Mathematical Analysis 1

Martin Klazar

(KAM MFF UK Praha)

Preliminary version of December 16, 2024. Chapters 1–8, plus a part of Chapter 9, of the fourteen planned chapters are complete.

dedicated to the memory of Jiří Matoušek (1963–2015)

Was sich überhaupt sagen lässt, lässt sich klar sagen; . . .[1]

L. Wittgenstein [41, Vorwort (Foreword)]

[1] The omitted conclusion of Wittgenstein's quote is more famous than the beginning: "Was sich überhaupt sagen lässt, lässt sich klar sagen; und wovon man nicht reden kann, darüber muss man schweigen." (Note that the complete quote rhymes.) This can be translated as "*What can be said at all, can be said clearly;* and what one cannot talk about, must be left in silence."

Introduction

This is my translation of my textbook in Czech of elementary mathematical analysis [24]. It contains fourteen chapters and one appendix. These chapters are based on the fourteen lectures in the course *Matematická analýza 1* (NMAI054) which I was teaching in Czech in School of Computer Science of Faculty of Mathematics and Physics of Charles University (in Prague) in winter and spring of 2024. I also used my fourteen lectures in the previous school year 2022/23, see

`https://kam.mff.cuni.cz/~klazar/MAI24.html` and
`https://kam.mff.cuni.cz/~klazar/MAI23.html`.

Time is flying. First time I lectured on mathematical analysis on October 5, 2004, see

`https://kam.mff.cuni.cz/%7Eklazar/MA04.html`.

Each chapter begins with a summary and contains a number of exercises illustrating topics covered. Solutions or hints to solutions of them can be found in Appendix A which takes over 11% of the textbook. The content of every chapter and every section should be clear from the title, see pages viii to ix. This text is a minimal version of the more ambitious *Mathematical Analysis 1*$^+$ which is in preparation; so far only in Czech but I am switching to English. It will contain six additional chapters and three additional appendices.

I dedicate my textbook to the memory of Jiří Matoušek who was my colleague in the Department of Applied Mathematics of MFF UK. He was one of the greatest contemporary Czech mathematicians and computer scientists. With enthusiasm he had prepared lectures for the analysis course in 2014/15, but the fateful illness did not allow him to deliver them.

Praha and Louny, August to December 2024 Martin Klazar

Some highlights

Many lecture notes, textbooks and monographs on elementary mathematical analysis exist, both on paper and electronic. I cite only [2, 5, 9, 15, 19, 20, 21, 26, 30, 33, 35, 39, 42]. Why to add another text? For example, did not already *Nicolas Bourbaki (?–?)* write it all in [5] (I consulted it in the form of [6])? During twenty years of teaching mathematical analysis I gradually realized that there is surprisingly much room to say things better. Below I list some points of interest in this text.

1 Paradoxes. Real numbers. In Definition 1.2.5 we attempt to introduce sets. Definition 1.2.7 describes four kinds of lists of elements used to define sets. Definition 1.2.23 of ordered k-tuples is better than the standard one, see Exercise 1.2.24. Definitions 1.3.2 and 1.3.3 treat functions $f\colon A \to B$ in this form common in mathematical analysis. In Theorem 1.5.12 we show that fractions form an ordered field. In Theorem 1.5.19 we prove by infinite descend that $\sqrt{2}$ is irrational. Theorem 1.6.12 is an urtheorem of theorems on limits of monotone sequences. In Theorem 1.6.14 we prove completeness of $\mathbb{R}$. In Corollary 1.7.17 we deduce uncountability of $\mathbb{R}$ from Cantor's Theorem 1.7.7 which says that no set can be mapped onto its power set.

2 Existence of limits. In Theorem 2.1.8 we show that except for the existence of (additive and multiplicative) inverses and the axiom of shift, the arithmetic of infinities in $\mathbb{R}^*$ satisfies all other axioms of an ordered field. Theorem 2.1.9 is a related result concerning division. One can make more of subsequences than it seems. In Theorem 2.2.5 we show that a sequence has no limit iff it has two subsequences with different limits, and that the negation of $\lim a_n = A$ is equivalent to the existence of a subsequence with limit different from A. In some proofs these equivalences are useful. In Corollaries 2.3.4 and 2.3.10 we give robust forms of theorems on limits of monotone and quasi-monotone sequences. Theorem 2.3.25 is known as Fekete's lemma. In Exercises 2.3.26–2.3.29 we present some applications of it in extremal combinatorics.

3 Arithmetic of limits. AK series. We prove in detail Theorem 3.1.2 on the arithmetic of limits of sequences, including infinite limits. Propositions 3.1.4 and 3.1.6, whose proofs are left as exercises, supplement the theorem. Our Theorem 3.3.1 on limits and ordering is stronger than the usual version. Section 3.5 introduces AK (absolutely convergent) series. An AK series is a map $r\colon X \to \mathbb{R}$, defined on an at most countable set X, such that for some $c \ge 0$ for every finite $Y \subset X$ we have $\sum_{x \in Y} |r(x)| \le c$. The sum of r is then for finite X the usual finite sum, and for infinite X the limit $\lim_{n\to\infty} \sum_{i=1}^{n} r(f(i))$ where $f\colon \mathbb{N} \to X$ is any bijection. In Theorems 3.5.3 and 3.5.6 we show that these sums are commutative and associative. AK series form a (proper) class $\mathfrak{S}$. We introduce addition and multiplication on them and in Theorem 3.5.18 show that these operations endow $\mathfrak{T} = \mathfrak{S}/{\sim}$ with the structure of a semiring. The equivalence $\sim$ of two AK series $r\colon X \to \mathbb{R}$ and $s\colon Y \to \mathbb{R}$ means that there is a bijection $f\colon X \to Y$ such that for every $x \in X$ we have $r(x) = s(f(x))$.

4 Infinite series. Elementary functions. Section 4.1 is devoted to classical infinite series. Proposition 4.1.5 shows that the sum of any series with only finitely many summands of one sign cannot be changed by reordering. Proposition 4.1.14 determines when it is possible to reorder a series to sum to $\pm\infty$. Proposition 4.1.16 considers reorderings with no sum. We prove Riemann's Theorem 4.1.17 on series that can be reordered to have any sum. Definition 4.2.6 introduces useful notation for functions. Here we work within the set $\mathcal{R}$ of functions of the type $f\colon M \to \mathbb{R}$, where $M \subset \mathbb{R}$, and use several operations on $\mathcal{R}$: addition $f + g$, multiplication $f \cdot g$, division f/g, restriction $f \mid X$, composition $f(g)$ and inverting f^{-1}; the last operation is defined only for injective f. Differentiation is introduced in lecture 7. In Theorem 4.3.4 we prove by means of AK series the exponential identity $\mathrm{e}^{x+y} = \mathrm{e}^x \cdot \mathrm{e}^y$. The runner Theorem 4.3.19 on the geometry of sine and cosine will be proven in *MA 1*$^+$. Precise Definition 4.4.5 of Elementary Functions is given. Definitions 4.5.1 and 4.5.6 introduce in non-standard ways polynomials and rational functions. In Propositions 4.5.2 and 4.5.11 we obtain for these functions canonical forms.

5 Limits of functions. Asymptotic notation. We recommend to the reader Theorem 5.3.1 on limits of monotone functions and Theorem 5.3.7 on the interplay of limits of functions and the linear order $(\mathbb{R}^*, <)$. Theorem 5.4.1 on limits of composite functions is an equivalence, which is stronger than the usual implication given in the literature. In Section 5.5 on asymptotic notation $(O, o, \sim, \dots)$ we also explain asymptotic expansions of functions, and give three examples of such expansions. Do you know what is the asymptotic expansion, for $n \to +\infty$, of the probability that the random graph with n vertices is connected? The initial term 1 is well known, but we describe all remaining terms of this asymptotic expansion.

6 Continuous functions. We state Blumberg's Theorem 6.1.12 which says that every function from $\mathbb{R}$ to $\mathbb{R}$ has a continuous restriction to a set dense in $\mathbb{R}$. We prove it in *MA 1*$^+$. Here we prove Theorem 6.2.3 which says that the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ is in bijection with $\mathbb{R}$. In Section 6.4 we discuss real compact sets. Theorem 6.5.6 says that every uniformly continuous function has a (unique) continuous extension to the closure of its definition domain. We prove Theorem 6.6.3 on continuity of power series. Theorem 6.6.11 on continuity of inverses is a climax of the chapter. It says that if a function $f\colon M \to \mathbb{R}$, $M \subset \mathbb{R}$, is injective and continuous then the inverse f^{-1} is continuous in five cases: if (i) M is compact, (ii) M is an interval, (iii) M is open, (iv) M is closed and f is monotone and (v) $M \subset (a, b)$ is dense in (a, b) and f is monotone and uniformly continuous.

7 Derivatives. What set can be the definition domain of a differentiated function? The literature, for example [35], prefers intervals. But for example [39] allows any definition domain. We adopt the latter approach. Then it is easy to get in Theorem 7.1.25 an example of a discontinuous derivative. In Theorem 7.1.8 we give for arbitrary definition domain a version of the well known criterion of local extremes. Definition 7.2.7 formalizes the common but never precised intuitive understanding of a tangent line as a limit of secants. Theorem 7.2.11 shows how to get a tangent line in a point lying outside the graph of a function. In Sections 7.3 and 7.4 we describe locally and globally interplays between derivatives on one side and arithmetic operations, compositions and inverses on the other side. In Theorem 7.5.1 we differentiate with the help of power series the exponential function, sine and cosine. In the final Theorem 7.6.3 we prove that a certain subset of Elementary Functions, so called Simple Elementary Functions, is closed to derivatives. The general case of Elementary Functions is currently left as Problem 7.6.1.

8 Applications of mean value theorems. In Section 8.1 we give three classical theorems of Rolle, Lagrange, and Cauchy on mean values of functions. In Section 8.2 we show by means of Rolle's theorem that the sequence $(\log n)$ is not P-recurrent; it satisfies no linear recurrence with polynomial coefficients. In Sections 8.3 and 8.4 we prove in two ways with the help of Lagrange's theorem that real transcendental numbers exist. Section 8.5 contains classical results on monotonicity of functions and l'Hospital rules. We made an effort to present them in fresh way. In Section 8.6 we introduce higher-order derivatives and treat the relation of f'' to convexity and concavity of f. In Theorem 8.6.9 we prove the classical result relating convexity and concavity to one-sided derivatives for general definition domains. In the last Section 8.7 *Drawing the graph of a function* we describe in 12 steps how to determine main geometric features of the graph of f; in step 0 we check if f is an elementary, or even a simple elementary, function. We work out in detail three examples: $\operatorname{sgn} x$, $\tan x$ and $\arcsin\left(\frac{2x}{1+x^2}\right)$. Drawing the graphs of the Riemann function $r(x)$ and the function x^x is left to exercises.

9 Taylor expansions. Primitives

Contents

Chapter 1

Paradoxes. Real numbers

This chapter is based on the lecture

`https://kam.mff.cuni.cz/~klazar/MAI24_pred1.pdf`

I gave on February 22, 2024. Section 1.1 presents two paradoxes concerning infinite sums. In Section 1.2 we give a survey of logical and set-theoretic notation; besides other we define ordered pairs and ordered triples. In Section 1.3 we introduce functions and list their basic types and operations with them. Section 1.4 is devoted to linear orders and to suprema and infima. In Section 1.5 we define rational numbers and ordered fields. Theorem 1.5.12 shows that rational numbers form an ordered field. In Theorem 1.5.19 we prove that in fractions the equation $x^2 = 2$ is unsolvable. By Corollary 1.5.21 the linear order of fractions is not complete, it contains nonempty and from above bounded sets lacking suprema. Section 1.6 introduces Cantor's real numbers. Important are Definition 1.6.3 of $\mathbb{R}$, Theorem 1.6.12 that is an urtheorem of theorems on limits of monotone sequences, and completeness of $\mathbb{R}$ in Theorem 1.6.14. In Section 1.7 we define finite, infinite, countable and uncountable sets. In Theorem 1.7.5 we prove that the set of fractions is countable. By Cantor's Theorem 1.7.7 no set can be mapped onto its power set. The Corollary 1.7.17 is that $\mathbb{R}$ is uncountable. Definition 1.7.13 and Theorem 1.7.16 (not proven here) describe decimal expansions of real numbers.

1.1 Two paradoxes

• *Mathematical analysis analyzes what?* Infinite operations and processes. Infinity is bound to produce paradoxes and we show two. Clearly,

$$S = 1 - 1 + \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{3} + \cdots + \tfrac{1}{n} - \tfrac{1}{n} + \cdots = 0$$

because the sequence of partial sums 1, $1-1=0$, $1-1+\frac{1}{2}=\frac{1}{2}$, $1-1+\frac{1}{2}-\frac{1}{2}=0$, $\dots$ is $(1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \dots)$ and goes to 0. By reordering the summands in S we change the sum: $S = 1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} - \frac{1}{n} + \cdots > 0$ because $\frac{1}{2n-1} + \frac{1}{2n} - \frac{1}{n} = \frac{1}{2n(2n-1)} > 0$.

Exercise 1.1.1 *Why after the reordering of S the sum is positive?*

Thus the definition of the infinite sum $S = a_1 + a_2 + \ldots$ as the limit $S = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$ of the sequence of partial sums $a_1 + a_2 + \cdots + a_n$, for limits of real sequences see Definition 2.1.15, is *not* very good because — in contrast with finite sums — the sum may depend on the order of summands. In Section 3.5 we introduce infinite sums that are, like finite sums, commutative and associative.

There is also a strange infinite table with entries -1, 0 and 1. On the main diagonal it has 1s, above it -1s and elsewhere 0s:

1	-1	0	0	0	...	$\sum = 0$
0	1	-1	0	0	...	$\sum = 0$
0	0	1	-1	0	...	$\sum = 0$
0	0	0	1	-1	...	$\sum = 0$
0	0	0	0	1	...	$\sum = 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\sum = 1$	$\sum = 0$	$\sum = 0$	$\sum = 0$	$\sum = 0$	...	$\sum = 1 \setminus 0$

The total sum of entries is 0 when we sum rows first, but it is 1 when we sum columns first!

Exercise 1.1.2 *With nonnegative entries this does not happen. Then the total sum, obtained by summing rows first and then by summing columns first, always comes out the same. It may be $+\infty$, though.*

We return to infinite sums in Sections 3.5 and 4.1.

1.2 Logical and set-theoretic notation

Exercise 1.2.1 *Can you pronounce and write letters*

$$\alpha,\ \beta,\ \Gamma,\ \gamma,\ \Delta,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \Theta,\ \theta,\ \vartheta,\ \iota,\ \kappa,\ \Lambda,\ \lambda,\ \mu,\ \nu,\ \Xi,\ \xi,\ o,\ \Pi,\ \pi,\ \rho,\ \Sigma,\ \sigma,$$

$$\tau,\ \Upsilon,\ \upsilon,\ \Phi,\ \phi,\ \varphi,\ \chi,\ \Psi,\ \psi,\ \Omega \text{ and } \omega$$

in the Greek alphabet?

Exercise 1.2.2 *Can you recognize Latin letters*

$$\mathfrak{a},\ \mathfrak{A},\ \mathfrak{b},\ \mathfrak{B},\ \mathfrak{c},\ \mathfrak{C},\ \mathfrak{d},\ \mathfrak{D},\ \mathfrak{e},\ \mathfrak{E},\ \mathfrak{f},\ \mathfrak{F},\ \mathfrak{g},\ \mathfrak{G},\ \mathfrak{h},\ \mathfrak{H},\ \mathfrak{i},\ \mathfrak{I},\ \mathfrak{j},\ \mathfrak{J},\ \mathfrak{k},\ \mathfrak{K},\ \mathfrak{l},\ \mathfrak{L},\ \mathfrak{m},\ \mathfrak{M},\ \mathfrak{n},\ \mathfrak{N}$$

$$\mathfrak{o},\ \mathfrak{O},\ \mathfrak{p},\ \mathfrak{P},\ \mathfrak{q},\ \mathfrak{Q},\ \mathfrak{r},\ \mathfrak{R},\ \mathfrak{s},\ \mathfrak{S},\ \mathfrak{t},\ \mathfrak{T},\ \mathfrak{u},\ \mathfrak{U},\ \mathfrak{v},\ \mathfrak{V},\ \mathfrak{w},\ \mathfrak{W},\ \mathfrak{x},\ \mathfrak{X},\ \mathfrak{y},\ \mathfrak{Y},\ \mathfrak{z} \text{ and } \mathfrak{Z}$$

in the Fraktur hand?

