Tutorial 2, October 10, 2019

1. Prove that for any field norm $\|\cdot\|$ the function $d(x, y) = \|x - y\|$ is a metric.

2. Prove that the trivial norm is a norm.

3. (HW) Prove that $\|x\| = |x|^c$, $x \in \mathbb{Q}$ and $c > 0$, is a norm if and only if $c \leq 1$.

4. In every normed field, $\|0_F\| = 0$, $\|1_F\| = \|-1_F\| = 1$, $\|x\| = \|-x\|$ and $\|1_F/x\| = 1/\|x\|$ (for $x \neq 0_F$).

5. Prove that the function $\mathrm{ord}_p(\cdot)$ is additive.

6. Prove that if one of the elements is zero then the strong triangle inequality holds.

7. Prove the product formula. Is it really an infinite product?

8. (HW) Prove that for any two coprime integers $m, n \in \mathbb{Z}$ there are integers a, b such that $am + bn = 1$.

9. (HW) Let (M, d) be a metric space, $a \in M$, and $r > 0$ be real. Prove that every ball $B(a, r) = \{x \in M \mid d(x, a) < r\}$ is an open set.

10. Let (M, d) be a metric space and $X \subset M$. Prove that X is closed if and only if for every convergent sequence $(a_n) \subset X$, $\lim a_n = a \in X$.