equivalence of two $\mathbb{Z}_2$-spaces is a homotopy equivalence as in Definition 1.2.2 where, moreover, the witnessing maps f and g are $\mathbb{Z}_2$-maps, and so are all the intermediate maps in the homotopies $g \circ f \sim \mathrm{id}$ and $f \circ g \sim \mathrm{id}$.

Second, two simplicial complexes K and L are said to be the same *simple-homotopy equivalent* (or to have the same *simple-homotopy type*) if K can be transformed into L by a finite sequence of elementary collapses and elementary expansions. Here an *elementary collapse* of a simplicial complex K is the following operation, quite important in combinatorial topology: Assuming that $F \subset G$ (proper inclusion!) are simplices of K such that G is the only inclusion-maximal simplex of K that contains F, the elementary collapse of F means the removal of all simplices H with $F \subseteq H \subseteq G$ from K. Elementary expansion is the inverse operation to elementary collapse. An elementary collapse preserves the homotopy type (it actually corresponds to a deformation retraction of the polyhedron), and hence simple-homotopy equivalent simplicial complexes are also homotopy equivalent, but not necessarily the other way round. The notion of simple-homotopy equivalence, developed by Whitehead in the 1950s, can be regarded as a "combinatorial version" of homotopy equivalence. Finally, the reader will easily guess what simple-$\mathbb{Z}_2$-homotopy equivalence means: The elementary collapses and expansions are performed in pairs so that the $\mathbb{Z}_2$-symmetry is preserved throughout the sequence.

It was also shown that box complexes are universal—they may "look like" a completely arbitrary simplicial $\mathbb{Z}_2$-complex. Namely, for every simplicial $\mathbb{Z}_2$-complex K there exists a graph G such that $\mathsf{B}(G)$ is simple-$\mathbb{Z}_2$-homotopy equivalent to K (we again refer to [Cso07] for this result and references to earlier work by Csorba and by Živaljević).

Hom complexes, introduced by Lovász, constitute a far-reaching generalization of box complexes of graphs, and they have been connected to some of the most exciting recent developments in topological combinatorics. For every two graphs F and G, the Hom complex $\mathrm{Hom}(F, G)$ is a simplicial complex[6] reflecting the structure of the set of all homomorphisms from F to G.

To introduce $\mathrm{Hom}(F, G)$, we first define a *multihomomorphism*[7] $F \to G$ as a mapping $\varphi\colon V(F) \to 2^{V(G)} \setminus \{\emptyset\}$ (i.e., associating a nonempty subset of vertices of G with every vertex of F) such that whenever $\{u_1, u_2\}$ is an edge of F, we have $\{v_1, v_2\} \in E(G)$ for every $v_1 \in \varphi(u_1)$ and every $v_2 \in \varphi(u_2)$.

[6] The original definition of $\mathrm{Hom}(F, G)$ actually yields a polyhedral cell complex. What we define here is the first barycentric subdivision of that complex; it is a simplicial complex and it seems easier to grasp with the background presented in this book.

[7] An example of a concept easier than its name.

Let us remark that while a graph homomorphism f is formally not a multihomomorphism, it can be regarded as one if needed (a vertex u is assigned the one-element set $\{f(u)\}$). Multihomomorphisms can also be composed: If φ is a multihomomorphism $F \to G$ and ψ a multihomomorphism $G \to H$, then the composition $\psi \circ \varphi$ is the multihomomorphism $F \to H$ that maps a vertex $u \in V(F)$ to the set $\bigcup_{v \in \varphi(u)} \psi(v)$.

Let $\mathrm{mhom}(F, G)$ denote the set of all multihomomorphisms $F \to G$. We introduce a partial ordering $\leq$ on $\mathrm{mhom}(F, G)$ by letting $\varphi \leq \psi$ if $\varphi(u) \subseteq \psi(u)$ for all $u \in V(F)$, and we define $\mathrm{Hom}(F, G)$ as the order complex the resulting poset.

Hom complexes are mostly considered with F a (small) fixed graph and G varying. In particular, $\mathrm{Hom}(K_2, G)$ is yet another disguise of the box complex of G (more precisely, $\mathsf{B}(G)$ and $\mathrm{Hom}(K_2, G)$ are simple-$\mathbb{Z}_2$-homotopy equivalent [Cso07]; also see Exercise 6). And even with very simple F, the complexes $\mathrm{Hom}(F, G)$ are complicated, beautiful, and fascinating. For example, there are only few graphs simpler than the complete graph K_m or the 5-cycle C_5, but it took a long effort of several authors to analyze the topology of $\mathrm{Hom}(C_5, K_n)$ (culminating in a proof by Schultz [Sch06b] that it is homeomorphic to the Stiefel manifold $V_{n-1,2}$, which was conjectured by Csorba).