These two kinds of alphabets are used in mathematical texts. The Hebrew letter aleph $\aleph$ is used for denoting cardinalities (generalized numbers of elements) of sets. Also some other Hebrew letters and some Cyrillic letters appear in mathematical notation but we stop here.

The definitoric equality $\equiv$ is used in expressions like $a \equiv b$ to define the new symbol a by the already known expression b. Sometimes a and b may exchange their roles. Elsewhere we encounter symbols $:=$ and $=:$ (or other) serving this purpose. We write iff to abbreviate "if and only if".

• *Logical notation.* Let φ, ψ, θ, $\dots$ be propositions, statements with unique truth value either truth T or falsity F (see [36]). We combine them by means of logical connectives $\varphi \vee \psi$ (or, disjunction), $\varphi \wedge \psi$ (and, conjunction), $\varphi \Rightarrow \psi$ (if ... then ..., implication), $\varphi \iff \psi$ (equivalence) and $\neg\varphi$ (negation). Truth tables of these connectives are well known. For example, for any truth values of φ and ψ, both composite propositions in

$$\neg(\varphi \vee \psi) \iff \neg\varphi \wedge \neg\psi \ \text{ and } \ \neg(\varphi \wedge \psi) \iff \neg\varphi \vee \neg\psi$$

hold, they always have the truth value T. We say that they are tautologies. We write composite propositions with the help of brackets. The convention is that $\neg$ bounds more strongly than $\vee$ and $\wedge$, and that these bound more strongly than $\Rightarrow$ and $\iff$. The corresponding brackets may be then omitted.

Exercise 1.2.3 *Show that* $(\varphi \Rightarrow \psi) \iff (\neg\psi \Rightarrow \neg\varphi)$ *is a tautology.*

Suppose that $\varphi(x)$ is a propositional form with the variable x. Replacing in it all occurrences of x with an arbitrary, but always the same, element a from the domain of $\varphi(x)$ we get the proposition $\varphi(a)$. The domain of $\varphi(x)$ is the collection of objects a for which one can decide if the proposition $\varphi(a)$ holds. The expressions

$$\forall\, x\, \big(\varphi(x)\big) \ \text{ and } \ \exists\, x\, \big(\varphi(x)\big)$$

use the general quantifier $\forall$ and the existential quantifier $\exists$. They respectively mean that for every element a in the domain of $\varphi(x)$ the proposition $\varphi(a)$ holds, and that in the domain there is at least one element a such that $\varphi(a)$ holds.

Exercise 1.2.4 *For any propositional form* $\varphi(x)$*, both propositions*

$$\neg\exists\, x\, \big(\varphi(x)\big) \iff \forall\, x\, \big(\neg\varphi(x)\big) \ \text{ and } \ \neg\forall\, x\, \big(\varphi(x)\big) \iff \exists\, x\, \big(\neg\varphi(x)\big)$$

are true.

We discuss truth in more detail in *MA 1*$^+$. We often omit $\forall$ and understand it implicitly. For example, in the domain of natural numbers the expression $m + n = n + m$ really means that $\forall\, m\, \big(\forall\, n\, \big(m + n = n + m\big)\big)$.

• *On sets.* Sets are a main tool of rigorous mathematics. To define them sensibly is hard. Here is our take on it.

Definition 1.2.5 (set) *A set x is a clearly defined collection of other sets y, called elements of x, that is an element of another set. We write $y \in x$, respectively $y \notin x$, to denote that the set y is, respectively is not, an element of the set x. By writing $x \ni y$, respectively $x \not\ni y$, we say the same. The empty set is the unique set with no elements; it is denoted by $\emptyset$. We denote sets by small and capital letters of the Latin alphabet, for example by x, y, z, $\dots$ or by A, B, C, $\dots$ or by other letters.*

Isn't it a circular definition? We say what is a set by referring to sets. To dispel possible misunderstanding we expand on the definition.

We observe and use a mathematical universe which is inhabited by abstract entities called sets. They mutually relate by two binary relations, the *membership* $\in$ and the *equality* $=$. We postulate rules for these relations in axioms of set theory and mathematical logic. The rules for $=$ are straightforward but the rules for $\in$ are less intuitive. To determine the latter we take inspiration in properties of idealized collections of physical objects. The primary purpose of sets in mathematics is *not* to describe and manipulate collections of mathematical objects like numbers and functions, but rather to build these objects, *to build the whole mathematics from sets*. Sets are viewed best as purely mathematical constructs. It is useful to populate the mathematical universe beside sets also by related entities called classes, see Definition 1.2.14 below. A class is also a clearly defined collection of sets, but unlike a set it need not be an element of another class or set. We say more about sets and classes in *MA 1*$^+$. For more information on them we recommend to the reader the books [12, 32] and the article [3].

Exercise 1.2.6 *Is there a set A such that $A \in A$?*

• *Defining sets by lists.* Any finite set can be, in theory, defined in finite time by listing its elements. But also some infinite sets can be determined by "lists" of their elements. We explain this matter in the next definition.

Definition 1.2.7 (four kinds of lists of elements) (a) *By a list of elements of a nonempty finite set we mean any finite expression like*

$$\{b,\, a,\, a,\, c\}\,.$$

The names (which may be lists of elements themselves) of all distinct elements of the set, here a, b and c, are given in some order, are separated by commas and are enclosed in curly brackets. The names may be repeated and their order does not matter.

(b) *A list of elements of a nonempty finite set with (ellipsis) $\dots$ is any finite expression of the form*

$$\{p,\, q,\, \dots,\, r\}\,.$$

Here p, q and r are names of three usually distinct elements of the set and "p, q, $\dots$, r" indicates, by some implicit rule, all elements of the finite set.

(c) *An* <u>*elliptical list of elements*</u> *of an infinite set is any finite expression of the form*

$$\{p,\ q,\ \dots\}\,.$$

Here p and q are names of two distinct elements of the set and "$p,\ q,\ \dots$" indicates, by some implicit rule, all elements of the infinite set.

(d) *Recall that the empty set is denoted by the symbol $\emptyset$.*

(a) For example, both lists of elements $\{3,1,5\}$ and $\{1,5,1,3,5,5\}$ denote the set with three (distinct) elements 1, 3 and 5. (b) For example, $\{2,3,\dots,11\}$ denotes the set of ten natural numbers from 2 to 11, thus

$$\{2,\ 3,\ \dots,\ 11\} = \{2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11\}\,.$$

We write "usually" not to exclude the definition $M \equiv \{p,q,\dots,r\}$ when M has less the three elements. (c) For example, $\{1,2,\dots\}$ means the set $\mathbb{N}$ of natural numbers and $\{-3,-2,\dots\}$ means the set of integers larger than -4. (d) Thus $\emptyset = \{\,\}$. In each case the meaning of the ellipsis $\dots$ has to be inferred from the context. It should be straightforward, but misunderstanding cannot be completely ruled out — L. Wittgenstein taught us that it is in the nature of language.

Another example of a list of elements of the first kind (without $\dots$) is

$$\{a,\ b,\ 2,\ b,\ \{\emptyset,\ \{\emptyset\}\},\ \{a\}\}\,.$$

Exercise 1.2.8 *How many elements does this set have?*

• *Hereditarily finite sets.* The situation with describing finite sets by lists of elements is more complicated than that. If we want a complete description of a set which one can go through in finite time then it does not suffice that the set is finite. Also every element of it has to be finite, as well as every element of its every element, and so on. The set $X = \{x_1,x_2,x_3\}$ is not completely determined until we know which equalities $x_1 = x_2$, $x_1 = x_3$ and $x_2 = x_3$ hold. Else we do not know how many elements X has. In set theory (i.e., mathematics) equalities $x = y$ are decided by the *axiom of extensionality.* It says that two sets are equal iff they have the same elements. We are able to go through these elements in finite time only if there are finitely many of them. Thus we arrive at the following definition. In it we understand finiteness intuitively, we define it precisely later.

Definition 1.2.9 (hereditary finiteness) *A set x is* <u>*hereditarily finite*</u>*, abbreviated* HF*, if for every $n \in \{0,1,\dots\}$ and every chain of sets*

$$x_n \in x_{n-1} \in \cdots \in x_0 = x$$

the set x_n is finite.

An example of an HF set is the number four in the set-theoretic form:

$$4 = \{\emptyset,\ \{\emptyset\},\ \{\emptyset,\ \{\emptyset\}\},\ \{\emptyset,\ \{\emptyset\},\ \{\emptyset,\ \{\emptyset\}\}\}\}\,.$$

Another HF set is $\{\{\{\{\{\{\emptyset\}\}\}\}\}\}$. The set $\{\{0, 1, \dots\}\}$ has just one element and is finite but is not HF. The *axiom of foundation*, which we state in *MA 1*$^+$, ensures that every chain of sets $x_0 \ni x_1 \ni \dots$ is finite.

Exercise 1.2.10 *Let $x_0 \ni x_1 \ni \cdots \ni x_n$ be a maximal chain of sets nested with respect to $\in$ (i.e., it cannot be prolonged). What is x_n?*

In *MA 1*$^+$ we show that HF sets are exactly the sets whose so called *lists of elements of elements* are finite.

• *Sets defined by properties.* Another method to define sets is via properties of their elements. The technical term for it is the *axiom schema of comprehension.*

Definition 1.2.11 (comprehension) *The comprehension definition of a set M has the form*

$$M \equiv \{x \in N :\ \varphi(x)\},\ \ \textit{respectively}\ \ M \equiv \{x :\ \varphi(x)\}\,,$$

where N is an already defined set and $\varphi(x)$ is a propositional form. Thus M consists of the sets x in N such that $\varphi(x)$ holds, respectively of the sets x in the domain of $\varphi(x)$ such that $\varphi(x)$ holds.

In formalized set theory properties and propositional forms are replaced with formulas. It is assumed that every element of N belong to the domain of $\varphi(x)$. Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of natural numbers. Then, for example, the set

$$M \equiv \{n \in \mathbb{N} :\ \exists\, m \in \mathbb{N}\, \big(n = 2 \cdot m\big)\}$$

contains exactly all even natural numbers. But where did $\mathbb{N}$ come from? We return to this question in *MA 1*$^+$.

Exercise 1.2.12 *Define by comprehension the set $\mathbb{P}$ of prime numbers.*

The reader probably knows that comprehension definitions of sets of the second kind are in general problematic. Not every propositional form $\varphi(x)$ has a clearly determined domain. In the definition of the first kind an already defined set N serves as a de facto domain of $\varphi(x)$. But in the definition $M \equiv \{x :\ \varphi(x)\}$ one takes x from anywhere. As we now remind, this may lead to contradictions.

• *Russel's paradox, classes,* GB*,* ZF *and* ZFC. The British mathematician and philosopher *Bertrand Russel (1872–1970)* pointed out that the set definition

$$M \equiv \{x :\ x \notin x\}\,,$$

where the domain of the propositional form $x \notin x$ is the universe of all sets, leads to a contradiction.

Exercise 1.2.13 *How? What contradiction?*

One way how to block Russel's paradox is to say that the above collection of sets M is a class, a very large collection of sets which is not an element of anything.

Definition 1.2.14 (class) *A class is a clearly defined collection of sets. Classes are denoted by capital Latin letters in the Fracture hand, for example* $\mathfrak{A}, \mathfrak{B}, \mathfrak{S}, \dots$ *(see Exercise 1.2.2). Like sets, classes mutually relate by* $\in$ *and* $=$*. Every set is a class and every class that is an element of a class is a set. A class that is not an element of another class is called proper class and is not a set.*

Set theory based on sets and classes is denoted by the acronym GB, after the Austrian-American mathematician *Kurt Gödel (1906–1978)*, who was the greatest mathematical logician of the 20th century and was born and grew up in Brno (Austria-Hungary, today Czechia, oops Moravia), and the Swiss mathematician *Paul Bernays (1888–1977)*.

Another way that blocks Russel's paradox, without classes by means of only sets, is the carefully built set theory denoted by the acronym ZF, after the German mathematician *Ernst Zermelo (1871–1953)* and the German-Israeli mathematician *Adolf Fraenkel (1891–1965)*. Finally, the acronym ZFC, which is used most often, refers to the ZF set theory with the axiom of choice added. In *MA* 1^+ we explain why this set-theoretic axiom is important, and there we write on GB, ZF and ZFC set theories more.

But what the hell is all this doing in an introductory text on analysis, some may ask. Classes were forced upon us by the development in Section 3.5. There we cannot keep silent about classes without being non-rigorous. Elsewhere in this text we work only with sets.

• *Relations between sets.* A set A is a subset of a set B, written $A \subset B$, if any $x \in A$ is also an element of B. Sets A and B are disjoint if they have no common element. Sets A and B are equal, written $A = B$, iff $\forall x\,\big(x \in A \iff x \in B\big)$. This is the axiom of extensionality.

Exercise 1.2.15 *The equivalence* $A = B \iff A \subset B \wedge B \subset A$ *holds.*

• *Set operations.* Let A and B be sets. Their union $A \cup B \equiv \{x:\ x \in A \vee x \in B\}$ and their intersection $A \cap B \equiv \{x \in A:\ x \in B\}$. The sum (of the set A) $\bigcup A \equiv \{x:\ \exists\, b \in A\,\big(x \in b\big)\}$. The intersection (of the set A) $\bigcap A \equiv \{x:\ \forall\, b \in A\,\big(x \in b\big)\}$, for $A \neq \emptyset$. The difference $A \setminus B \equiv \{x \in A:\ x \notin B\}$. The power set of A is $\mathcal{P}(A) \equiv \{X:\ X \subset A\}$. As we know from the passage about Russel's paradox, in formalized set theory the only correct definitions of these are for $A \cap B$ and for $A \setminus B$. Definitions of the form $X \equiv \{x:\ \dots\}$ are formally not allowed because it is forbidden to take x from anywhere, it can only be taken from an already defined set. In formalized set theory the "definitions" $X \equiv \{x:\ \dots\}$ are replaced by axioms. By them one introduces the union, the sum and the power set. By $|X|$ $(\in \mathbb{N}_0)$ we denote the number of elements of a finite set X.

Exercise 1.2.16 *Sets A a B are disjoint iff $A \cap B = \emptyset$.*

Exercise 1.2.17 *What is $\bigcap \emptyset$?*

Exercise 1.2.18 *Let A and B be finite sets. Is it always true that $|A \setminus B| = |A| - |B|$?*

Exercise 1.2.19 *What is $\mathcal{P}(\emptyset)$? If $n \in \{0, 1 \dots\}$, what is $|\mathcal{P}(\{1, 2, \dots, n\})|$?*

Exercise 1.2.20 (De Morgan formulas) *Let A and $B \neq \emptyset$ be sets. Then $A \setminus \bigcup B = \bigcap B_0$ and $A \setminus \bigcap B = \bigcup B_0$, where $B_0 \equiv \{A \setminus C : C \in B\}$.*

Augustus De Morgan (1806–1871) was a British mathematician.

• *Pairs and triples.* The next definition of ordered pair of two sets is due to the Polish mathematician *Kazimierz Kuratowski (1896–1980).*

Definition 1.2.21 (ordered pairs) *Let A and B be sets. The set*

$$(A,\, B) \equiv \{\{B,\, A\},\, \{A\}\}$$

is the (ordered) pair of A and B.

Exercise 1.2.22 *Using the axiom of extensionality prove the equivalence*

$$(A, B) = (C,\, D) \iff A = C \wedge B = D\ .$$

We define the ordered triple of sets A, B and C as

$$(A,\, B,\, C) \equiv \{(1,\, A),\, (2,\, B),\, (3,\, C)\}\ ,$$

and *not* as $(A, (B, C))$ or $((A, B), C)$, as one can often read, even in textbooks of set theory! Why? With the definition

$$(A,\, B,\, C) \equiv (A,\, (B,\, C))$$

it is not clear if the set (A, B, C) is an ordered triple of the sets A, B and C or an ordered pair of the sets A and (B, C).

Definition 1.2.23 (ordered k-tuples) *Let $k \in \mathbb{N}$ and let $A_1, \dots, A_k$ be sets. Their ordered k-tuple is the set*

$$\langle A_1,\, \dots,\, A_k\rangle \equiv \{(1,\, A_1),\, (2,\, A_2),\, \dots,\, (k,\, A_k)\}\ .$$

For $k = 3$ we have that $\langle A_1, A_2, A_3\rangle = (A_1, A_2, A_3)$. For $k = 2$ the sets $\langle A, B\rangle$ and (A, B) are different but function in the same way. We shall use the latter.

