Tutorial 7, November 14, 2019

1. For $x \in \mathbb{R}$ and $\delta > 0$ we denote by $P(x, \delta)$ the deleted δ-neighborhood of x. Why is $[0, 1] \setminus P(x, \delta)$ compact?

2. (HW 3 pts.) Let (M, d) be a compact metric space and $f \colon M \to \mathbb{R}$ be a continuous function. Prove that f is uniformly continuous.

3. (HW 3pts.) Let $f \colon [0, 1] \to \mathbb{R}$ be a broken line (i.e. a piecewise linear continuous function) and let $S = \max |s|$, taken over all slopes s of the straight segments of the graph of f. Prove that then for the slope t of any secant line of the graph of f we have $|t| \leq S$.

4. Let $f \colon [a, b] \to \mathbb{R}$ be a linear function. Show that then for every $x \in [a, b]$ one has $\min(f(a), f(b)) \leq f(x) \leq \max(f(a), f(b))$.

5. Prove that for every $a, b \in \mathbb{R}$ one has $|a + b| \geq |a| - |b|$.

6. Prove that if $a, b \in \mathbb{R}$ with $0 < a < 1$ then the function $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n x) \colon \mathbb{R} \to \mathbb{R}$ is continuous.

7. Determine the intervals (or subsets) of the definition domains, on which the following sequences of functions converge pointwisely, uniformly, and locally uniformly. What are the limit functions?

   (a) $f_n(x) = \frac{1}{x+n}$, defined on $\mathbb{R}$.
   (b) $f_n(x) = x^n - x^{3n}$, defined on $[0, 1]$.
   (c) $f_n(x) = x^{n+1} - x^{n-1}$, defined on $[0, 1]$.
   (d) (HW 3 pts.) $f_n(x) = x^n - x^{n+1}$, defined on $\mathbb{R}$.