The Babson–Kozlov–Lovász theorem. The main source of interest in the Hom complexes has been the following conjecture of Lovász:

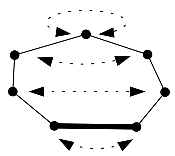
If $\mathrm{Hom}(C_{2r+1}, G)$ *is* k*-connected, then* $\chi(G) \geq k+4$*, for all graphs* G *and all* $r \geq 1$.

It is instructive to compare this to Theorem 5.9.4, the Lovász 1978 bound, which can equivalently be rephrased as "$\mathrm{Hom}(K_2, G)$ being k-connected implies $\chi(G) \geq k + 3$." The Lovász conjecture thus says that odd cycles as "test graphs" are better by 1 than K_2.

The conjecture was proved in a *tour de force* of Babson and Kozlov [BK07] by complicated computations with spectral sequences (and so it may now be called the Babson–Kozlov–Lovász theorem). Later Živaljević [Živ05] discovered a simple proof of "half" of the result (for all k odd, that is), and then Schultz [Sch06a] extended Živaljević's idea to a simple proof of the full conjecture. Kozlov [Koz06] found another very short proof.

We will outline the main ideas of Schultz's proof and some conclusions following from it. First we note a kind of generalization of the functorial property of the box complex $\mathsf{B}(.)$ used in the main text: A graph homomorphism $g\colon G \to G'$ induces, for every F, a simplicial map $\mathrm{Hom}(F, g)\colon \mathrm{Hom}(F, G) \to \mathrm{Hom}(F, G')$, and almost the same construction works if g is a multihomomorphism (Exercise 7). There is a dual construction as well: A (multi)homomorphism $f\colon F \to F'$ induces, for every G, a simplicial map $\mathrm{Hom}(f,G)\colon \mathrm{Hom}(F',G) \to \mathrm{Hom}(F,G)$; note that here the direction of the map is reversed! These constructions also behave "properly" with respect to map composition—briefly speaking, $\mathrm{Hom}(.,.)$ is functorial in both arguments.

For suitable graphs F, including K_2 and odd cycles, $\mathrm{Hom}(F,G)$ can be made into a simplicial $\mathbb{Z}_2$-complex by the following construction. The assumption is that F must possess an involutive automorphism i_F that flips some edge; that is, i_F is an isomorphism of F with itself such that $i_F \circ i_F = \mathrm{id}$ and there is an edge $\{u,v\} \in E(F)$ with $i_F(u) = v$ and $i_F(v) = u$. For $F = K_2$, i_{K_2} swaps the two vertices of K_2, and for $F = C_{2r+1}$, the following $i_{C_{2r+1}}$ is used (depicted for C_7, with the flipped edge marked bold):



Then, as a particular case of the construction noted above, i_F induces a simplicial map $\nu := \mathrm{Hom}(i_F, G)$ of $\mathrm{Hom}(F,G)$ into itself, and it is easy to check that this ν is a free $\mathbb{Z}_2$-action on $\mathrm{Hom}(F,G)$. Moreover, if $g\colon G \to G'$ is a graph homomorphism, then $\mathrm{Hom}(F,g)$ becomes a simplicial $\mathbb{Z}_2$-map $\mathrm{Hom}(F,G) \to \mathrm{Hom}(F,G')$ (Exercise 8).

Following the proof pattern used for the box complex in this section, one can easily see that for proving the Lovász conjecture, it suffices to prove "only" that no $(m-4)$-connected $\mathbb{Z}_2$-space can be $\mathbb{Z}_2$-mapped into $\mathrm{Hom}(C_{2r+1}, K_m)$. For this, in turn, it is enough to show

$$\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{Hom}(C_{2r+1}, K_m)) \leq m-3, \tag{5.2}$$

which then yields $\chi(G) \geq \mathrm{coind}_{\mathbb{Z}_2}(\mathrm{Hom}(C_{2r+1}, G)) + 3$, a slightly stronger form of the Lovász conjecture.

A natural approach to (5.2) would be to exhibit a $\mathbb{Z}_2$-map of $\|\mathrm{Hom}(C_{2r+1}, K_m)\|$ into S^{m-3}, i.e., to actually bound $\mathrm{ind}_{\mathbb{Z}_2}$—but here the troubles start: While (5.2) does indeed hold, $\mathrm{ind}_{\mathbb{Z}_2}(\mathrm{Hom}(C_{2r+1}, K_m)) \leq m-3$ actually *fails* for some values of r and m, and thus the proof must go differently.