Exercise 1.2.24 *Let $k, l \in \mathbb{N}$, $A \equiv \langle A_1, \dots, A_k\rangle$ be an ordered k-tuple, $B \equiv \langle B_1, \dots, B_l\rangle$ be an ordered l-tuple and let $A = B$. Prove that then $k = l$ and for every $i \in \{1, 2, \dots, k\}$ it holds that $A_i = B_i$.*

Standard ordered k-tuples are often defined as iterated pairs, that is, for example as

$$(A_1,\, (A_2,\, (\dots,\, (A_{k-1},\, A_k)\dots)))\ .$$

They obviously do not enjoy the property stated in Exercise 1.2.24.

1.3 Functions and relations

We treat functions set-theoretically. (So no "a function is a rule that …".) We compose any two functions, as we need it for the use of elementary functions in Chapter 4.

• *Functions and congruence of functions.* Let A and B be sets. Their Cartesian product is the set

$$A \times B \equiv \{(a,\, b):\ a \in A,\, b \in B\}\,.$$

Any set $C \subset A \times B$ is then a (binary) relation between A and B. Instead of $(a, b) \in C$ we write aCb, for example $2 < 5$. When $A = B$, we speak of a relation on A.

Definition 1.3.1 (functional relation) *A relation C between A and B is functional if for every $a \in A$ there is exactly one $b \in B$ such that aCb.*

Definition 1.3.2 (functions and graphs) *A function (a map) from a set A to a set B is any ordered triple (A, B, f) such that f is a functional relation between A and B. We denote it by the symbol*

$$f\colon A \to B\,.$$

For any $a \in A$ we denote by $f(a)$ the unique $b \in B$ such that afb. We often abuse notation and denote the triple (A, B, f) simply by f; it has a good reason, see the next definition and exercise. Then we denote the third component f in (A, B, f) by G_f and call it the graph of f.

Thus $G_f = \{(x, f(x)):\ x \in A\} = f$. Instead of $f\colon A \to B$ we sometimes write also $A \ni a \mapsto f(a) \in B$. The set A is the definition domain of f and B is the range of f. We denote the definition domain of f by $M(f)$. In $f(a) = b$ the element b is the value of f in the argument a.

When some mathematical objects are considered, one should precisely describe conditions when two of these objects are regarded as identical although they may differ as sets. The relation resulting from these conditions is called congruence of the considered objects. Well known examples of congruences are isomorphisms of various algebraic and combinatorial structures. Congruence is usually an equivalence relation (see Definition 1.3.17). Congruence of functions works as follows.

Definition 1.3.3 (congruent functions) *Let (A, B, f) and (C, D, g) be functions. They are congruent, that is for practical purposes identical, if $f = g$ (as sets of ordered pairs).*

Exercise 1.3.4 *Two functions (A, B, f) and (C, D, g) are congruent $\iff$ $A = C$ and $f = g$.*

In other words, congruent functions may differ only in their ranges.

For a function $f\colon A \to B$ and any set C we define the sets

$$\begin{aligned} f[C] &\equiv \{f(a):\ a \in C \cap A\}\ (\subset B)\ \ldots \text{ the image of } C \text{ by } f \text{ and} \\ f^{-1}[C] &\equiv \{a \in A:\ f(a) \in C\}\ (\subset A)\ \ldots \text{ the preimage of } C \text{ by } f\,. \end{aligned}$$

Note that the set C is arbitrary. The set $f[A]$ is the image of f.

Exercise 1.3.5 *Is it true that $f^{-1}[f[C]] = C$ and that $f[f^{-1}[C]] = C$?*

• *Species of functions.* As we know, $\mathbb{N} = \{1, 2, \ldots\}$. We set $\mathbb{N}_0 \equiv \{0, 1, \ldots\}$. For $n \in \mathbb{N}$ let $[n] \equiv \{1, 2, \ldots, n\}$ and let $[0] \equiv \emptyset$. Let X be any set and $n \in \mathbb{N}_0$. Three important families of functions are sequences, words and operations.

$$\begin{aligned} a\colon \mathbb{N} \to X &\quad \ldots \quad a \text{ is a sequence (in } X)\,, \\ u\colon [n] \to X &\quad \ldots \quad u \text{ is a word (over } X) \text{ and} \\ o\colon X \times X \to X &\quad \ldots \quad o \text{ is a (binary) operation (on } X)\,. \end{aligned}$$

For a sequence a in X we write a_n instead of $a(n)$ and invoke it by writing $(a_n) \subset X$. A word u over X is written as $u_1 u_2 \ldots u_n$ where $u_i = u(i)$ for $i \in [n]$. For $n = 0$ we have the empty word $u = \emptyset$. The value of an operation $o((a, b)) = c$ is recorded as $a\,o\,b = c$, for example $1 + 1 = 2$. Any function $o\colon X \to X$ is called a unary operation (on X).

A function $f\colon X \to Y$ is

$$\begin{aligned} \text{injective} &\equiv \text{always } f(x) = f(x') \Rightarrow x = x'\,, \\ \text{surjective (onto)} &\equiv f[X] = Y\,, \\ \text{bijective (a bijection)} &\equiv f \text{ is onto and injective}\,, \\ \text{constant} &\equiv \exists\, c \in Y\ \forall\, x \in X\ \big(f(x) = c\big) \text{ and} \\ \text{identical} &\equiv X \subset Y \wedge \forall\, x \in X\ \big(f(x) = x\big)\,. \end{aligned}$$

In a more narrow sense we understand by an identical function on X the function $\mathrm{id}_X : X \to X$, $\mathrm{id}_X(x) = x$.

Exercise 1.3.6 *When is the identical function from X to Y bijective?*

Exercise 1.3.7 *Is it true that if two functions are congruent (in the sense of Definition 1.3.3), then they both are, or both are not, injective? Same question for surjectivity, bijectivness, constantness and identicalness.*

• *Operations on functions.* We introduce three. They are motivated by elementary functions in Chapter 4. The first operation is unary and partial (sometimes it is not defined). Let $f\colon X \to Y$ be injective. Its inverse function (its inverse) f^{-1} is the function

$$f^{-1}\colon f[X] \to X \text{ where } f^{-1}(y) = x \iff f(x) = y\,.$$

If f is not injective then f^{-1} is not defined.

Exercise 1.3.8 *The notation $f^{-1}[A]$ may be ambiguous, it may mean the preimage of A by f and also the image of A by f^{-1}. Is it a problem?*

Exercise 1.3.9 *Let f be injective. Are f and $(f^{-1})^{-1}$ congruent? Are they equal as triples? Same question for f^{-1} and $((f^{-1})^{-1})^{-1}$.*

Composition is a binary operation. For $g\colon X \to Y$ and $f\colon A \to B$ their composition $f \circ g = f(g)\colon X' \to B$ has for $x \in X'$ values $f(g)(x) = f(g(x))$ and

$$X' \equiv \{x \in X :\ g(x) \in A\}\ (= g^{-1}[A])\,.$$

In $f(g)$ we call the function g inner, and f outer.

Exercise 1.3.10 *Let f_1 and f_2, respectively g_1 and g_2, be congruent. Are then $f_1(g_1)$ and $f_2(g_2)$ congruent?*

The third operation on functions is in fact a system of unary operations. For $f\colon X \to Y$ and any set Z we define the restriction of f to Z to be the function $f \mid Z\colon X \cap Z \to Y$ with the values

$$(f \mid Z)(x) \equiv f(x), \quad x \in X \cap Z\,.$$

Then f is an extension of $f \mid Z$. It is clear that if f_1 and f_2 are congruent then for any Z the restrictions $f_1 \mid Z$ and $f_2 \mid Z$ are congruent. A function (A, B, f) is a restriction of another function (X, Z, g) if $f \subset g$, i.e. $f = g \mid A = g \mid M(f)$. If f and g are functions, X is any set and $f \mid X$ and $g \mid X$ are congruent, then we write that $f = g$ on X.

Exercise 1.3.11 *Any composition of two injections is an injection. Composition $f(g)$ of the surjections $g\colon X \to Y$ and $f\colon Y \to B$ is surjective. In general composition of two surjective maps need not be surjective.*

Exercise 1.3.12 *For any three functions f, g and h it holds that $f \circ (g \circ h) = (f \circ g) \circ h$ as triples.*

Exercise 1.3.13 *For every map $h\colon X \to Z$ there exist a set Y and functions $g\colon X \to Y$ and $f\colon Y \to Z$ such that $h = f \circ g$, g is onto and f is injective.*

Exercise 1.3.14 *$f\colon X \to Y$ is bijective iff there is a function $g\colon Y \to X$ such that $f(g)$ is id_Y and $g(f)$ is id_X.*

Exercise 1.3.15 *Which constant functions can be inverted?*

• *Sum (union) and intersection of a set system.* Let I be a set. A set system $\{A_i :\ i \in I\}$ indexed by I, denoted also by A_i, $i \in I$, is simply any function (I, Y, A) where for $i \in I$ the value $A(i)$ is denoted by the symbol A_i. Then

$$\textstyle\bigcup_{i \in I} A_i = \bigcup\{A_i :\ i \in I\} \equiv \bigcup A[I]$$

and, for $A[I] \neq \emptyset$,

$$\textstyle\bigcap_{i \in I} A_i = \bigcap\{A_i :\ i \in I\} \equiv \bigcap A[I]\,.$$

Exercise 1.3.16 *Explain the meaning of the notation* $\bigcup_{i=1}^{\infty} A_i$ *and* $\bigcap_{i=0}^{\infty} A_i$.

• *Equivalence relations.* A relation R on A is reflexive (respectively irreflexive) if always aRa (respectively never aRa). R is symmetric if always aRb implies bRa. It is transitive if always aRb and bRc imply aRc.

Definition 1.3.17 (equivalence relation) *Equivalence relation is a reflexive, symmetric and transitive relation on a set.*

Definition 1.3.18 (partition) *Let A and B be sets. A is a partition of B if the elements of A are nonempty, pairwise disjoint and $\bigcup A = B$. The elements in A are then called blocks of the partition A.*

Let R be an equivalence relation on A. For $a \in A$ we define the sets

$$[a]_R \equiv \{b \in A : \ bRa\} \quad (\subset A)$$

and call them blocks of R. It is clear that if aRb then $[a]_R = [b]_R$. We set

$$A/R \equiv \{[a]_R : \ a \in A\} .$$

Exercise 1.3.19 *If R is an equivalence relation on A then A/R is a partition of A. Elements $b, c \in A$ lie in one block of A/R iff bRc.*

Let X be a partition of Y. We define the relation Y/X on Y by

$$x\,(Y/X)\,y \equiv \exists\, Z \in X\,\big(x,\, y \in Z\big) .$$

Exercise 1.3.20 *If X is a partition of Y then Y/X is an equivalence relation on Y. Elements $x, y \in Y$ lie in one block of X iff $x(Y/X)y$.*

Exercise 1.3.21 *If R is an equivalence relation on A and B is a partition of A then*

$$A/(A/R) = R \ \text{ and } \ A/(A/B) = B .$$

1.4 Suprema and infima

Let R be a relation on A. It is trichotomic if for every $a, b \in A$ it is true that aRb or bRa or $a = b$.

• *Linear orders* belong to fundamental mathematical structures.

Definition 1.4.1 (linear order) *A linear order, shortly* LO*, is an irreflexive, transitive and trichotomic relation on a set.*

To invoke a LO on A, we use notation like $(A, <)$. For $a, b \in A$ notation $a \le b$ means that $a < b$ or $a = b$. Notation $a > b$ means that $b < a$. Similarly for $a \ge b$. We say that the relations $<$ and $>$ are *strict*, and that $\le$ and $\ge$ are *non-strict*.

Exercise 1.4.2 *If* $(A, <)$ *is a* LO *then the relations* $\leq$ *and* $\geq$ *are reflexive, transitive and trichotomic.*

Exercise 1.4.3 *In any* LO $(A, <)$ *for* $a, b \in A$ *exactly one of* $a < b$, $b < a$ *and* $a = b$ *holds.*

• *Suprema and infima.* Let $(A, <)$ be a LO and $B \subset A$. The set B is bounded from above if there is an $h \in A$ such that for every $b \in B$ we have $b \leq h$. Then h is an upper bound of B. $H(B)$ $(\subset A)$ is the set of upper bounds of B. We similarly define boundedness from below, lower bounds and the set $D(B)$ $(\subset A)$ of lower bounds of B. An element $m \in B$ is the maximum of B if $m \in H(B)$. We similarly define the minimum of B. We denote these elements as $\max(B)$ and $\min(B)$. For $B = \emptyset$ they are not defined.

Exercise 1.4.4 *Show that maxima and minima are unique.*

Exercise 1.4.5 *Any nonempty finite subset in any* LO *has both maximum and minimum.*

In the theory of real numbers suprema and infima of their sets, which we now define more generally in any LO, play important roles.

Definition 1.4.6 (sup and inf) *Let* $(A, <)$ *be a* LO *and* $B \subset A$. *Then the elements*

$$\sup(B) \equiv \min(H(B)) \ (\in A) \ \text{ and } \ \inf(B) \equiv \max(D(B)) \ (\in A)\,,$$

if they exist, are respectively called the supremum and the infimum of B.

In contrast to maxima and minima, suprema and infima may lie outside the considered set.

Exercise 1.4.7 *Suprema and infima are unique.*

Exercise 1.4.8 *Prove the following result. State and prove the analogous result for infima.*

Proposition 1.4.9 (approximating suprema) *Let* $(A, <)$ *be a* LO, $B \subset A$ *and* $c \in A$. *Then* $c = \sup(B)$ *iff for every* $b \in B$ *we have* $b \leq c$ *and for every* $a \in A$ *with* $a < c$ *there is a* $b \in B$ *such that* $a < b$.

Thus $\sup(B)$ can be approximated from below arbitrarily tightly (in the sense of $<$) by elements of B.

1.5 Rational numbers

Real numbers are built from rational ones, which we now recall. Natural numbers and integers will be constructed in *MA 1*$^+$.

- *Rational numbers (fractions).* Let

$$\mathbb{Z} \equiv \{\ldots,\ -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots\}$$

(this is a "list" of elements of a kind we did not consider in Definition 1.2.7) be the integers, with their usual operations of addition $+$ and multiplication $\cdot$, the usual neutral elements 0 a 1 and the usual LO $<$. We set

$$Z \equiv \left\{\tfrac{m}{n}:\ m \in \mathbb{Z},\, n \in \mathbb{Z} \setminus \{0\}\right\},$$

where $\frac{m}{n}$ denotes the ordered pair (m, n). The elements of Z are so called protofractions. The congruence relation $\sim$ on Z is defined by

$$\frac{k}{l} \sim \frac{m}{n} \equiv kn = lm\ .$$

Exercise 1.5.1 *Show that $\sim$ is an equivalence relation on Z.*

Definition 1.5.2 (rational numbers) *We set $\mathbb{Q} \equiv Z/\!\sim$. The blocks $[\frac{m}{n}]_\sim$ in $\mathbb{Q}$ are called rational numbers or fractions. We usually write them simply as $\frac{m}{n}$ or m/n.*

Due to the embedding $\mathbb{Z} \ni m \mapsto [\frac{m}{1}]_\sim \in \mathbb{Q}$ we view the integers as a subset, actually a subring, of fractions.

For the next exercise recall that $m, n \in \mathbb{Z}$ are coprime, if their only common divisors are -1 and 1. A protofraction $\frac{m}{n}$ is in lowest terms if $n > 0$ and m and n are coprime. We denote the set of such protofractions by Z_{z} $(\subset Z)$.

Exercise 1.5.3 (representing fractions) *There exists exactly one bijection $f\colon \mathbb{Q} \to Z_{\mathrm{z}}$ such that always $f([m/n]_\sim) \in [m/n]_\sim$.*

Thus fractions have a (canonical) system of distinct representatives by protofractions in lowest terms.

- *Ordered fields.* We begin with the definition of a field.

