Tutorial 6, November 7, 2019

1. Prove that $f_n \rightrightarrows f$ on M iff $\|f_n - f\| \to 0$ for $n \to \infty$.

2. Prove that $f_n \overset{\text{loc}}{\rightrightarrows} f$ on $[0,1)$, where $f_n(x) = x^n$ and f is the pointwise limit of the functions f_n on $[0,1)$, the zero function.

3. (HW 3 pts.) Prove that for $f_n(x) = \frac{nx}{1+n^2x^2}$ the sequence of functions $f_n \overset{\text{loc}}{\not\rightrightarrows} \equiv 0$ on $\mathbb{R}$.

4. Prove that the convergence of the $f_n(x)$ in the previous exercise is uniform on $(-\infty, -\delta) \cup (\delta, +\infty)$ for every $\delta > 0$.

5. Prove that the set of functions $X = \{f_1, f_2, \dots\} \subset N$, where $f_n(1/n) = 1$ and $f_n(x) = 0$ for $x \neq 1/n$ and $N = \{f \mid f\colon [0,1] \to \mathbb{R} \text{ and } f \text{ is bounded}\}$, is a closed and bounded but not compact subset of the metric space N.

6. Prove that the normed space of all real functions defined on a nonempty set is complete.

7. How does exactly follow from the theorem in the lecture — each Cauchy sequence (f_n) of continuous functions $f_n\colon M \to \mathbb{R}$ defined on a metric space M has a uniform limit that is a continuous function on M — that the uniform limit of continuous functions is a continuous function?

8. Prove that for finite M, $f_n \to f$ on M implies $f_n \rightrightarrows f$ on M.

9. (HW 4 pts.) Let $f_n \rightrightarrows f$ on M and $g_n \rightrightarrows f$ on M. Determine if then also $f_n + g_n \rightrightarrows f + g$ on M.

10. (HW 4 pts.) Let $f_n \rightrightarrows f$ on M and $g_n \rightrightarrows g$ on M. Determine if then also $f_n g_n \rightrightarrows fg$ on M.