bisected by h: at most $\frac{1}{2}|A|$ points in each of the open half-spaces).
- Mass partition theorems. For example, a mass distribution in $\mathbb{R}^2$ can be dissected into 4 equal parts by 2 lines. Generalization to 2^d pieces by d hyperplanes in $\mathbb{R}^d$ fails for $d \geq 5$ (moment curve).
- Akiyama–Alon: n-point sets $A_1, \ldots, A_d$ in general position in $\mathbb{R}^d$ can be partitioned into n rainbow d-tuples with disjoint convex hulls.
- Any necklace with d kinds of stones can be divided between two thieves by d cuts (place the necklace on the moment curve in $\mathbb{R}^d$ and use ham sandwich). Continuous version (Hobby–Rice).
- Kneser graph $\mathrm{KG}(\mathcal{F})$: edges = pairs of disjoint sets. Chromatic number $\chi(G)$. **Lovász–Kneser theorem:** $\chi\big(\mathrm{KG}(\binom{[n]}{k})\big) = n-2k+2$, $n \geq 2k$. Important examples of graphs with a large chromatic number.
- First proof of the Lovász–Kneser theorem: Let $d := n-2k+1$, suppose a d-coloring exists. Choose n points in general position on S^d. For $i \in [d]$, let A_i consist of the $\boldsymbol{x} \in S^d$ such that the open hemisphere centered at $\boldsymbol{x}$ contains a k-tuple of color i; A_{d+1} is the rest. Apply Lyusternik–Shnirel'man ("open-or-closed" version).
- Dol'nikov: $\chi(\mathrm{KG}(\mathcal{F})) \geq \mathrm{cd}_2(\mathcal{F})$; here $\mathrm{cd}_2(\mathcal{F})$ is the minimum number of white points in a red–blue–white coloring with no $F \in \mathcal{F}$ completely red or completely blue. Proof as before, $d := \chi(\mathrm{KG}(\mathcal{F}))$.
- Gale's lemma: $d+2k$ points can be placed on S^d so that every open hemisphere contains at least k of them. Bárány's proof of the Lovász–Kneser theorem: Dimension one lower than before, $d := n-2k$; supposing that $(d+1)$-coloring exists, place points as in Gale's lemma, define A_i as in the previous proof (but now $i \in [d+1]$), and apply Lyusternik–Shnirel'man for open sets.

- Schrijver's theorem: The subgraph of $\mathrm{KG}(\binom{[n]}{k})$ induced by all stable sets (independent sets in the cycle of length n) has chromatic number $n-2k+2$, same as the whole graph. Proof: Gale's lemma in a stronger form, with every open hemisphere containing a stable set.

Chapter 4

- Quotient space. Sum. Wedge. Contracting a contractible subcomplex is a homotopy equivalence.
- **Join**. For simplicial complexes: $\mathsf{K} * \mathsf{L} := \{F \uplus G : F \in \mathsf{K}, G \in \mathsf{L}\}$, where $F \uplus G = (F \times \{1\}) \cup (G \times \{2\})$. For spaces: $X * Y$ is a quotient space of $X \times Y \times [0,1]$. Both give the same ($\|\mathsf{K} * \mathsf{L}\| \cong \|\mathsf{K}\| * \|\mathsf{L}\|$); use geometric interpretation (X and Y placed in skew affine subspaces; take all segments $\boldsymbol{xy}$, $\boldsymbol{x} \in X$, $\boldsymbol{y} \in Y$). Points written $tx \oplus (1-t)y$ (formal convex combinations).
- ***k*-connected space** (every map from S^i, $i \le k$, is nullhomotopic). S^n is $(n-1)$-connected and not n-connected.
- k-connected $\Leftrightarrow$ 1-connected and zero homology up to dimension k. (k-connected)$*$(ℓ-connected) is $(k+\ell+2)$-connected. Nerve theorem: If subcomplexes cover K and all of their intersections are contractible or empty, then their nerve is homotopy equivalent to $\|\mathsf{K}\|$.
- CW-complex. Cellular map.