Definition 1.5.4 (field) *A field X_{F} is any structure*

$$X_{\mathrm{F}} \equiv \langle X,\, 0_X,\, 1_X,\, +,\, \cdot \rangle$$

such that X is a set (or a class), $0_X, 1_X \in X$ are distinct elements, $+$ and $\cdot$ are operations on X (the "addition" and "multiplication") and that for every $\alpha, \beta, \gamma \in X$ the following axioms hold. First, $\alpha + 0_X = \alpha \cdot 1_X = \alpha$ which means *that both 0_X and 1_X is a* *neutral element* *to the respective operation. Further, it holds that*

$$\begin{aligned}
\alpha + \beta = \beta + \alpha &\quad \ldots \quad \textit{commutativity of } + \,, \\
\alpha \cdot \beta = \beta \cdot \alpha &\quad \ldots \quad \textit{commutativity of } \cdot \,, \\
(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) &\quad \ldots \quad \textit{associativity of } + \,, \\
(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma) &\quad \ldots \quad \textit{associativity of } \cdot \ \textit{ and} \\
\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma) &\quad \ldots \quad \textit{distributivity of } \cdot \textit{ to } k + \,.
\end{aligned}$$

Finally, it holds that ($\alpha, \beta, \gamma \in X$)

$$\begin{aligned}
\forall\, \alpha\ \exists\, \beta\ \big(\alpha + \beta = 0_X\big) &\quad \ldots \quad \underline{\textit{additive inverses}}\ \textit{always exist and} \\
\forall\, \alpha \neq 0_X\ \exists\, \gamma\ \big(\alpha \cdot \gamma = 1_X\big) &\quad \ldots \quad \underline{\textit{multiplicative inverses}}\ \textit{almost always exist}\,.
\end{aligned}$$

The additive inverse β to α is denoted by $-\alpha$, and the multiplicative inverse γ by $\frac{1}{\alpha}$ or $1/\alpha$ or α^{-1}.

A ring is any structure $R_{\mathrm{R}} \equiv \langle R, 0_R, 1_R, +, \cdot \rangle$ satisfying all above field axioms, with the possible exception of existence of multiplicative inverses. For example, the integers form a ring but do not form a field.

Exercise 1.5.5 *Neutral elements in any field are unique. The same holds for additive and multiplicative inverses.*

Exercise 1.5.6 *Describe a field with two elements. Is it unique? Is there a field with just one element?*

Definition 1.5.7 (ordered field) *An ordered field is any structure $T_{\mathrm{OF}} \equiv \langle T, 0_T, 1_T, +, \cdot, < \rangle$ such that $\langle T, 0_T, 1_T, +, \cdot \rangle$ is a field, $(T, <)$ is a* LO *and that for every $\alpha, \beta, \gamma \in T$ two axioms of order hold, namely*

$$\begin{aligned}
\alpha < \beta \Rightarrow \alpha + \gamma < \beta + \gamma &\quad \ldots \quad \textit{the } \underline{\textit{axiom of shift}} \textit{ and} \\
\alpha > 0_T \wedge \beta > 0_T \Rightarrow \alpha \cdot \beta > 0_T &\quad \ldots \quad \textit{the } \underline{\textit{axiom of positivity}}\,.
\end{aligned}$$

Exercise 1.5.8 *Every ordered field is infinite.*

Definition 1.5.9 *An ordered field T_{OF} is Archimedean if every element $x \in T$ has an upper bound of the form $x \le 1_T + 1_T + \cdots + 1_T$.*

This term refers to *Archimedes of Syracuse (cca 287–cca 212 before our era).*

• *Fractions form an ordered field.* We remind the arithmetic of fractions and then prove that it makes $\mathbb{Q}$ an ordered field. We take the ordered integral domain of integers $\langle \mathbb{Z}, 0, 1, +, \cdot, < \rangle$ for granted. We set $0_{\mathbb{Q}} \equiv [0/1]_\sim$ and $1_{\mathbb{Q}} \equiv [1/1]_\sim$ (recall that $\sim$ is the congruence on the set of protofractions Z).

Exercise 1.5.10 $\frac{0}{1} \not\sim \frac{1}{1}$, *so that $0_{\mathbb{Q}} \neq 1_{\mathbb{Q}}$.*

Addition and multiplication of fractions are well known:

$$[a/b]_\sim + [c/d]_\sim \equiv [(ad+cb)/bd]_\sim \text{ and } [a/b]_\sim \cdot [c/d]_\sim \equiv [ac/bd]_\sim .$$

As for comparison, for fractions $[a/b]_\sim$ and $[c/d]_\sim$ with $b, d > 0$ (which can be assumed) we set

$$[a/b]_\sim < [c/d]_\sim \equiv ad < cb$$

where on the right side $<$ is the standard LO on $\mathbb{Z}$.

Exercise 1.5.11 *Why can we assume that the denominators b and d are positive?*

Theorem 1.5.12 ($\mathbb{Q}$ is an OF) *The structure $\mathbb{Q}_{\mathrm{OF}} \equiv \langle \mathbb{Q}, 1_\mathbb{Q}, 0_\mathbb{Q}, +, \cdot, < \rangle$ is an ordered field.*

Proof. First we check that $+$, $\cdot$ and $<$ do not depend on the representations by protofractions. Let $\frac{a}{b} \sim \frac{a'}{b'}$ and $\frac{c}{d} \sim \frac{c'}{d'}$, so that $ab' = a'b$ and $cd' = c'd$. Then $\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd} \sim \frac{a'd'+c'b'}{b'd'} = \frac{a'}{b'} + \frac{c'}{d'}$ because $(ad+cb)b'd' = a'bdd' + c'dbb' = (a'd' + c'b')bd$. Similarly $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \sim \frac{a'c'}{b'd'} = \frac{a'}{b'} \cdot \frac{c'}{d'}$ because $acb'd' = a'bc'd = a'c'bd$. Finally, we may assume (by Exercise 1.5.11) that $b, d, b', d' > 0$. Then $\frac{a}{b} < \frac{c}{d}$ iff $ad < cb$ iff $adb'd' = a'bdd' < c'dbb' = cbb'd'$ iff $a'd' < c'b'$ (since $bd, b'd' > 0$) iff $\frac{a'}{b'} < \frac{c'}{d'}$.

Thus the arithmetic operations and the comparison are defined correctly. Since $\frac{a}{b} + \frac{0}{1} = \frac{a\cdot 1 + 0 \cdot b}{b \cdot 1} = \frac{a}{b}$, the fraction $0_\mathbb{Q}$ is neutral to $+$. Similarly for $1_\mathbb{Q}$. Commutativity of addition and multiplication is immediate from commutativity of these operations in $\mathbb{Z}$. The same holds for associativity of multiplication. As for associativity of addition, we indeed have that

$$\left(\tfrac{a}{b} + \tfrac{c}{d}\right) + \tfrac{e}{f} = \tfrac{(ad+cb)f+ebd}{bdf} = \tfrac{adf+(cf+ed)b}{bdf} = \tfrac{a}{b} + \left(\tfrac{c}{d} + \tfrac{e}{f}\right)$$

where we additionally used the distributive law in $\mathbb{Z}$. As for the distributive law, it holds too because

$$\tfrac{a}{b} \cdot \left(\tfrac{c}{d} + \tfrac{e}{f}\right) = \tfrac{a(cf+ed)}{bdf} \sim \tfrac{acbf+aebd}{b^2df} = \tfrac{a}{b} \cdot \tfrac{c}{d} + \tfrac{a}{b} \cdot \tfrac{e}{f}$$

where we used also the congruence $\sim$ on Z. The additive inverse to $\frac{a}{b}$ is $\frac{-a}{b}$ because $\frac{a}{b} + \frac{-a}{b} = \frac{ab+(-a)b}{b^2} \sim \frac{0}{1}$. Similarly for multiplicative inverses: the multiplicative inverse to $\frac{a}{b} \not\sim \frac{0}{1}$, i.e. when $a \neq 0$, is $\frac{b}{a}$ because $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} \sim \frac{1}{1}$.

It remains to show that $(\mathbb{Q}, <)$ is a LO and that the two order axioms hold. Irreflexivity of $<$ is clear. Let $\frac{a}{b} < \frac{c}{d}$ and $\frac{c}{d} < \frac{e}{f}$, with $b, d, f > 0$. Thus $ad < cb$ and $cf < ed$. Multiplying the first inequality by f and the second one by b we get that $adf < edb$. Hence $af < eb$ and $\frac{a}{b} < \frac{e}{f}$. The transitivity of $<$ is proven. If $\frac{a}{b}$ and $\frac{c}{d}$ are two (proto)fractions with $b, d > 0$, then one of $ad < cb$, $ad > cb$ and $ad = cb$ holds. In the first case we have that $\frac{a}{b} < \frac{c}{d}$, in the second case that $\frac{a}{b} > \frac{c}{d}$ and in the third case that $\frac{a}{b} \sim \frac{c}{d}$. This proves that $<$ is trichotomic. Suppose that $\frac{a}{b} < \frac{c}{d}$ with $b, d > 0$, thus $ad < cb$, and that $\frac{e}{f}$ is any fraction.

Then also $\frac{a}{b} + \frac{e}{f} < \frac{c}{d} + \frac{e}{f}$: $(af + eb)df < (cf + ed)bf$ iff $adf^2 < cbf^2$ iff $ad < cb$. Finally, suppose that $\frac{0}{1} < \frac{a}{b}, \frac{c}{d}$, where $b, d > 0$. Then $0 < a, c$, so that $0 < ac$. Hence $\frac{0}{1} < \frac{ac}{bd} = \frac{a}{b} \cdot \frac{c}{d}$. □

Exercise 1.5.13 *The ordered field $\mathbb{Q}_{\mathrm{OF}}$ is Archimedean.*

• *Incompleteness of fractions.* $\mathbb{Q}_{\mathrm{OF}}$ lacks an important property which real numbers have. It is completeness.

Definition 1.5.14 (completeness) *A* LO *is complete if every nonempty and from above bounded set in it has a supremum. An ordered field is complete if its* LO *is complete.*

Exercise 1.5.15 *What is the supremum of $\emptyset$? What is the supremum of a set that is not bounded from above?*

Exercise 1.5.16 *Show that in any complete ordered field any nonempty and from below bounded set has an infimum.*

Exercise 1.5.17 *Show that every complete ordered field is Archimedean.*

It follows from the next theorem that the LO $(\mathbb{Q}, <)$ is not complete. In the proof we use the axiom of induction. In *MA 1*$^+$ we show that this axiom follows from more fundamental axioms of set theory.

Axiom 1.5.18 (axiom of induction) *In the standard linear order* $(\mathbb{N}, <)$ *every nonempty set of natural numbers has minimum.*

We show that the number $\sqrt{2}$ is irrational. But the real numbers are not yet defined; we put it in the following form.

Theorem 1.5.19 ($\sqrt{2} \notin \mathbb{Q}$) *The equation $x^2 = 2$ has no solution in $\mathbb{Q}$.*

Proof. We proceed by contradiction. Let $a, b \in \mathbb{N}$ satisfy $(a/b)^2 = 2$. So $a^2 = 2b^2$ and a, b is a solution of the equation $x^2 = 2y^2$. By the axiom of induction we can take a to be minimum one. The number a^2 is even. Hence a is even, $a = 2c$ for some $c \in \mathbb{N}$. But then

$$(2c)^2 = 2b^2 \rightsquigarrow 4c^2 = 2b^2 \rightsquigarrow b^2 = 2c^2\,.$$

We got a new solution b, c ($\in \mathbb{N}$) of $x^2 = 2y^2$. But $b < a$, contradicting the minimality of the solution a, b. □

One can generalize rational insolubility of $x^2 - 2 = 0$ as follows.

Exercise 1.5.20 *If the equation $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ satisfies that $n \in \mathbb{N}$, $n \geq 2$, $a_i \in \mathbb{Z}$, a prime number p divides every coefficient a_i but p^2 does not divide a_0, then in $\mathbb{Q}$ the equation has no solution.*

Corollary 1.5.21 (incompleteness of $\mathbb{Q}$) *The* LO $(\mathbb{Q}, <)$ *is not complete. For example, the set* $X = \{x \in \mathbb{Q} : x^2 < 2\}$ *(*$\subset \mathbb{Q}$*) is nonempty and bounded from above but* $\sup(X)$ *does not exist.*

Proof. The first two properties of X are clear as $1 \in X$ and $x < 2$ for every $x \in X$. For contradiction let $s \equiv \sup(X) \in \mathbb{Q}$. Clearly, $s > 0$. If $s^2 > 2$, there is an $r \in \mathbb{Q}$ such that $0 < r < s$ and $(s-r)^2 > 2$. But then for every $x \in X$ we have $s - r > x$. This contradicts the fact that s is the smallest upper bound of X. If $s^2 < 2$, there is a fraction $r > 0$ such that $(s+r)^2 < 2$. Thus $s + r \in X$ and we have a contradiction with the fact that s is an upper bound of X. By the trichotomy, $s^2 = 2$. But this is not possible by the previous theorem. □

Exercise 1.5.22 *Find for the fractions* r *some specific values. They are functions of* s*.*

1.6 Cantor's real numbers

We present Cantor's definition of real numbers. In it real numbers are equivalence blocks of Cauchy sequences of fractions. Each of these blocks is an uncountable set. In *MA 1*$^+$ we describe Dedekind's construction of real numbers. In it every real number is a hereditarily at most countable set. The German mathematician *Georg Cantor (1845–1918)* is worldwide famous as a creator of set theory. The French mathematician *Augustin-Louis Cauchy (1789–1857)* belongs to the founders of mathematical analysis, especially in the complex domain. The German mathematician *Richard Dedekind (1831–1916)* made in 1858 an epochal mathematical discovery: real numbers can be rigorously and relatively straightforwardly built from the natural numbers.

• *Real numbers.* A sequence $(a_n) \subset \mathbb{Q}$ is called Cauchy if for every $k \in \mathbb{N}$ there is an $n_0 \in \mathbb{N}$ such that

$$m, n \ge n_0 \Rightarrow |a_m - a_n| \le \tfrac{1}{k}\,.$$

We define the set $C \equiv \{(a_n) : (a_n) \subset \mathbb{Q} \text{ and is Cauchy}\}$. Like on Z, we define on C the congruence $\sim$. We set $(a_n) \sim (b_n) \equiv$ for every k there is an n_0 such that

$$n \ge n_0 \Rightarrow |a_n - b_n| \le \tfrac{1}{k}\,.$$

Exercise 1.6.1 *Show that* $\sim$ *is an equivalence relation on* C*.*

Exercise 1.6.2 *Prove that* $(a_n) \sim (b_n)$ *iff for every* k *there is an* n_0 *such that*

$$m, n \ge n_0 \Rightarrow |a_m - b_n| \le \tfrac{1}{k}\,.$$

Definition 1.6.3 (Cantor's reals) *We define* $\mathbb{R} \equiv C/\sim$*. We call this set of blocks of the equivalence relation* $\sim$ *the set of real numbers.*

• *Arithmetic of real numbers.* We set $0_{\mathbb{R}} \equiv [(0,0,\dots)]_\sim$ and $1_{\mathbb{R}} \equiv [(1,1,\dots)]_\sim$. These are the zero and the unity in $\mathbb{R}$. The sum and the product of real numbers $[(a_n)]_\sim$ and $[(b_n)]_\sim$ is defined, respectively, by

$$[(a_n)]_\sim + [(b_n)]_\sim \equiv [(a_n + b_n)]_\sim \ \text{ and } \ [(a_n)]_\sim \cdot [(b_n)]_\sim \equiv [(a_n \cdot b_n)]_\sim \,.$$

The order on $\mathbb{R}$ is defined by $[(a_n)]_\sim < [(b_n)]_\sim \equiv$ there exist a k and an n_0 such that

$$n \ge n_0 \Rightarrow a_n \le b_n - \tfrac{1}{k}\,.$$

Next three propositions show correctness of these definitions. Clearly, $(0,0,\dots)$ and $(1,1,\dots)$ are in C. Hence $0_{\mathbb{R}}, 1_{\mathbb{R}} \in \mathbb{R}$.

Proposition 1.6.4 (correctness of + and · 1) *Let (a_n) and (b_n) be in C. Then $(a_n + b_n), (a_n b_n) \in C$ as well.*

Proof. Let (a_n) and (b_n) be as stated and let a k be given. Then for every large m a n we have that $|a_m - a_n|, |b_m - b_n| \le \frac{1}{2k}$. So for the same m and n,

$$|(a_m + b_m) - (a_n + b_n)| \le |a_m - a_n| + |b_m - b_n| \le \tfrac{1}{2k} + \tfrac{1}{2k} = \tfrac{1}{k}\,.$$

Hence $(a_n + b_n)$ is Cauchy.

Cauchy sequences are bounded and so there is an l such that for every n one has that $|a_n|, |b_n| \le l$. For every large m and n it holds that $|a_m - a_n|, |b_m - b_n| \le \frac{1}{2lk}$. Thus for the same m and n we have

$$|a_m b_m - a_n b_n| \le |a_m| \cdot |b_m - b_n| + |b_n| \cdot |a_m - a_n| \le \tfrac{l}{2lk} + \tfrac{l}{2lk} = \tfrac{1}{k}\,.$$

Hence $(a_n b_n)$ is Cauchy. □

The sum and the product of two real numbers is therefore a real number.

