LECTURE 2

8/10/2020

METRIC TSP

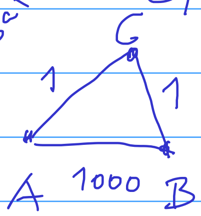
$V = \{1, 2, \dots, n\}$

$d : V \times V \to \mathbb{R}_0^+ : \forall u, v \in V : \quad d(u,v) = d(v,u)$

$\forall u, v, z \in V : \quad d(u,z) \le d(u,v) + d(v,z)$ — triangle inequality

$\forall u, v \in V \quad d(u,v) = 0 \iff u = v$

does not satisfy ...



TSP-MST ALGORITHM

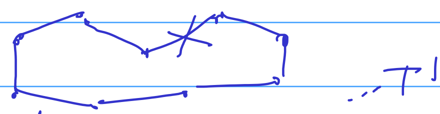
- Compute an MST T of G
- Find an Eulerian tour in "$2T$" ← every edge in T replaced by 2 parallel edges
- obtain a Hamiltonian tour from the Eulerian tour by shortcutting

Eul. tour $\quad \boxed{i_0, i_1}\ \boxed{\cancel{i_2, i_3, \dots}}\ i_j \dots \quad i_e, \cancel{i_0}$

Note: the resulting sequence is a Hami. tour

Cost?

$$cost(T) \le OPT$$



💡 Hamilt. tour without any single edge is a spanning tree

$$cost(T) \le cost(T') \le OPT$$

$$cost(\text{Eulerian tour}) \le 2\,cost(T) \le 2\,OPT$$

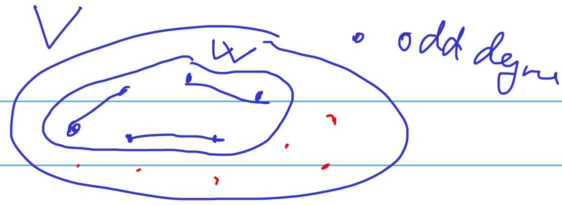
→ 💡 $\forall i_1, i_2, \dots, i_k : d(i_1, i_k) \le d(i_1, i_2) + d(i_2, i_3) + \dots + d(i_{k-1}, i_k)$

for $k = 2$ ... Δ-inequality

$k > 2$ ... by induction

Applying this observation to the shortcutting procedure, we conclude $A(I) \le 2\,cost(T) \le 2\,OPT$

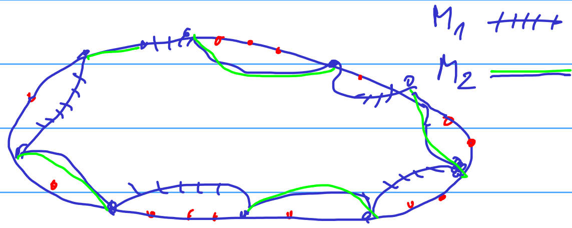
THM: TSP-MST is a 2-approximation algorithm.



CHRISTOFIDES ALGORITHM

- Compute MST T
- Let W be the vertices of odd degree in T
- Find a min cost perfect matching M on W in
- Find an Eulerian tour in "$T \cup M$"
- Shortcut the Eulerian tour as in TSP-MST

the size of W is even.



Correctness ✓

Cost ?

$$\mathrm{Cost}(T) \le OPT$$

$$\mathrm{Cost}(M) \le \frac{1}{2}\, OPT$$

shortcut the optimal tour
to a tour through W
split it into 2 matchings M_1, M_2 on W
note: $\mathrm{cost}(M_1) + \mathrm{cost}(M_2) \le OPT$

$$\Rightarrow \exists i \ \text{ s.t. } \ \mathrm{cost}(M_i) \le \tfrac{1}{2}\, OPT$$

$$\mathrm{cost}(M) \le \mathrm{cost}(M_i) \le \tfrac{1}{2}\, OPT$$

THM: The approximation ratio is $\frac{3}{2}$ of CHRISTOFIDES ALG.

DISCRETE PROBABILITY SPACE (Ω, P) where

Ω is a finite or countable set of elementary events

P is a probability function $P : \Omega \to [0,1]$

satisfying $\sum_{\omega \in \Omega} P[\omega] = 1$

E.g. Tossing a coin : $\Omega = \{H, T\}$

$P[H] = P[T] = \frac{1}{2}$

Rolling a die : $\Omega = \{1,2,3,4,5,6\}$ $P[1] = P[2] = \dots = \frac{1}{6}$

$\Omega = \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}$

n tosses of a coin : $\Omega = \{H, T\}^n$, $\forall \omega \in \Omega : P[\omega] = \frac{1}{2^n}$

Uniform probability space (for finite spaces)

every elementary event has probability $\frac{1}{|\Omega|}$

Geometric probability space : $\Omega = \{1, 2, \dots\}$

for $i \in \Omega$: $P[i] = \frac{1}{2^i}$

RANDOM VARIABLE on a probability space (Ω, P)

is an arbitrary function $X : \Omega \to \mathbb{R}$

e.g. • for n tosses of a coin : the number of heads

• die : the number of dots

the expectation of a random variable X

$$E[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P[\omega]$$

An EVENT is a subset $A \subseteq \Omega$

e.g. die : $A = \{2,4,6\}$ $B = \{1,3,5\}$ $C = \{3,6\}$

probability of A : $P[A] = \sum_{\omega \in A} P[\omega]$

INDICATOR VARIABLE of AN EVENT A

$$I_A(\omega) = \begin{cases} 1 & \text{for } \omega \in A \\ 0 & \omega \notin A \end{cases}$$

$$\mathbb{E}[I_A] = P[A] \qquad \text{H.W}$$

LINEARITY OF EXPECTATION

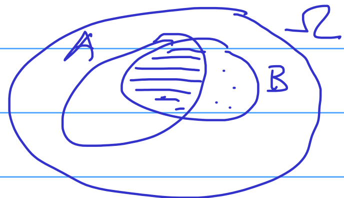
Lemma: for any two random variables X, Y:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$\forall \alpha \qquad \mathbb{E}[\alpha \cdot X] = \alpha \cdot \mathbb{E}[X]$$

CONDITIONAL PROBABILITY of an event A assuming an event B has occurred is defined as

$$P[A|B] = \frac{P[A \cap B]}{P[B]}, \text{ for } P[B] > 0$$



Note: $P[\cdot\,|B]$ is a new probability measure on Ω:

for $\omega \notin B$: $P[\omega|B] = 0$

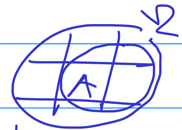
$\omega \in B$: $P[\omega|B] = \frac{P[\omega]}{P[B]}$

$$\sum_{\omega \in \Omega} P[\omega|B] = \sum_{\omega \in B} P[\omega|B] = 1$$

THM about complete probability: if $B_1, \dots, B_\ell$ are disjoint events s.t. $\bigcup_{i=1}^{\ell} B_i = \Omega$, then for an event $A \subseteq \Omega$

$$P[A] = P[A|B_1] \cdot P[B_1] + P[A|B_2] \cdot P[B_2] + \dots + P[A|B_\ell] \cdot P[B_\ell]$$

proof: $P[A] = P[A \cap \bigcup_{i=1}^{\ell} B_i] = \sum_{i=1}^{\ell} P[A \cap B_i] = \sum P[A|B_i] \cdot P[B_i]$



$\square$

INDEPENDENT EVENTS

A, B are independent if $P[A \cap B] = P[A] \cdot P[B]$

equivalently: if $P[B] = 0$ or $P[A] = P[A|B]$