Chapter 5

- **$\mathbb{Z}_2$-space** (X, ν), ν a homeomorphism $X \to X$, $\nu^2 = \mathrm{id}$. Free $\mathbb{Z}_2$-space (ν has no fixed points). **$\mathbb{Z}_2$-map** $(f \circ \nu = \omega \circ f)$.
- **$\mathbb{Z}_2$-index** $\mathrm{ind}_{\mathbb{Z}_2}(X) := \min\{n : X \xrightarrow{\mathbb{Z}_2} S^n\}$. Main properties:
  - $\mathrm{ind}_{\mathbb{Z}_2}(X) > \mathrm{ind}_{\mathbb{Z}_2}(Y) \Rightarrow X \overset{\mathbb{Z}_2}{\not\to} Y$.
  - If X is $(n-1)$-connected, then $\mathrm{ind}_{\mathbb{Z}_2}(X) \ge n$.
  - For *free* K, we have $\mathrm{ind}_{\mathbb{Z}_2}(\mathsf{K}) \le \dim(\mathsf{K})$.
  - (Sarkaria's inequality) If L_0 is an invariant subcomplex of a $\mathbb{Z}_2$-complex L, then $\mathrm{ind}_{\mathbb{Z}_2}(\mathsf{L}_0) \ge \mathrm{ind}_{\mathbb{Z}_2}(\mathsf{L}) - \mathrm{ind}_{\mathbb{Z}_2}(\Delta(\mathsf{L} \setminus \mathsf{L}_0)) - 1$.
- **Deleted join** $\mathsf{K}^{*2}_{\Delta} := \{F \uplus G : F, G \in \mathsf{K}, F \cap G = \emptyset\}$ (a free $\mathbb{Z}_2$-space). $\mathrm{ind}_{\mathbb{Z}_2}((\sigma^n)^{*2}_{\Delta}) = n$. Deleted join $(\mathbb{R}^d)^{*2}_{\Delta} := (\mathbb{R}^d)^{*2} \setminus \{\frac{1}{2}\boldsymbol{x} \oplus \frac{1}{2}\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^d\}$. $\mathrm{ind}_{\mathbb{Z}_2}((\mathbb{R}^d)^{*2}_{\Delta}) = d$.
- **Nonembeddability theorem:** If $\mathrm{ind}_{\mathbb{Z}_2}(\mathsf{K}^{*2}_{\Delta}) > d$, then any map $\|\mathsf{K}\| \to \mathbb{R}^d$ identifies two points with disjoint support, and in particular, K is not realizable in $\mathbb{R}^d$. Proof: For a map $f\colon \|\mathsf{K}\| \to \mathbb{R}^d$ with $f(\boldsymbol{x}_1) \ne f(\boldsymbol{x}_2)$ whenever $\mathrm{supp}(\boldsymbol{x}_1) \ne \mathrm{supp}(\boldsymbol{x}_2)$, the map f^{*2} can be regarded as a $\mathbb{Z}_2$-map $\|\mathsf{K}^{*2}_{\Delta}\| \to (\mathbb{R}^d)^{*2}_{\Delta}$, and the right-hand side has $\mathbb{Z}_2$-index d.
- Topological Radon theorem: Any continuous map of the $(d+1)$-simplex into $\mathbb{R}^d$ identifies two points with disjoint supports.

- **Van Kampen–Flores theorem:** Let K denote the d-skeleton of the $(2d{+}2)$-simplex; then K cannot be realized in $\mathbb{R}^{2d}$ ($d = 1$ is the nonplanarity of K_5). Needed: $\mathrm{ind}_{\mathbb{Z}_2}(\mathsf{K}^{*2}_{\Delta}) > 2d$.
- First proof of Van Kampen–Flores: Bier spheres; shows that $\|\mathsf{K}^{*2}_{\Delta}\| \cong S^{2d+1}$.
- The construction of the Bier spheres also gives that most triangulations of S^n are nonpolytopal.
- Kneser colorings and index of the deleted join: Let $\mathcal{F}$ be the system of inclusion-miniman nonfaces of a simplicial complex K on $[n]$. Then $\mathrm{ind}_{\mathbb{Z}_2}(\mathsf{K}^{*2}_{\Delta}) \geq n{-}\chi(\mathrm{KG}(\mathcal{F})){-}1$. Method: Sarkaria's inequality; an m-coloring of $\mathrm{KG}(\mathcal{F})$ provides a $\mathbb{Z}_2$-embedding of the order complex of $(\sigma^{n-1})^{*2}_{\Delta} \setminus \mathsf{K}^{*2}_{\Delta}$ into S^{m-1}.
- Implies the Lovász–Kneser theorem, as well as Dol'nikov's theorem.
- **Neighborhood complex** $\mathsf{N}(G)$ of a graph (vertex set $V(G)$, maximal simplices = neighborhoods of vertices). Lovász's theorem: If $\mathsf{N}(G)$ is k-connected, then $\chi(G) \geq k{+}3$.
- Box complex $\mathsf{B}(G)$ of a graph (vertex set $V \times [2]$, simplices $A \uplus B$, where A and B are color classes of a complete bipartite subgraph of G). Graph homomorphism; induces a $\mathbb{Z}_2$-map of the box complexes. Implies $\chi(G) \geq \mathrm{ind}_{\mathbb{Z}_2}(\mathsf{B}(G)) + 2$. Lovász's theorem can be proved from this.
- Generalized Mycielski construction $M_r(G)$. $\mathsf{N}(M_r(G)) \simeq \mathrm{susp}(\mathsf{N}(G))$; yields a lower bound on the chromatic number for iterated $M_r(\cdot)$.