Proposition 1.6.5 (correctness of + and · 2) *Let $(a_n), (b_n), (c_n) \in C$ and $(a_n) \sim (c_n)$. Then*

$$(a_n + b_n) \sim (c_n + b_n) \ \text{ and } \ (a_n b_n) \sim (c_n b_n)\,.$$

Proof. Let (a_n), (b_n) and (c_n) be as stated and let a k be given. As before we take an l such that for every n it holds that $|b_n| \le l$. For every large n we have that $|a_n - c_n| \le \frac{1}{k}$. For these n one has that

$$|(a_n + b_n) - (c_n + b_n)| = |a_n - c_n| \le \tfrac{1}{k}\,.$$

Hence $(a_n + b_n) \sim (c_n + b_n)$. For every large n also $|a_n - c_n| \le \frac{1}{lk}$. Thus

$$|a_n b_n - c_n b_n| = |a_n - c_n| \cdot |b_n| \le \tfrac{l}{lk} = \tfrac{1}{k}\,.$$

Hence $(a_n b_n) \sim (c_n b_n)$. □

From this we easily deduce that if (a_n), (a'_n), (b_n) and (b'_n) are in C and $(a_n) \sim (a'_n)$ and $(b_n) \sim (b'_n)$, then

$$[(a_n + b_n)]_\sim = [(a'_n + b'_n)]_\sim \text{ and } [(a_n b_n)]_\sim = [(a'_n b'_n)]_\sim \,.$$

The sum and the product in $\mathbb{R}$ are therefore independent of the representing Cauchy sequences.

Proposition 1.6.6 (correctness of $<$) *Let $(a_n), (b_n), (c_n), (d_n) \in C$, let $(a_n) \sim (c_n)$ and $(b_n) \sim (d_n)$. Then*

$$[(a_n)]_\sim < [(b_n)]_\sim \iff [(c_n)]_\sim < [(b_n)]_\sim$$

and

$$[(a_n)]_\sim < [(b_n)]_\sim \iff [(a_n)]_\sim < [(d_n)]_\sim$$

Proof. Let (a_n), (b_n), (c_n) and (d_n) be as stated. Suppose that the first comparison holds. Thus for some k and every large n it holds that $a_n \le b_n - \frac{2}{k}$. For every large n we have also that $|a_n - c_n| \le \frac{1}{k}$. Thus for every large n we have that

$$c_n \le a_n + \tfrac{1}{k} \le b_n + \tfrac{1}{k} - \tfrac{2}{k} = b_n - \tfrac{1}{k} \,.$$

Hence $[(c_n)]_\sim < [(b_n)]_\sim$. Suppose that the second comparison holds. Then for some k and every large n one has that $c_n \le b_n - \frac{2}{k}$. Then for every large n we have that

$$a_n \le c_n + \tfrac{1}{k} \le b_n - \tfrac{2}{k} + \tfrac{1}{k} = b_n - \tfrac{1}{k} \,.$$

Hence $[(a_n)]_\sim < [(b_n)]_\sim$. The second equivalence is proven similarly. □

Hence if (a_n), (a'_n), (b_n) and (b'_n) are in C, $(a_n) \sim (a'_n)$ and $(b_n) \sim (b'_n)$, then

$$[(a_n)]_\sim < [(b_n)]_\sim \iff [(a'_n)]_\sim < [(b'_n)]_\sim \,.$$

Comparison of two real numbers therefore does not depend on representing Cauchy sequences.

Thus the arithmetic on $\mathbb{R}$ is correctly defined. We show that it makes $\mathbb{R}$ an ordered field. We begin by showing that it makes $\mathbb{R}$ a ring.

Proposition 1.6.7 ($\mathbb{R}$ is a ring) $\mathbb{R}_{\mathrm{R}} \equiv \langle \mathbb{R}, 0_\mathbb{R}, 1_\mathbb{R}, +, \cdot \rangle$ *is a ring.*

Proof. By Theorem 1.5.12 the structure $\mathbb{Q}_{\mathrm{R}} \equiv \langle \mathbb{Q}, 0_\mathbb{Q}, 1_\mathbb{Q}, +, \cdot \rangle$ is a ring. Let Q be the set of sequences $(a_n) \subset \mathbb{Q}$. We consider the structure

$$Q_{\mathrm{R}} \equiv \langle Q,\, 0_Q,\, 1_Q,\, +,\, \cdot \rangle$$

with $0_Q \equiv (0_\mathbb{Q}, 0_\mathbb{Q}, \dots)$, $1_Q \equiv (1_\mathbb{Q}, 1_\mathbb{Q}, \dots)$ and

$$(a_n) + (b_n) \equiv (a_n + b_n) \text{ and } (a_n) \cdot (b_n) \equiv (a_n \cdot b_n) \,.$$

Clearly Q_R is a ring — we have the same ring $\mathbb{Q}_\mathrm{R}$ in every coordinate and satisfaction of ring axioms in every coordinate implies their satisfaction in Q_R.

But $C \subset Q$ and we have shown above that C is closed to constant operations 0_Q and 1_Q and to operations of component-wise addition and multiplication. Thus $C_\mathrm{R} \equiv \langle C, 0_Q, 1_Q, +, \cdot \rangle$ is a subring of Q_R. We have the onto map

$$F\colon C \to \mathbb{R}, \quad F(a_n) \equiv F((a_n)) \equiv [(a_n)]_\sim\,.$$

In it $F(a_n) = F(b_n)$ iff $(a_n) \sim (b_n)$. In the above Propositions 1.6.4 and 1.6.5 we have shown that the operations $+$ and $\cdot$ in C_R are lifted by F to operations in $\mathbb{R}_\mathrm{R}$. Since this preserves ring axioms, $\mathbb{R}_\mathrm{R}$ is a ring. □

$\mathbb{Q}_\mathrm{R}$ is actually a field but neither Q_R nor C_R is a field, the equality

$$(0,\, 1,\, 0,\, 0,\, 0,\, \dots) \cdot (1,\, 0,\, 0,\, 0,\, 0,\, \dots) = (0,\, 0,\, 0,\, 0,\, 0,\, \dots) = 0_Q\,,$$

shows that in C_R the product of two nonzero elements may be zero. The lifting by F removes this problem and makes $\mathbb{R}_\mathrm{R}$ a field, even an ordered one.

Theorem 1.6.8 ($\mathbb{R}$ is an OF) *The above defined structure*

$$\mathbb{R}_\mathrm{OF} \equiv \langle \mathbb{R},\, 0_\mathbb{R},\, 1_\mathbb{R},\, +,\, \cdot,\, < \rangle$$

is an ordered field.

Proof. By the previous proposition $\mathbb{R}_\mathrm{R}$ is a ring. It remains to show that (i) $\mathbb{R}_\mathrm{R}$ has multiplicative inverses, that (ii) $(\mathbb{R}, <)$ is a LO and that (iii) both order axioms hold in $\mathbb{R}_\mathrm{OF}$.

(i) Let $\alpha = [(a_n)]_\sim \in \mathbb{R}$ be nonzero, so that $(a_n) \not\sim (0, 0, \dots)$. Thus for some l and infinitely many n we have that $|a_n| \ge \frac{2}{l}$. Since $(a_n) \in C$, for every large n one has that $|a_n| \ge \frac{1}{l}$. For $a_n \ne 0_\mathbb{Q}$ we define $b_n \equiv 1/a_n$ and for $a_n = 0_\mathbb{Q}$ we set $b_n \equiv 0_\mathbb{Q}$. We show that $(b_n) \in C$. Let a k be given. For every large m and n it holds that $|a_m - a_n| \le \frac{1}{l^2 k}$. Then for every large m and n we have that

$$|b_m - b_n| = \tfrac{|a_n - a_m|}{|a_m a_n|} \le \tfrac{l^2}{l^2 k} = \tfrac{1}{k}\,.$$

Hence $(b_n) \in C$. We set $\beta \equiv [(b_n)]_\sim \in \mathbb{R}$ and get that $\alpha \cdot \beta = 1_\mathbb{R}$, because $(a_n b_n) \sim (1, 1, \dots)$. Hence $\beta = \alpha^{-1}$.

(ii) Irreflexivity and transitivity of $<$ are immediate. We prove trichotomy. Let $\alpha = [(a_n)]_\sim$ and $\beta = [(b_n)]_\sim$ be two distinct real numbers. So $(a_n) \not\sim (b_n)$ and for example for some k and infinitely many n it holds that $a_n - b_n \ge \frac{2}{k}$ (the other possibility with $a_n - b_n \le -\frac{2}{k}$ is resolved similarly). Since $(a_n), (b_n) \in C$, for every large n it holds that $a_n - b_n \ge \frac{1}{k}$. Thus $b_n \le a_n - \frac{1}{k}$ for these n and $\beta < \alpha$.

(iii) Let $\alpha = [(a_n)]_\sim$, $\beta = [(b_n)]_\sim$ and $\gamma = [(c_n)]_\sim$ be tree real numbers. If $\alpha < \beta$ then for some k and every large n one has that $a_n \le b_n - \frac{1}{k}$. For these n we have that also $a_n + c_n \le b_n + c_n - \frac{1}{k}$ and $\alpha + \gamma < \beta + \gamma$. If $0_\mathbb{R} < \alpha, \beta$ then for some k and every large n one has that $1/\sqrt{k} \le a_n, b_n$. Then for these n also $\frac{1}{k} \le a_n b_n$ and $0_{\mathbb{R}} < \alpha\beta$. So both axioms of order hold. □

$\mathbb{R}_{\mathrm{OF}}$ contains in a natural way $\mathbb{Q}_{\mathrm{OF}}$.

Definition 1.6.9 (rational reals) *A real number α is rational if for some fraction a it holds that $\alpha = [(a, a, \dots)]_{\sim}$. We denote this real number by $\overline{a}$.*

Exercise 1.6.10 *Show that the map $f\colon \mathbb{Q} \to \mathbb{R}$ given by $f(a) = \overline{a}$ is an isomorphism between $\mathbb{Q}_{\mathrm{OF}}$ and the ordered subfield of rational real numbers in $\mathbb{R}$.*

Thus if we want to work with a fraction b as a real number, we understand it as a rational real number $\overline{b}$.

• *Remark on non-strict inequalities.* We make an effort to use in computations in $\mathbb{R}_{\mathrm{OF}}$, whenever it is possible, non-strict inequalities $\le$ and $\ge$ instead of the strict $<$ and $>$. So above in the definition of the LO $(\mathbb{R}, <)$ we write $n \ge n_0 \Rightarrow a_n \le b_n - \frac{1}{k}$, and not $n > n_0 \Rightarrow a_n < b_n - \frac{1}{k}$. *Non-strict inequalities are safer than strict ones.* For if $x, y \in \mathbb{R}$ then the inequality $x \le y$ is preserved when it is multiplied by any real $z \ge 0$, but the strict inequality $x < y$ need not be preserved.

Exercise 1.6.11 *Why?*

• *Completeness of real numbers.* The extension of $\mathbb{Q}$ in $\mathbb{R}$ brings completeness. However, in the next section we pay a price for it. We need the following interesting result which we call an urtheorem on limits of monotone sequences.

Theorem 1.6.12 (ULMS) *For $n \in \mathbb{N}$ let $a_n, b \in \mathbb{Q}$ be such that $a_1 \ge a_2 \ge \dots \ge b$. Then the sequence (a_n) is Cauchy.*

Proof. Let (a_n) and b be as stated. Then for every two indices $m \le n$ we have that $0 \le a_m - a_n \le a_1 - b$. Suppose that (a_n) is not Cauchy. Then there is a k and a sequence of natural numbers $m_0 < m_1 < \dots$ such that for every $i \in \mathbb{N}$ it holds that $a_{m_i} - a_{m_{i-1}} \ge \frac{1}{k}$. We take any $n \in \mathbb{N}$ such that $\frac{n}{k} > a_1 - b$ and get the contradiction that $a_{m_n} - a_{m_0} = \sum_{i=1}^{n}(a_{m_i} - a_{m_{i-1}}) \ge \frac{n}{k} > a_1 - b$. □

Thus any weakly decreasing and from below bounded sequence of fractions defines a real number. For this form of Theorem 2.3.3 on limits of monotone sequences (of real numbers) we even do not need real numbers.

Exercise 1.6.13 *State and prove the variant of the theorem for weakly increasing sequences of fractions.*

Now we prove that the LO $(\mathbb{R}, <)$ is complete.

Theorem 1.6.14 (completeness of $\mathbb{R}$) *The linear order $(\mathbb{R}, <)$ is complete.*

Proof. Suppose that $B \subset \mathbb{R}$ is nonempty and bounded from above. We define two sequences $(a_n), (b_n) \subset \mathbb{Q}$. We take any rational real upper bound $\overline{a_1}$ of B and set $b_1 \equiv 1_{\mathbb{Q}}$. Suppose that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathbb{Q}$ are defined. If $\overline{a_n} - \overline{b_n}$ is an upper bound of B, we set $a_{n+1} \equiv a_n - b_n$ and $b_{n+1} \equiv b_n$. If it is not, so that $\beta > \overline{a_n} - \overline{b_n}$ for some $\beta \in B$, we set $a_{n+1} \equiv a_n$ and $b_{n+1} \equiv b_n/2$. We repeat this indefinitely. The sequence (a_n) weakly decreases and every $\overline{a_n}$ is an upper bound of B. Since $B \neq \emptyset$, (a_n) is bounded from below. Thus by Theorem 1.6.12 we have $(a_n) \in C$. We show that $\alpha \equiv [(a_n)]_{\sim} = \sup(B)$.

Let $\alpha < \beta$ for some $\beta = [(c_n)]_{\sim} \in B$. Then for some k and every large n we have that $a_n \le c_n - \frac{1}{k}$. Since $(c_n) \in C$, we can take an m such that for every large n we have that $a_m \le c_n - \frac{1}{2k}$. But then $\overline{a_m} < \beta$, contradicting that $\overline{a_m}$ is an upper bound of B. Thus α is an upper bound of B. Let a $\gamma \in \mathbb{R}$ with $\gamma < \alpha$ be given. Since $B \neq \emptyset$, the step $b_n \rightsquigarrow b_n/2$ is performed infinitely many times. It follows that there exist an $m \in \mathbb{N}$ and a $\beta \in B$ such that $\beta > \overline{a_m} - \overline{b_m} \ge \alpha - (\alpha - \gamma) = \gamma$. Hence $\beta > \gamma$ and it follows that α is the smallest upper bound of B. □

By Exercise 1.5.17 the ordered field $\mathbb{R}_{\mathrm{OF}}$ is Archimedean as a consequence of its completeness.

Exercise 1.6.15 *Show in a simple way that $\mathbb{R}_{\mathrm{OF}}$ is Archimedean.*

Corollary 1.6.16 ($\sqrt{2} \in \mathbb{R}$) *The equation $x^2 = 2$ is solvable in the field $\mathbb{R}$.*

Proof. We take the set $X \equiv \{a \in \mathbb{R} :\ a^2 < 2\}$. Using Theorem 1.6.14 we set $s \equiv \sup(X) \in \mathbb{R}$. As in the proof of Corollary 1.5.21 we show that neither of the cases $s^2 < 2$ and $s^2 > 2$ occurs. Hence $s^2 = 2$. □

We generalize this argument in the next theorem which we prove later in Corollary 6.3.7. Continuity of functions means — we define it precisely later — that a small change in the argument causes a small change in the value.

Theorem 1.6.17 (Bolzano–Cauchy) *Let $u \le v$ be in $\mathbb{R}$ and $f\colon [u, v] \to \mathbb{R}$ be a continuous function with the property that $f(u)f(v) \le 0$. Then there exists a $w \in [u, v]$ such that $f(w) = 0$.*

Bernard Bolzano (1781–1848) was a Czech-Italian-German priest, philosopher and mathematician.

1.7 $\mathbb{R}$ is uncountable

We begin with the definition of finite and infinite sets.

• *Finite and infinite sets.* We take the set $\mathbb{N} = \{1, 2, \ldots\}$ of natural numbers for granted and define whether a given set X is finite or infinite by using maps from $\mathbb{N}$ to X.

Definition 1.7.1 (finite and infinite) *A set is* *infinite* *if there is an injective map from* $\mathbb{N}$ *to the set. Else the set is* *finite*.

Exercise 1.7.2 *If* X *is finite then there is an onto map* $f\colon \mathbb{N} \to X$.

Exercise 1.7.3 *For every finite* X *there is a number* $n \in \mathbb{N}_0$ *and a bijection* $f\colon [n] \to X$.

• *Countable and uncountable sets.* We shall consider really large sets.