Schultz derived (5.2) from the following more general inequality:

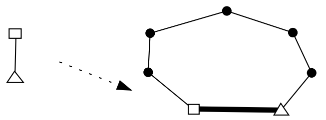
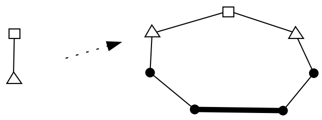
$$\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{Hom}(C_{2r+1}, G)) \leq \mathrm{ind}_{\mathbb{Z}_2}(\mathrm{Hom}(K_2, G)) - 1 \tag{5.3}$$

for all G (then (5.2) follows by setting $G := K_m$ and using $\mathrm{ind}_{\mathbb{Z}_2}(\mathrm{Hom}(K_2, K_m)) = m-2$).

We should remark that (5.3) is somewhat disappointing: The hope originally attached to the Lovász conjecture was that it might sometimes establish a stronger bound on $\chi(G)$ than the "old" Lovász inequality $\chi(G) \geq \mathrm{ind}_{\mathbb{Z}_2}(\mathrm{Hom}(K_2, G)) + 2$, but (5.3) shows that it is never the case (of course, it might happen by some miracle that for some G, $\mathrm{coind}_{\mathbb{Z}_2}(\mathrm{Hom}(C_{2r+1}, G))$ is easier to determine than $\mathrm{ind}_{\mathbb{Z}_2}(\mathrm{Hom}(K_2, G))$).

We now discuss the proof of (5.3). We have seen that every homomorphism $f\colon F \to F'$ yields the simplicial map $\mathrm{Hom}(f, G)$, and hence a continuous map $\|\mathrm{Hom}(F', G)\| \to \|\mathrm{Hom}(F, G)\|$. Now each point $\boldsymbol{x}$ of the polyhedron $\mathrm{Hom}(F, F')$ can be regarded as a (very) generalized homomorphism $F \to F'$; namely, it is a formal convex combination of multihomomorphisms $F \to F'$ (and the multihomomorphisms involved in such a convex combination have to form a chain under $\leq$). A key observation in Schultz's proof is that such an $\boldsymbol{x} \in \|\mathrm{Hom}(F, F')\|$ also yields a continuous map $\mathrm{Hom}(\boldsymbol{x}, G)\colon \|\mathrm{Hom}(F', G)\| \to \|\mathrm{Hom}(F, G)\|$. The construction is a bit demanding to write down explicitly, but conceptually it is very easy—since $\boldsymbol{x}$ is a formal convex combination of multihomomorphisms, we just take the corresponding convex combination of the images under these multihomomorphisms. Moreover, we also need that $\mathrm{Hom}(\boldsymbol{x}, G)$ varies continuously with $\boldsymbol{x}$, and in particular, if $\boldsymbol{a}, \boldsymbol{b} \in \|\mathrm{Hom}(F, F')\|$ are points connected by a path, then the corresponding maps $\mathrm{Hom}(\boldsymbol{a}, G)$ and $\mathrm{Hom}(\boldsymbol{b}, G)$ are homotopic.

To establish (5.3), we use this construction with $F = K_2$ and $F' = C_{2r+1}$. The multihomomorphisms of an edge into an odd cycle are easy to describe, and it turns out that $\mathrm{Hom}(K_2, C_{2r+1})$ is an S^1 with the antipodal $\mathbb{Z}_2$ action (geometrically it is the boundary of a $(4r{+}2)$-gon). We now consider two particular vertices $\boldsymbol{a}$ and $\boldsymbol{b}$ of $\mathrm{Hom}(K_2, C_{2r+1})$, corresponding to the multihomomorphisms depicted below:

$\boldsymbol{a}$:  $\boldsymbol{b}$: 

