Tutorial 5, October 31, 2019

1. (HW, 3 pts) Prove that every closed ball is a closed set.

2. Prove that for every $a \in M$ and every positive $s < r$, $\overline{B}(a, s) \subset B(a, r)$.

3. In the statement of the Baire theorem we have the countable union $\bigcup_{n=1}^{\infty}$. How does follow from this that the theorem holds for finite unions too?

4. (HW, 4 pts) Here is an attempt to construct a countable and closed set $X \subset \mathbb{R}$ without isolated points. Let $X_0 = \{0\}$. This is a closed and at most countable set but it has an isolated point. Let $X_1 = \{1/n \mid n \in \mathbb{N}\}$. The set $X_0 \cup X_1$ is clearly countable and closed and no point in X_0 is isolated. But every point in X_1 is isolated. But we can add for every $b \in X_1$ in a similar way a sequence of points converging to b. Let X_2 be the union of these sequences over all $b \in X_1$. Then $X_0 \cup X_1 \cup X_2$ is countable and closed and no point in $X_0 \cup X_1$ is isolated. We can similarly remove isolation of points in X_2 by adding a countable set X_3 and so on. The result $\bigcup_{n=0}^{\infty} X_n$ is countable and closed and none of its points is isolated. But this contradicts the Baire theorem. What is wrong?

5. Prove that the union of two sparse (meager) sets is a sparse set.

6. Is sparseness of a set $X \subset M$ a relative or an absolute property?

7. For a metric space (M, d) and $X \subset M$, we call the set X *dense (in M)* if for every ball $B \subset M$, $X \cap B \neq \emptyset$. Is it true that the complement of a sparse (meager) set is a dense set?

8. Is it true that the complement of a dense set is a sparse (meager) set?

9. Is the intersection of two dense sets a dense set?

10. (HW, 4 pts) Let $f_n(x) = x^n \colon [0, 1] \to \mathbb{R}$, $n = 1, 2, \ldots$, and let f be the pointwise limit of f_n ($f(x) = 0$ for $0 \leq x < 1$ and $f(1) = 1$). Prove that the set $S = \{X \subset [0, 1] \mid X \neq \emptyset, f_n \rightrightarrows f \text{ on } X\}$ does not have maximal elements with respect to inclusion. What are the minimal elements?