Definition 1.7.4 (uncountable sets) *A set is* *countable* *if there is a bijection from* $\mathbb{N}$ *to the set. A set is* *at most countable* *if it is finite or countable. A set is* *uncountable* *if it is not at most countable.*

$\mathbb{N}$ is obviously countable. Are there any uncountable sets? Shortly we prove that the set of real numbers $\mathbb{R} = C/\sim$ is uncountable. But first we show that the set of fractions $\mathbb{Q}$ is countable.

Theorem 1.7.5 ($\mathbb{Q}$ is countable) *The set of fractions* $\mathbb{Q}$ *is countable.*

Proof. Recall that Z_{z} $(\subset Z)$ are protofractions $\frac{m}{n}$ in lowest terms. By Exercise 1.5.3 it suffices to show that Z_{z} is countable. For $\frac{m}{n} \in Z_{\mathrm{z}}$ and $j \in \mathbb{N}$ we define the norm $\|\frac{m}{n}\| \equiv |m| + n$ $(\in \mathbb{N})$ and the list

$$Z(j) \equiv \big(z_{1,\,j} < z_{2,\,j} < \cdots < z_{k_j,\,j} :\ z_{i,\,j} \in Z_{\mathrm{z}},\ \|z_{i,\,j}\| = j\big)\,.$$

For example,

$$Z(5) = \big(\tfrac{-4}{1} < \tfrac{-3}{2} < \tfrac{-2}{3} < \tfrac{-1}{4} < \tfrac{1}{4} < \tfrac{2}{3} < \tfrac{3}{2} < \tfrac{4}{1}\big)\,,$$

$k_5 = 8$ and $\frac{0}{5} \notin Z(5)$ because the protofraction $\frac{0}{5}$ is not in lowest terms. Clearly, $j \neq j' \Rightarrow Z(j) \cap Z(j') = \emptyset$, every set $Z(j)$ is finite (and nonempty) and $\bigcup_{j\in\mathbb{N}} Z(j) = Z_{\mathrm{z}}$. We define the map $f\colon \mathbb{N} \to Z_{\mathrm{z}}$ by

$$f(1) \equiv z_{1,\,1},\ f(2) \equiv z_{2,\,1},\ \ldots, f(k_1) \equiv z_{k_1,\,1},\ f(k_1+1) \equiv z_{1,\,2},\ \ldots$$

— the values of f first run through the k_1 ordered protofractions in $Z(1)$, then through k_2 ordered protofractions in $Z(2)$, and so on. For $j \in \mathbb{N}$ the general value equals

$$f(k_1 + k_2 + \cdots + k_{j-1} + i) = z_{i,\,j},\ i \in [k_j]\,.$$

For $j = 1$ the argument of f is defined as i. We see that f is a bijection. □

Exercise 1.7.6 *The set* $\mathbb{Z}$ *of integers is countable.*

• *Cantor's theorem.* We deduce the uncountability of $\mathbb{R}$ from the next basic set-theoretic result due to G. Cantor: for no set there is an onto map from the set to its power set.

Theorem 1.7.7 (Cantor's on $\mathcal{P}(X)$) *For no set X there is a surjective map $f\colon X \to \mathcal{P}(X)$.*

Proof. Let X be a set and $f\colon X \to \mathcal{P}(X)$ be any map. Let $Y \equiv \{x \in X : x \notin f(x)\}$. Then $Y \subset X$ but we show that $Y \notin f[X]$. Suppose that for some $y \in X$ we have $f(y) = Y$. This at once gives a contradiction: if $y \in Y$ then $y \notin f(y) = Y$, and if $y \notin Y = f(y)$ then $y \in Y$. Hence f is not onto. □

Exercise 1.7.8 *Determine Y when $X = \emptyset$.*

Exercise 1.7.9 *Prove the following proposition.*

Proposition 1.7.10 (injective Cantor's theorem) *For no set X there is an injective map $f\colon \mathcal{P}(X) \to X$.*

Let $\{0,1\}^{\mathbb{N}}$ be the set of sequences $(a_n) \subset \{0,1\}$. We show that it is uncountable.

Corollary 1.7.11 (on $\{0,1\}^{\mathbb{N}}$) *There is no onto map $f\colon \mathbb{N} \to \{0,\,1\}^{\mathbb{N}}$.*

Proof. The map $g\colon \{0,1\}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ given by $g((a_n)) \equiv \{n \in \mathbb{N} : \ a_n = 1\}$ is clearly a bijection. If the stated onto map f existed, the composition $g \circ f$ would go from $\mathbb{N}$ onto $\mathcal{P}(\mathbb{N})$, in contradiction with Theorem 1.7.7. □

Exercise 1.7.12 *Show that g is a bijection.*

• *Decimal expansions.* In practice we write real numbers as "infinite" decimal expansions.

Definition 1.7.13 (decimal expansion) *A decimal expansion ρ is any sequence*

$$\rho = a_n\, a_{n-1}\, \ldots\, a_0\, .\, a_{-1}\, a_{-2}\, \ldots$$

such that $n \in \mathbb{N}$, $a_n \in \{+,-\}$, for every $m \le n-1$ the digit $a_m \in \{0,1,\ldots,9\}$ and if $a_{n-1} = 0$ then $n = 1$.

For example, in the decimal expansion $\pi = 3.1415\ldots$ we have $n = 1$, $a_1 = +$, $a_0 = 3$, $a_{-1} = 1$, $a_{-2} = 4$ and so on; the $+$ sign is by convention omitted. We denote the set of decimal expansions by R.

There is a map $F\colon R \to \mathbb{R}$ such that $F(\rho)$ is the real number

$$\big[\big(\varepsilon a_{n-1}10^{n-1},\, \varepsilon(a_{n-1}10^{n-1} + a_{n-2}10^{n-2}),\\ \varepsilon(a_{n-1}10^{n-1} + a_{n-2}10^{n-2} + a_{n-3}10^{n-3}),\, \ldots\big)\big]_{\sim} \quad (\in \mathbb{R})$$

where for $a_n = +$ (respectively $-$) we set $\varepsilon \equiv 1$ (respectively -1). We say that ρ is a mostly-nine expansion if there is an m such that $a_m = a_{m-1} = a_{m-2} = \cdots = 9$ and either $m = n-1$, or $m < n-1$ and $a_{m+1} < 9$. Its successor expansion arises by changing the 9s in $a_m, a_{m-1}, \ldots$ to 0s and then if

$m < n-1$ by increasing the digit a_{m+1} by 1, and if $m = n-1$ by increasing n by 1 and adding the new digit $a_{n-1} \equiv 1$. The sign and other digits are not changed.

Definition 1.7.14 (associated expansions) *Two decimal expansions $\rho \ne \rho'$ are associated if $\{\rho, \rho'\} = \{+0.000\dots, -0.000\dots\}$ or ρ and ρ' are a mostly-nine expansion and its successor.*

Exercise 1.7.15 *The set of pairs of associated expansions is countable.*

For instance,

$$\{-23.56999\dots,\ -23.57000\dots\} \text{ and } \{+999.999\dots,\ +1000.000\dots\}$$

are two pairs of associated expansions. Another example is the well known "paradoxical" (see [1]) pair of associated expansions

$$\{\kappa,\ \kappa'\} = \{+0.999\dots,\ +1.000\dots\}\,.$$

By the next theorem simply $F(\kappa) = F(\kappa') = 1_{\mathbb{R}}$, although $\kappa \ne \kappa'$.

Theorem 1.7.16 (R and $\mathbb{R}$) *The map $F\colon R \to \mathbb{R}$ is onto and for every two decimal expansions ρ and ρ' it holds that $F(\rho) = F(\rho')$ iff $\rho = \rho'$ or $\{\rho, \rho'\}$ is a pair of associated expansions.*

We prove this theorem in *MA 1*$^+$.

• $\mathbb{R}$ *is uncountable.* In the culmination and conclusion of Chapter 1 we prove that the set of real numbers is uncountable.

Corollary 1.7.17 (price for completeness) *There is no map from $\mathbb{N}$ onto $\mathbb{R}$. Thus the set of real numbers is uncountable.*

Proof. Using the previous theorem we represent real numbers by decimal expansions. We consider the set of expansions

$$X \equiv \{+0\,.\,a_{-1}\,a_{-2}\,\dots\,a_{-m}\,\cdots\colon\ a_{-m} \in \{0,\,1\},\, m \in \mathbb{N}\}\,.$$

By the previous theorem the restriction $F \,|\, X$ is injective. For $y \in Y = F[X]$ $(\subset \mathbb{R})$ we denote by $F^{-1}(y)$ the unique $x \in X$ such that $F(x) = y$. Then $F^{-1}\colon Y \to X$ is a bijection.

We assume for the contrary that $f\colon \mathbb{N} \to \mathbb{R}$ is surjective. Then we have also an onto map $f_0\colon \mathbb{N} \to Y$. It is clear that we have also a bijection $g\colon X \to \{0,1\}^{\mathbb{N}}$. Thus we have also the onto map

$$h = g(F^{-1}(f_0))\colon \mathbb{N} \to \{0,1\}^{\mathbb{N}}\,,$$

which contradicts Corollary 1.7.11. □

Exercise 1.7.18 *Why is there a map onto f_0? Why is there the bijection g? Why is the map h surjective?*

We show another method to prove the uncountability of $\mathbb{R}$, also due to G. Cantor. We know that it suffices to show that there is no map from $\mathbb{N}$ onto $\{0,1\}^{\mathbb{N}}$.

Theorem 1.7.19 (Cantor's diagonalization) *Let $F\colon \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ be any map. Then the sequence $f\colon \mathbb{N} \to \{0,1\}$, given by*

$$f(n) \equiv 1 - (F(n))(n)\,,$$

is not in $F[\mathbb{N}]$ and therefore F is not surjective.

Proof. The map f is not in the image of F because $f \neq F(n)$ for every n. And this holds because for every n we have $f(n) \neq (F(n))(n)$. □

The interesting article [28] considers the uncountability of $\mathbb{R}$ from the perspective of the so-called reverse mathematics ([34]).

Chapter 2

Existence of limits of sequences

In Section 2.1 of this chapter, which is based on the lecture

`https://kam.mff.cuni.cz/~klazar/MAI24_pred2.pdf`

presented on February 29, 2024, we introduce arithmetic of infinities. We work in the extended reals

$$\mathbb{R}_{\mathrm{EX}} \equiv \langle \mathbb{R}^*,\, 0_{\mathbb{R}},\, 1_{\mathbb{R}},\, +,\, \cdot,\, <,\, /,\, \pm(\cdot)\rangle \ \text{ where } \ \mathbb{R}^* \equiv \mathbb{R} \cup \{-\infty,\, +\infty\}\,.$$

In Proposition 2.1.6 we show that in the LO $(\mathbb{R}^*, <)$ every set has infimum and supremum. Theorems 2.1.8 and 2.1.9 describe arithmetic in $\mathbb{R}_{\mathrm{EX}}$. Then we define neighborhoods of points and infinities. Definition 2.1.15 introduces both finite and infinite limits of real sequences. Definition 2.1.18 introduces robustness of properties of real sequences.

Section 2.2 deals mainly with subsequences. In Theorem 2.2.5 we show that every real sequence has a subsequence that has a limit. In this theorem we also characterize by limits of subsequences the sequences that (i) have no limit or that (ii) have no limit or the limit differs from the given A $(\in \mathbb{R}^*)$. In Proposition 2.2.10 we show that $\lim n^{1/n} = 1$. The proof uses the binomial theorem which is reminded in Exercise 2.2.9.

Section 2.3 presents five existential theorems on limits of real sequences. By Theorem 2.3.3 every monotone sequences has a limit. Theorem 2.3.9 generalizes it to quasi-monotone sequences. Corollaries 2.3.4 and 2.3.10 are robust versions of these theorems. The Bolzano–Weierstrass Theorem 2.3.15 says that every bounded sequence has a convergent subsequence. By Theorem 2.3.20 convergent real sequences coincide with the Cauchy sequences. Fekete's lemma (Theorem 2.3.25) shows that subadditivity or superadditivity of a sequence (a_n) suffices for the existence of the limit $\lim \frac{a_n}{n}$.

2.1 Infinities, neighborhoods, limits

• *Notation.* Recall the logical and set-theoretic notation of Section 1.2. Recall the meaning of the symbols $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{N}_0$. We use letters i, j, k, l, m, m_0, m_1, $\dots$ and n, n_0, n_1, $\dots$, possibly with primes, to denote natural numbers. By a, b, c, d, e, δ, ε and θ, possibly with indices and primes, we denote real numbers. The numbers δ, ε and θ are always positive. We refer to the elements of $\mathbb{R}$ as (real) numbers or points (on the real axis). A sequence $(a_n) \subset \mathbb{R}$ is a function $a\colon \mathbb{N} \to \mathbb{R}$. Sets of real numbers are denoted by M and N.

Exercise 2.1.1 *Negate in words the proposition*

$$\forall\,\varepsilon\;\exists\,\delta\;\forall\,a,\,b \in M\,\big(|a-b| < \delta \Rightarrow |f(a)-f(b)| < \varepsilon\big)\,.$$

Exercise 2.1.2 (triangle inequality) *For every real numbers a_1, a_2, $\dots$, a_n it holds that*

$$|a_1 + a_2 + \cdots + a_n| \le |a_1| + |a_2| + \cdots + |a_n|\,.$$

We speak briefly of Δ-inequality. In Theorem 4.1.19 we give an infinite version.

• *Computing with infinities.* We add to $\mathbb{R}$ two new different elements, the infinities $+\infty$ and $-\infty$. We get the extended reals

$$\mathbb{R}^* \equiv \mathbb{R} \cup \{+\infty,\,-\infty\}\,.$$

The elements in $\mathbb{R}^*$ are denoted by A, B, K and L. For the next definition we assume that the arithmetic in the ordered field $\mathbb{R}_{\mathrm{OF}}$ is known. In an expression containing k infinities $\pm\infty$ or $\mp\infty$ (or some k symbols with signs), the selection of equal signs means the selection of all upper, or all lower, signs. The selection of independent signs means the selection of all 2^k combinations of signs.

Definition 2.1.3 (operations in $\mathbb{R}_{\mathrm{EX}}$) *We define the structure*

$$\mathbb{R}_{\mathrm{EX}} \equiv \langle \mathbb{R}^*,\, 0_{\mathbb{R}},\, 1_{\mathbb{R}},\, +,\, \cdot,\, <,\, /,\, \pm(\cdot)\rangle\,.$$

The first three items $\mathbb{R}^*$, $0_{\mathbb{R}}$ and $1_{\mathbb{R}}$ are clear.
1. *For $A, B \in \mathbb{R}$ the sum $A + B$ is as in $\mathbb{R}_{\mathrm{OF}}$. For $A = \pm\infty$ and $B \in \mathbb{R}$ we set $A + B \equiv A$, and similarly if A and B are exchanged. If $A = B = \pm\infty$ then $A + B \equiv A$. The two sums $(\pm\infty) + (\mp\infty)$ (equal signs) are not defined and are labeled as indefinite expressions.*
2. *For $A, B \in \mathbb{R}$ the product $A \cdot B = AB$ is as in $\mathbb{R}_{\mathrm{OF}}$. If $A, B \ne 0$ and at least one of them is an infinity then AB is infinity with the corresponding sign (explained below). The four products $0 \cdot (\pm\infty)$ and $(\pm\infty) \cdot 0$ are indefinite expressions.*
3. *For $A, B \in \mathbb{R}$ the comparison $A < B$ is as in $\mathbb{R}_{\mathrm{OF}}$. We add the comparisons $-\infty < a$, $a < +\infty$ and $-\infty < +\infty$, for any $a \in \mathbb{R}$.*
4. *The division operation / is extended from $\mathbb{R}_{\mathrm{OF}}$ by $\frac{a}{\pm\infty} \equiv 0$ (for any $a \in \mathbb{R}$) and for any $a \in \mathbb{R} \setminus \{0\}$ by setting $\frac{\pm\infty}{a}$ to infinity with the corresponding sign.*

The four ratios $\frac{\pm\infty}{\pm\infty}$ (independent signs) are indefinite expressions. So are the ratios $\frac{A}{0}$ (for any $A \in \mathbb{R}^*$).
5. *Finally, we extend from $\mathbb{R}_{\mathrm{OF}}$ the change of sign operation by $-(\pm\infty) \equiv \mp\infty$ (equal signs).*

Neither additive nor multiplicative inverses of infinities are defined, although $-(\pm\infty) = \mp\infty$ and $\frac{1}{\pm\infty} = 0$. The corresponding sign is given by the rule that the product, or the ratio, of two equal (respectively different) signs is the sign $+$ (respectively $-$). In contrast with $\mathbb{R}_{\mathrm{OF}}$ the operations in $\mathbb{R}_{\mathrm{EX}}$ are partial, are not always defined. They are undefined exactly on the indefinite expressions which we now repeat, they are

$$\pm\infty + (\mp\infty),\ 0 \cdot (\pm\infty),\ (\pm\infty) \cdot 0,\ \tfrac{\pm\infty}{\pm\infty} \text{ and } \tfrac{A}{0}$$

where in the first expression signs are equal, in the fourth one are independent and $A \in \mathbb{R}^*$. In $\mathbb{R}_{\mathrm{OF}}$ division and sign change can be expressed by other operations and inverses, $a/b = a \cdot b^{-1}$ and $-a$ is the additive inverse to a, but in $\mathbb{R}_{\mathrm{EX}}$ this is not possible. More generally, the subtraction in $\mathbb{R}_{\mathrm{OF}}$ can be expressed as $a - b \equiv a + (-b)$. In $\mathbb{R}_{\mathrm{EX}}$ we define it in the same way.