So $\boldsymbol{a}$ is a homomorphism sending K_2 onto the edge of C_{2r+1} that is flipped by $i_{C_{2r+1}}$, while $\boldsymbol{b}$ is a multihomomorphism. We note that $\boldsymbol{a}$ commutes with i_{K_2} and $i_{C_{2r+1}}$, i.e., $\boldsymbol{a} \circ i_{K_2} = i_{C_{2r+1}} \circ \boldsymbol{a}$, while $\boldsymbol{b}$ is invariant under $i_{C_{2r+1}}$, i.e., $i_{C_{2r+1}} \circ \boldsymbol{b} = \boldsymbol{b}$. This translates to the induced maps as follows: $f_{\boldsymbol{a}} := \mathrm{Hom}(\boldsymbol{a}, G)\colon \|\mathrm{Hom}(C_{2r+1}, G)\| \to \|\mathrm{Hom}(K_2, G)\|$ is a $\mathbb{Z}_2$-map, while $f_{\boldsymbol{b}} := \mathrm{Hom}(\boldsymbol{b}, G)$ is an *even* map between the same spaces, where a map $f\colon X \to Y$ of a $\mathbb{Z}_2$-space (X, ν) into a space Y is called even if $f(\nu(x)) = f(x)$ for all $x \in X$. At the same time, $f_{\boldsymbol{a}}$ and $f_{\boldsymbol{b}}$ are homotopic, since $\mathrm{Hom}(K_2, C_{2r+1})$ is path-connected (and it is fun to visualize the path from $\boldsymbol{a}$ to $\boldsymbol{b}$ by "walking" the K_2 around the cycle).

With this preparation, (5.3) follows from the next claim: *If (X, ν) and (Y, ω) are $\mathbb{Z}_2$-spaces and there exist two homotopic maps $f, g\colon X \to$*

Y, where f is a $\mathbb{Z}_2$-map and g is an even map, then

$$\mathrm{coind}_{\mathbb{Z}_2}(X) \leq \mathrm{ind}_{\mathbb{Z}_2}(Y) - 1. \tag{5.4}$$

This can easily be proved assuming the notion of even and odd degree of a map of spheres (see Section 2.4). Indeed, if (5.4) didn't hold, there would be $\mathbb{Z}_2$-maps $S^n \xrightarrow{\mathbb{Z}_2} X$ and $Y \xrightarrow{\mathbb{Z}_2} S^n$ for some n, and by composing these with f and g, respectively, we would get two homotopic maps $f^*, g^*: S^n \to S^n$, where f^* is a $\mathbb{Z}_2$-map and g^* is even. But this is impossible, since a $\mathbb{Z}_2$-map has odd degree and an even map has even degree, and such maps cannot be homotopic. (I don't know of a way of deriving the claim directly from the Borsuk–Ulam theorem.) We have seen a more or less complete proof of the Babson–Kozlov–Lovász theorem.

The proof of Schultz, as well as that of Kozlov [Koz06] actually establish the stronger inequality $\text{cohom-ind}_{\mathbb{Z}_2}(\mathrm{Hom}(C_{2r+1}, K_m)) \leq m-3$, where $\text{cohom-ind}_{\mathbb{Z}_2}(.)$ is the cohomological index mentioned in the notes to Section 5.3 (the Babson–Kozlov original proof gave this only for m odd). Interestingly, the counterpart of (5.4), under the same assumptions on X and Y, is $\text{cohom-ind}_{\mathbb{Z}_2}(X) \leq \text{cohom-ind}_{\mathbb{Z}_2}(Y) - 1$, with the same kind of index on both sides. This is interesting because both $\mathrm{ind}_{\mathbb{Z}_2}(X) \leq \mathrm{ind}_{\mathbb{Z}_2}(Y) - 1$ and $\mathrm{coind}_{\mathbb{Z}_2}(X) \leq \mathrm{coind}_{\mathbb{Z}_2}(Y) - 1$ are generally *false*, as can be shown by examples, and so $\text{cohom-ind}_{\mathbb{Z}_2}$ comes out as a better-behaved index.

When investigating Hom complexes, one doesn't have to stop at odd cycles, of course. Let us call a graph T a *test graph* if for every G, the k-connectedness of $\mathrm{Hom}(T, G)$ implies $\chi(G) \geq k + 1 + \chi(T)$ (so Theorem 5.9.4 says that K_2 is a test graph, and the Babson–Kozlov–Lovász theorem tells us that C_{2r+1} is one). There are many more test graphs known besides these (see, e.g., [Sch06a]), but as shown by Hoory and Linial [HL05], not all graphs are test graphs. Here we leave the exciting domain of Hom complexes, referring, e.g., to Kozlov [Koz07] for a much more thorough treatment of this topic and to recent work, partially still in progress, by Babson, Čukić, Dochtermann, Kozlov, and others for fresh news.

Generalized Mycielski graphs. Gyárfás et al. [GJS04] used the generalized Mycielski construction as an auxiliary step in a solution to a problem of Harvey and Murty: For every k, they construct a k-chromatic graph possessing a k-coloring such that the neighborhood of each color class is an independent set (in other words, if u has a neighbor of color i and v has a neighbor of color i, then u and v are not connected by an edge).

Distance-α graphs on spheres. Lovász [Lov83] used Theorem 5.9.4 in a solution of a problem of Erdős and Graham. He considered the graphs