Chapter 6

- ***G*-space**, G-action $\Phi = (\varphi_g)_{g\in G}$; ***G*-map.** Free G-action (no φ_g has a fixed point for $g \neq e$) vs. fixed-point free G-action (no point fixed by all φ_g).
- The only group acting freely on S^{2n} is $\mathbb{Z}_2$. Every $\mathbb{Z}_q$ acts freely on S^1 (rotation) and on S^{2n-1} (join of S^1's).
- E_nG space (G finite): n-dimensional, $(n{-}1)$-connected, finite, free simplicial G-complex. Canonical example: $G^{*(n+1)}$.
- ***G*-index** $\mathrm{ind}_G(X) := \min\{n : X \xrightarrow{G} E_nG\}$. Properties analogous to the $\mathbb{Z}_2$-index. Nontrivial part: $E_nG \not\xrightarrow{G} E_{n-1}G$ (generalized Borsuk–Ulam).
- $X^n_{\Delta(k)}$ (n-fold k-wise deleted product of a space; delete from X^n all n-tuples in which some k components coincide). We need $(\mathbb{R}^d)^p_{\Delta(p)}$ for prime p, which has $\mathbb{Z}_p$-index $d(p{-}1){-}1$ (deformation retraction to $S^{d(p-1)-1}$).
- $X^{*n}_{\Delta(k)}$ (n-fold k-wise deleted join of a space; delete from X^{*n} all $\frac{1}{n}x_1 \oplus \frac{1}{n}x_2 \oplus \cdots \oplus \frac{1}{n}x_n$, where some k of the x_i coincide). Again, we need only $(\mathbb{R}^d)^{*p}_{\Delta(p)}$ for prime p; $\mathbb{Z}_p$-index is $(d{+}1)(p{-}1){-}1$.
- $\mathsf{K}^{*n}_{\Delta(k)}$ (n-fold k-wise deleted join of a simplicial complex; delete from K^{*n} all n-tuples of simplices in which some k simplices share a vertex). We use mainly $k = 2$.

- Necklace for q thieves with d kinds of stones: $N := d(q-1)$ stones suffice. Proof: For prime q, encode divisions by points of $(\sigma^N)^{*q}_{\Delta(2)}$; if no just division existed, the shares of the thieves would yield a $\mathbb{Z}_q$-map $(\sigma^N)^{*q}_{\Delta(2)} \to (\mathbb{R}^d)^q_{\Delta(q)}$.
- Tverberg's theorem: Any $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ can be partitioned into r groups whose convex hulls all have a common point. **Topological Tverberg theorem:** Any continuous map $f\colon \|\sigma^N\| \to \mathbb{R}^d$ identifies some p points with disjoint supports into a single point, where $N = (d+1)(p-1)$. Proof: A bad f would yield a $\mathbb{Z}_p$-map $(\sigma^N)^{*p}_{\Delta(2)} \to (\mathbb{R}^d)^{*p}_{\Delta(p)}$. Works only for p prime; the theorem is also known for prime powers but not in general!
- A lower bound for the number of Tverberg partitions follows by a similar method (find many smaller invariant subcomplexes of $(\sigma^N)^{*p}_{\Delta(2)}$ that still have $\mathbb{Z}_p$-index N).
- Kneser hypergraph $\mathrm{KG}_r(\mathcal{F})$. Chromatic number of a hypegraph (no monochromatic edge). Embeddability of $\|\mathsf{K}\|$ into $\mathbb{R}^d$ without p-fold points, with *p prime*, can be related to $\chi(\mathrm{KG}_r(\mathcal{F}))$, where $\mathcal{F}$ consists of the minimal nonfaces of K, as in the $\mathbb{Z}_2$-case.
- **Colored Tverberg theorem:** If we have points of $d+1$ colors in $\mathbb{R}^d$, sufficiently many points of each color, then we can select r disjoint rainbow subsets whose convex hulls all have a common point. Proof: Let r be a prime. Consider the simplicial complex K with $(d+1)$-element rainbow sets as maximal simplices, and produce a good coloring of $\mathrm{KG}_r(\mathcal{F})$. Result: $2r-1$ points of each color suffice.