Exercise 2.1.4 *Compute: $\frac{-\infty}{-2}$, $(-\infty) - (+\infty)$, $-\infty + 10$ and $\frac{\pm\infty}{0}$.*

Exercise 2.1.5 *Show that $(\mathbb{R}^*, <)$ is a* LO.

• *Properties of the arithmetic in $\mathbb{R}_{\mathrm{EX}}$.* We describe them in the next proposition and two theorems.

Proposition 2.1.6 (inf and sup in $\mathbb{R}_{\mathrm{EX}}$) *In the* LO $(\mathbb{R}^*, <)$ *every set $X \subset \mathbb{R}^*$ has infimum and supremum.*

Proof. We prove the existence of supremum, for infimum we would argue similarly. Recall that $H(X)$ $(\subset \mathbb{R}^*)$ denotes the set of upper bounds of X. We have $\sup(\emptyset) = \min(H(\emptyset)) = \min(\mathbb{R}^*) = -\infty$ and $\sup(\{-\infty\}) = \min(H(\{-\infty\})) = \min(\mathbb{R}^*) = -\infty$. If $+\infty \in X$ then $\sup(X) = \min(H(X)) = \min(\{+\infty\}) = +\infty$.

We resolve the remaining case when $X \neq \emptyset, \{-\infty\}$ and $+\infty \notin X$. Let $X' \equiv X \setminus \{-\infty\}$. Then $X' \neq \emptyset$ and $X' \subset \mathbb{R}$. If X' is not bounded from above in $(\mathbb{R}, <)$ then $\sup(X) = \min(H(X)) = \min(\{+\infty\}) = +\infty$. Finally, if X' is bounded from above in $(\mathbb{R}, <)$ then the supremum $\sup_{(\mathbb{R}^*,\, <)}(X) = \sup_{(\mathbb{R},\, <)}(X')$ $(\in \mathbb{R})$ exists by Theorem 1.6.14. □

Exercise 2.1.7 *Find all sets $X \subset \mathbb{R}^*$ such that* $\sup(X) = -\infty$.

Theorem 2.1.8 (OF axioms in $\mathbb{R}_{\mathrm{EX}}$) *With the exception of existence of additive and multiplicative inverses and the axiom of shift, the structure $\mathbb{R}_{\mathrm{EX}}$ satisfies all other axioms of ordered fields as long as involved arithmetic expressions are defined.*

Proof. As for the violated axioms, infinities do not have inverses and no inequality $a < b$ is preserved by adding an infinity to it.

Other OF axioms are not violated. Since $0 + (\pm\infty) = (\pm\infty) + 0 = \pm\infty$ and $1 \cdot (\pm\infty) = (\pm\infty) \cdot 1 = \pm\infty$ (equal signs), the elements 0 and 1 remain neutral. Addition and multiplication are introduced in Definition 2.1.3 in a commutative way and remain commutative.

Let $A, B, K \in \mathbb{R}^*$. We may assume that at least one of them is infinity. We prove associativity and check the equalities

$$(A + B) + K \overset{(1)}{=} A + (B + K) \;\text{ and }\; (A \cdot B) \cdot K \overset{(2)}{=} A \cdot (B \cdot K)\;.$$

(1) If two of A, B and K are infinities with different signs then neither side is defined. Else (1) equates the same infinity. (2) If one of A, B and K is zero then neither side is defined. Else (2) equates the same infinity with the sign equal to the product of the signs of A, B and K.

We prove the distributive law and check the equality

$$A \cdot (B + K) \overset{(3)}{=} (A \cdot B) + (A \cdot K)\;.$$

Suppose that $A = \pm\infty$. We may assume that $B, K \ne 0$ and have the same sign s (else the right side is not defined). Then (3) equates the same infinity with the sign equal to the product of the sign of A and the sign s. Suppose that $A \in \mathbb{R}$. We may assume that $A \ne 0$ and that $B + K$ is not the indefinite expression $\pm\infty + (\mp\infty)$. It follows that then (3) equates the same infinity.

The order. We know from Exercise 2.1.5 that $(\mathbb{R}^*, <)$ is a LO. It is easy to see that the axiom of positivity holds. □

Theorem 2.1.9 (division in $\mathbb{R}_{\mathrm{EX}}$) *For every $A, B, K, L \in \mathbb{R}^*$ it holds that*

$$\frac{A}{K} + \frac{B}{K} \overset{(1)}{=} \frac{A + B}{K} \quad \textit{and} \quad \frac{A}{K} \cdot \frac{B}{L} \overset{(2)}{=} \frac{A \cdot B}{K \cdot L}$$

if the involved arithmetic expressions are defined.

Proof. Let $A, B, K, L \in \mathbb{R}^*$. We again assume that one of them is infinity. We begin with (1). If $K = \pm\infty$ then the left side is not defined or (1) holds as $0 + 0 = 0$. Let $K \in \mathbb{R} \setminus \{0\}$. We may assume that $A + B$ is not $\pm\infty + (\mp\infty)$ and see that (1) equates the same infinity.

We turn to (2). We may assume that $K, L \ne 0$. If A or B is infinity then we may assume that $A, B \ne 0$ and $K, L \in \mathbb{R}$. Then (2) equates the same infinity. Let $A, B \in \mathbb{R}$. Then (2) holds as $0 = 0$. □

By [7, p. 214 of volume 1] and [10, p. 19] the symbol ∞ for infinity was introduced in 1655 by the English mathematician *John Wallis (1616–1703).*

- *Neighborhoods of points and infinities.* Recall notation for real intervals:

$$(a,\, b] = \{x \in \mathbb{R}:\; a < x \le b\},\; (-\infty,\, a) = \{x \in \mathbb{R}:\; x < a\}$$

etc. Unfortunately, one can still encounter notation for intervals using reverse square brackets. In it the previous intervals would be written as $]a, b]$ and $(-\infty, a[$. Sometimes brackets $\langle$ and $\rangle$ are used.

Definition 2.1.10 (neighborhoods of points and infinities) *For $b \in \mathbb{R}$ we define the ε-neighborhood of the point b as $U(b, \varepsilon) \equiv (b - \varepsilon, b + \varepsilon)$. We define ε-neighborhoods of infinities as*

$$U(-\infty,\, \varepsilon) \equiv (-\infty,\, -1/\varepsilon) \;\; \textit{and} \;\; U(+\infty,\, \varepsilon) \equiv (1/\varepsilon,\, +\infty)\,.$$

Four exercises give basic properties of neighborhoods.

Exercise 2.1.11 *If $c \in U(A, \varepsilon)$ and $c < b < A + \varepsilon$ or $A - \varepsilon < b < c$ then also $b \in U(A, \varepsilon)$.*

For $M, N \subset \mathbb{R}$ notation $M < N$ means that always $x \in M$, $y \in N \Rightarrow x < y$.

Exercise 2.1.12 *If $A < B$ then there is an ε such that $U(A, \varepsilon) < U(B, \varepsilon)$, in particular $U(A, \varepsilon) \cap U(B, \varepsilon) = \emptyset$.*

Exercise 2.1.13 *Always $\varepsilon \le \delta \Rightarrow U(A, \varepsilon) \subset U(A, \delta)$.*

Exercise 2.1.14 *Always $\bigcap_{k=1}^{\infty} U(b,\, 1/k) = \{b\}$ and $\bigcap_{k=1}^{\infty} U(\pm\infty, 1/k) = \emptyset$.*

• *Limits of sequences.* By (a_n), (b_n) and (c_n) we denote sequences of real numbers. We call them real sequences or just sequences. The next definition is fundamental.

Definition 2.1.15 (limit of a sequence) *Let $(a_n) \subset \mathbb{R}$ and $L \in \mathbb{R}^*$. If for every $\varepsilon > 0$ there is an n_0 such that $n \ge n_0 \Rightarrow a_n \in U(L, \varepsilon)$, we write that $\lim a_n = L$ or $\lim_{n\to\infty} a_n = L$ or $a_n \to L$, and say that the sequence (a_n) has the limit L.*

If $L \in \mathbb{R}$, we say that (a_n) has a finite limit or that it converges. If $L = \pm\infty$ then (a_n) has an infinite limit. A sequence diverges if it has no limit or an infinite limit. Finite limit $\lim a_n = a$ means that for every real (and no mater how small) number $\varepsilon > 0$ there is an index $n_0 \in \mathbb{N}$ such that for every index $n \ge n_0$ the distance between a_n and a is smaller than ε, i.e. $|a_n - a| < \varepsilon$. The infinite limit $\lim a_n = -\infty$ means that for every (and no mater how negative) number c there is an index n_0 such that for every index $n \ge n_0$ it holds that $a_n \le c$. The opposite inequality gives the limit $+\infty$. The eventually constant sequence (a_n) with $a_n = a$ for every $n \ge n_0$ is a convergent sequence. It has the limit $\lim a_n = a$. From now on we write, following the remark in Section 1.6, the non-strict $|a_n - a| \le \varepsilon$ instead of $|a_n - a| < \varepsilon$.

Exercise 2.1.16 *Explain why the condition $\forall\varepsilon \ldots |a_n - a| \le \varepsilon \ldots$ is equivalent to the condition $\forall\varepsilon \ldots |a_n - a| < \varepsilon \ldots$.*

Proposition 2.1.17 (uniqueness of limits) *Limits of sequences are unique, if* $\lim a_n = K$ *and* $\lim a_n = L$ *then* $K = L$.

Proof. Let (a_n), K and L be as stated, and ε be arbitrary. By Definition 2.1.15 there is an n_0 such that $n \geq n_0 \Rightarrow a_n \in U(K,\varepsilon)$ and $a_n \in U(L,\varepsilon)$. Thus for every ε we have that $U(K,\varepsilon) \cap U(L,\varepsilon) \neq \emptyset$. By Exercise 2.1.12, $K = L$. □

• *Robust properties of sequences.* Let $\mathbb{R}^{\mathbb{N}}$ be the set of real sequences. Any set $V \subset \mathbb{R}^{\mathbb{N}}$ is called a property of real sequences.

Definition 2.1.18 (robust property) *We say that a property* $V \subset \mathbb{R}^{\mathbb{N}}$ *is robust if for any sequences* (a_n) *and* (b_n) *such that* $a_n \neq b_n$ *for only finitely many indices* n*, the equivalence*

$$(a_n) \in V \iff (b_n) \in V$$

holds.

Exercise 2.1.19 *Which of the properties* $V_1, \dots, V_4$ *of sequences* (a_n) *are robust?* $V_1:$ (a_n) *converges,* $V_2:$ (a_n) *diverges,* $V_3:$ $\lim a_n = -\infty$ *and* $V_4:$ $a_1 = 0$.

Interestingly, the notion of robustness is itself robust, as the next exercise shows.

Exercise 2.1.20 *Prove the following.*
1. *If* $V \subset \mathbb{R}^{\mathbb{N}}$ *is robust, then so is* $\mathbb{R}^{\mathbb{N}} \setminus V$.
2. *If* $X \subset \mathcal{P}(\mathbb{R}^{\mathbb{N}})$ *is such that every* $Y \in X$ *is robust, then so is* $\bigcup X$.
3. *If* $X \subset \mathcal{P}(\mathbb{R}^{\mathbb{N}})$, $X \neq \emptyset$, *is such that every* $Y \in X$ *is robust, then so is* $\bigcap X$.

• *Two limits.* We show that $\lim \frac{1}{n} = 0$. Which is clear, for a given ε and every $n \geq n_0 \equiv \lceil 1/\varepsilon \rceil$ we have that

$$0 < \frac{1}{n} \leq \underbrace{\frac{1}{\lceil 1/\varepsilon \rceil}}_{\geq 1/\varepsilon} \leq \frac{1}{1/\varepsilon} = \varepsilon \rightsquigarrow |1/n - 0| \leq \varepsilon\,.$$

Here $\lceil a \rceil \in \mathbb{Z}$ denotes the upper integer part of a real number a, it is the smallest number $v \in \mathbb{Z}$ such that $v \geq a$. Similarly the lower integer part $\lfloor a \rfloor$ of a is the largest $v \in \mathbb{Z}$ such that $v \leq a$.

In the second example we compute that

$$\sqrt[3]{n} - \sqrt{n} \to -\infty\,.$$

Let a $c < 0$ be given. Then for every $n \geq n_0 \geq \max(\{4c^2, 2^6\})$ it holds that

$$\overbrace{\sqrt[3]{n} - \sqrt{n}}^{\text{nontrivial}} = \overbrace{n^{1/2} \cdot \underbrace{(n^{-1/6} - 1)}_{\cdots \leq -1/2}}^{\text{trivial}} \leq \underbrace{-n^{1/2}}_{\cdots \leq -2|c|} /2 \leq -2|c|/2 = c\,.$$

The first upper bracket says that in this form the limit is non-trivial, it looks like the indefinite expression $+\infty - (+\infty)$. Using a simple algebraic transformation we get the trivial form $(+\infty) \cdot (0 - 1) = -\infty$. Lower brackets show the upper bound for the enclosed expression when $n \geq n_0$.

One does not need to find the optimum value of the index n_0 as a function of ε or c. This is possible only in the simplest cases like $\lim \frac{1}{n}$. Usually we are content with any explicit value n_0 such that for $n \geq n_0$ the inequality in the definition of limit holds.

Exercise 2.1.21 *Find the limit* $\lim_{n\to\infty} \frac{\sqrt[3]{n}-\sqrt{n}}{\sqrt[4]{n}}$.

2.2 Subsequences and $\lim \sqrt[n]{n}$

• *Subsequences of sequences.* We obtain a subsequence of a sequence by omitting several terms from it so that still an infinite sequence remains. The formal definition is below. In Definition 2.2.6 we introduce so called weak subsequences.

Definition 2.2.1 (subsequence) *We say that* (b_n) *is a* <u>*subsequence*</u> *of* (a_n) *if for some* $(m_n) \subset \mathbb{N}$ *such that* $m_1 < m_2 < \dots$ *it holds for every* n *that* $b_n = a_{m_n}$. *We denote this relation of* (b_n) *and* (a_n) *by* $(b_n) \preceq (a_n)$.

If there is an m such that $(b_n) = (a_m, a_{m+1}, \dots)$, we call the subsequence (b_n) the <u>tail</u> of (a_n).

Exercise 2.2.2 *The relation* $\preceq$ *on* $\mathbb{R}^{\mathbb{N}}$ *is reflexive and transitive.*

Exercise 2.2.3 *Find sequences* $(a_n) \neq (b_n)$ *such that* $(a_n) \preceq (b_n)$ *and* $(b_n) \preceq (a_n)$.

Proposition 2.2.4 ($\preceq$ preserves limits) *Let* $(b_n) \preceq (a_n)$ *and* $\lim a_n = L$. <u>*Then*</u> $\lim b_n = L$.

Proof. This follows from Definitions 2.1.15 and 2.2.1, the sequence (m_n) in the latter definition satisfies for every n that $m_n \geq n$. □

We prove the first part of the next theorem later.

Theorem 2.2.5 (on subsequences) *Let* $(a_n) \subset \mathbb{R}$. *The following hold.*

1. *There exists a sequence* (b_n) *such that* $(b_n) \preceq (a_n)$ *and* $\lim b_n$ *exists.*
2. *The limit* $\lim a_n$ *does not exist* $\iff$ (a_n) *has two subsequences with different limits.*
3. *It is not true that* $\lim a_n = A$ $\iff$ (a_n) *has a subsequence whose limit differs from* A.

Proof. 1. We prove it later.

2. The implication $\neg \Rightarrow \neg$ follows from the last proposition. We prove the implication $\Rightarrow$. Suppose that (a_n) does not have a limit. By part 1 there is a $(b_n) \preceq (a_n)$ with $\lim b_n = B$. Since B is not a limit of (a_n), there exists an ε and a sequence $(c_n) \preceq (a_n)$ such that for every n it holds that $c_n \notin U(B, \varepsilon)$. By part 1 there is a $(d_n) \preceq (c_n)$ such that $\lim d_n = K$. Then $(d_n) \preceq (a_n)$ and $K \neq B$. Hence (b_n) and (d_n) are the required subsequences.

3. The implication $\neg \Rightarrow \neg$ again follows from the last proposition. We prove implication $\Rightarrow$. Suppose that $\neg(\lim a_n = A)$. Hence there is an ε and a $(b_n) \preceq (a_n)$ such that for every n one has that $b_n \notin U(A, \varepsilon)$. By part 1 there is a $(c_n) \preceq (b_n)$ such that $\lim c_n = B$. Then $(c_n) \preceq (a_n)$ and $B \neq A$. Hence (c_n) is the required subsequence. □

Thus it is always possible to prove that the given sequence does not have a limit by presenting two subsequences of it with different limits. For example,

$$(a_n) = ((-1)^n) = (-1, 1, -1, 1, -1, \dots)$$

does not have a limit because $(1, 1, \dots) \preceq (a_n)$ and $(-1, -1, \dots) \preceq (a_n)$ and these constant subsequences have different limits 1 and -1.

Definition 2.2.6 (weak subsequence) *Let $(a_n) \subset \mathbb{R}$. We say that (b_n) is a <u>weak subsequence</u> of (a_n) if there is an $(m_n) \subset \mathbb{N}$ such that $\lim m_n = +\infty$ and it holds for every n that $b_n = a_{m_n}$. Then we write $(b_n) \preceq^* (a_n)$.*

Exercise 2.2.7 *Generalize Proposition 2.2.4: if $(b_n) \preceq^* (a_n)$ and $\lim a_n = L$ then also $\lim b_n = L$.*

Exercise 2.2.8 *If $(b_n) \preceq^* (a_n)$ then there is a (c_n) such that $(c_n) \preceq (b_n)$ and $(c_n) \preceq (a_n)$.*

• *The limit of n-th root of n.* A limit is <u>nontrivial</u> if it involves limit of an indefinite (arithmetic nor power) expression. Else it is <u>trivial</u>. For example, the limits $\lim(2^n + 3^n)$ and $\lim \frac{4}{5n-3}$ are trivial, but $\lim(2^n - 3^n)$ and $\lim \frac{4n+7}{5n-3}$ are nontrivial. Nontrivial limits can be often computed by transforming them algebraically to trivial limits, like in the above $\lim(\sqrt[3]{n} - \sqrt{n})$. The next limit of $n^{1/n}$ is nontrivial because $n \to +\infty$, $\frac{1}{n} \to 0$ and $(+\infty)^0$ is an indefinite power expression (we describe these precisely later). As we see in a minute, the exponent prevails and $n^{1/n} \to 1$. We revisit limits of arithmetic-power expressions in *MA 1⁺*. We remind the well known <u>binomial theorem</u>.

Exercise 2.2.9 (binomial theorem) *For every $a, b \in \mathbb{R}$ and $n \in \mathbb{N}_0$,*

$$(a+b)^n = \textstyle\sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j} \,.$$

Here for $j \in \mathbb{N}$ we have $\binom{n}{j} \equiv \frac{n(n-1)\dots(n-j+1)}{j!}$ and $\binom{n}{0} \equiv 1$.

Proposition 2.2.10 ($n^{1/n} \to 1$) *It is true that*

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \sqrt[n]{n} = 1\,.$$

Proof. Always $n^{1/n} \ge 1$. If $n^{1/n} \not\to 1$, there would be a $c > 0$ and a sequence $(n_i) \subset \mathbb{N}$ such that $2 \le n_1 < n_2 < \dots$ and for every i it holds that $n_i^{1/n_i} \ge 1 + c$ (Exercise 2.2.11). Raising this inequality to the power n_i and using Exercise 2.2.9 we get for every index i that

$$\begin{aligned} n_i &\ge (1+c)^{n_i} = \textstyle\sum_{j=0}^{n_i} \binom{n_i}{j} c^j = 1 + \binom{n_i}{1}c + \binom{n_i}{2}c^2 + \dots + \binom{n_i}{n_i}c^{n_i} \\ &\ge \binom{n_i}{2}c^2 = \tfrac{n_i(n_i-1)}{2} \cdot c^2\,. \end{aligned}$$

Then for every i we have

$$n_i \ge \tfrac{n_i(n_i-1)}{2} \cdot c^2 \leadsto 1 + 2/c^2 \ge n_i\,.$$

This is impossible, the sequence $n_1 < n_2 < \dots$ is not bounded from above. □

Exercise 2.2.11 *Explain why there is the sequence* (n_i) *with the stated property.*

2.3 Five theorems

We present and prove five theorems on existence of limits of real sequences, namely Theorem 2.3.3, 2.3.9, 2.3.15, 2.3.20 and 2.3.25.

• *Monotonicity and boundedness.* We say that a sequence (a_n) weakly increases (respectively weakly decreases) if for every n it holds that $a_n \le a_{n+1}$ (respectively $a_n \ge a_{n+1}$). It increases (respectively decreases) if for every n it holds that $a_n < a_{n+1}$ (respectively $a_n > a_{n+1}$). It is monotone if it weakly decreases or weakly increases. It is strictly monotone if it decreases or increases.

We say that (a_n) is bounded from above if there is a c such that for every n it holds that $a_n \le c$. Reverting the inequality we get boundedness from below. A sequence (a_n) is bounded if it is bounded both from above and from below.

Exercise 2.3.1 *A sequence* (a_n) *is bounded iff there is a* c *such that for every* n *it holds that* $|a_n| \le c$.

Exercise 2.3.2 *Which of these nine properties of sequences are robust?*

• *Limits of monotone sequences.* One can often prove that a sequence has limit by means of the following theorem and corollary.

Theorem 2.3.3 (limits of MS) *Suppose that* $(a_n) \subset \mathbb{R}$ *weakly increases, respectively weakly decreases. Then*

$$\lim a_n = \sup(\{a_n:\ n \in \mathbb{N}\}),\ \ \textit{respectively}\ \ \lim a_n = \inf(\{a_n:\ n \in \mathbb{N}\})\,.$$

The supremum and infimum are taken in the linear order $(\mathbb{R}^*, <)$.

Proof. We assume that (a_n) weakly decreases (the other case is similar) and denote the stated infimum by A $(\in \mathbb{R}^*)$. Let an ε be given. We take a $c > A$ with $c \in U(A,\varepsilon)$. By the definition of infimum there is an m such that $a_m < c$. Then for every $n \ge m$ we have $A \le a_n \le a_m < c$. By Exercise 2.1.11 also $a_n \in U(A,\varepsilon)$. Hence $\lim a_n = A$. □

A drawback of the theorem is that monotonicity is not a robust property of sequences. We fix it in the corollary.

Corollary 2.3.4 (robust generalization) *Suppose that $(a_n) \subset \mathbb{R}$ has a tail $(a_m, a_{m+1}, \dots)$ that weakly increases, respectively weakly decreases. Then*

$$\lim a_n = \sup(\{a_n :\ n \ge m\}), \ \ \textit{respectively} \ \ \lim a_n = \inf(\{a_n :\ n \ge m\})\,.$$

The supremum and infimum are again taken in the linear order $(\mathbb{R}^*, <)$.

Proof. Let (a_n) and m be as stated. It is clear that the limit of the tail $(a_m, a_{m+1}, \dots)$ equals to the limit of the whole sequence. □

Exercise 2.3.5 *The assumption in the corollary is a robust property of sequences.*

• *Limits of quasi-monotone sequences.* We generalize monotone sequences. A sequence $(a_n) \subset \mathbb{R}$ goes up (respectively goes down) if for every index n the set of indices m such that $a_m < a_n$ (respectively $a_m > a_n$) is finite. We say that (a_n) is quasi-monotone if it goes up or down.

Exercise 2.3.6 *Every monotone sequence is quasi-monotone.*

Exercise 2.3.7 *Find a quasi-monotone sequence (a_n) such that for no m the tail $(a_m, a_{m+1}, \dots)$ is monotone.*

Exercise 2.3.8 *Express quasi-monotonicity only by quantifiers, logical connectives, brackets, variables and inequalities between natural and real numbers.*

In the next theorem we use the quantities lim sup and lim inf of a sequence. They are always defined and have values in $\mathbb{R}^*$. We introduce them in the next lecture.

Theorem 2.3.9 (limits of QMS). *If (a_n) goes up, respectively down, then* $\lim a_n = \limsup a_n$, *respectively* $\lim a_n = \liminf a_n$.

Proof. Suppose that (a_n) goes up (the other case is similar), $A \equiv \limsup a_n$ and that an ε is given. Then (a_n) has a subsequence (a_{m_n}) with $\lim a_{m_n} = A$ and for every $n \ge n_0$ we have that $a_n < A + \varepsilon$. We take an n' such that $a_{m_{n'}} \in U(A,\varepsilon)$. Since (a_n) goes up, we can take an $n_1 \ge n_0$ such that for every $n \ge n_1$,

$$a_{m_{n'}} \le a_n < A + \varepsilon\,.$$

By Exercise 2.1.11 it holds that $a_n \in U(A, \varepsilon)$. Hence $\lim a_n = A$. $\Box$

Here is the robust strengthening.

Corollary 2.3.10 (robust generalization) *Suppose that $(a_n) \subset \mathbb{R}$ has a tail $(a_m, a_{m+1}, \dots)$ that goes up, respectively down. Then*

$$\lim\, a_n = \limsup a_n, \ \textit{respectively} \ \lim\, a_n = \liminf a_n\,.$$

Proof. Clearly, the limit of the tail $(a_m, a_{m+1}, \dots)$ equals $\lim a_n$ and the same holds for the limsup and liminf. $\Box$

Exercise 2.3.11 *The assumption in the corollary defines a robust property of sequences.*

Now we see that also in Theorem 2.3.3 and Corollary 2.3.4 supremum and infimum can be replaced with limsup and liminf. Quasi-monotone sequences were introduced by the British mathematician *Godfrey H. Hardy (1877–1947)*. We revisit them in *MA 1*$^+$.

• *The Bolzano–Weierstrass theorem.* We begin with an auxiliary proposition which is of interest by itself.

Proposition 2.3.12 (existence of MSS) *Every real sequence has a monotone subsequence.*

Proof. For any (a_n) we set

$$M \equiv \{n:\ \forall\, m\, \big(n \le m \Rightarrow a_n \ge a_m\big)\}\,.$$

If $M = \{m_1 < m_2 < \dots\}$ is infinite, we have the weakly decreasing subsequence (a_{m_n}). If M is finite, we take a number $m_1 > \max(M)$ (if $M = \emptyset$ then m_1 is arbitrary). Then $m_1 \notin M$ and there is a number $m_2 > m_1$ such that $a_{m_1} < a_{m_2}$. Since $m_2 \notin M$, there is an $m_3 > m_2$ such that $a_{m_2} < a_{m_3}$. And so on, we get the increasing subsequence (a_{m_n}). $\Box$

Exercise 2.3.13 *Generalize this to any* LO.

From Theorem 2.3.3 and from the previous proposition we get two corollaries. The first one is part 1 of Theorem 2.2.5. The second one is the Bolzano–Weierstrass theorem.

Corollary 2.3.14 (part 1 of Theorem 2.2.5) *Every real sequence has a subsequence with limit.*

Proof. Let $(a_n) \subset \mathbb{R}$. By the previous proposition there is a monotone $(b_n) \preceq (a_n)$. By Theorem 2.3.3 the sequence (b_n) has a limit. $\Box$

Theorem 2.3.15 (Bolzano–Weierstrass) *Every bounded real sequence has a convergent subsequence.*

Proof. Let (a_n) be bounded and $(b_n) \preceq (a_n)$ be a monotone subsequence guaranteed by Proposition 2.3.12. Then (b_n) is bounded and by Theorem 2.3.3 it has a finite limit. $\square$

Exercise 2.3.16 *Let $a \le b$ be real numbers. Then every sequence $(a_n) \subset [a, b]$ has a subsequence (a_{m_n}) such that $\lim a_{m_n} \in [a, b]$.*

Karl Weierstrass (1815–1897) was a German mathematician.

• *Cauchy sequences.* We met rational Cauchy sequences in the definition of $\mathbb{R}$. Real Cauchy sequences are defined in the same way.

Definition 2.3.17 (real Cauchy sequence) *A sequence $(a_n) \subset \mathbb{R}$ is* <u>*Cauchy*</u> *if for every ε there is an n_0 such that $m, n \ge n_0 \Rightarrow |a_m - a_n| \le \varepsilon$.*

Exercise 2.3.18 *Cauchy sequences form a robust property of sequences.*

Exercise 2.3.19 *Every Cauchy sequence is bounded.*

Theorem 2.3.20 (on Cauchy sequences) *A real sequence converges iff it is Cauchy.*

Proof. Implication $\Rightarrow$. Let $\lim a_n = a$ and an ε be given. Then for every large n one has that $|a_n - a| \le \frac{\varepsilon}{2}$. Using Exercise 2.1.2 we have for every large m and n that

$$|a_m - a_n| \le |a_m - a| + |a - a_n| \le \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon\,.$$

Hence (a_n) is Cauchy.

Implication $\Leftarrow$. Let (a_n) be Cauchy. By Exercise 2.3.19 (a_n) is bounded. By the Bolzano–Weierstrass theorem it has a convergent subsequence (a_{m_n}) with the limit a. Thus for a given ε we have for every large m and n that $|a_{m_n} - a| \le \frac{\varepsilon}{2}$ and $|a_m - a_n| \le \frac{\varepsilon}{2}$. Always $m_n \ge n$, so that again we have by the Δ-inequality that for every large n,

$$|a_n - a| \le |a_n - a_{m_n}| + |a_{m_n} - a| \le \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon\,.$$

Hence $\lim a_n = a$. $\square$

Interestingly, A.-L. Cauchy was living for several years in political exile in Prague.

Exercise 2.3.21 *Show that there is a Cauchy sequence $(a_n) \subset \mathbb{Q}$ such that* $\lim a_n \notin \mathbb{Q}$.

Thus the previous theorem does not hold in $\mathbb{Q}_{\mathrm{OF}}$. This is not surprising, we know that this ordered field is not complete

Exercise 2.3.22 *Where the completeness of* $\mathbb{R}$ *was used in the previous proof?*

• *Fekete's lemma.* We state this lemma as a theorem. It is due to the Hungarian-Israeli mathematician *Michael Fekete (1886–1957).*

Exercise 2.3.23 *"fekete" means ...?*

Exercise 2.3.24 *Solve the previous exercise by means available in 1984.*

A sequence (a_n) is superadditive, respectively subadditive if for every two indices m and n it holds that $a_{m+n} \ge a_m + a_n$, respectively $a_{m+n} \le a_m + a_n$.

Theorem 2.3.25 (Fekete's lemma) *Let* $(a_n) \subset \mathbb{R}$ *and let* $M \equiv \{\frac{a_n}{n} : n \in \mathbb{N}\}$. *If* (a_n) *is superadditive, respectively subadditive, then* $\lim \frac{a_n}{n} = \sup(M)$, *respectively* $\lim \frac{a_n}{n} = \inf(M)$. *The supremum and infimum are taken in the* LO $(\mathbb{R}^*, <)$.

Proof. Suppose that (a_n) is superadditive (for subadditive sequence we argue similarly) and that an ε is given. We take a $c < \sup(M)$ such that $c \in U(\sup(M), \varepsilon)$. Then there is an m such that $\frac{a_m}{m} > c$. Let $n \ge m$. We express n as $n = km + l$, where $k \in \mathbb{N}$, $l \in \mathbb{N}_0$ and $0 \le l < m$ (so we divide n by m with remainder). From the superadditivity of (a_n) it follows that

$$\frac{a_n}{n} \ge \frac{ka_m}{km+l} + \frac{a_l}{n} = \frac{a_m/m}{1+l/km} + \frac{a_l}{n}\,.$$

For $n \to \infty$ also $k \to \infty$, thus $1 + \frac{l}{km} \to 1$, $\frac{a_l}{n} \to 0$ and for every δ it holds for large n that $\frac{a_n}{n} \ge \frac{a_m}{m} - \delta$. Thus there is an $n_0 \ge m$ such that for every $n \ge n_0$ we have that $c < \frac{a_n}{n} \le \sup(M)$. Using Exercise 2.1.11 we get that $\frac{a_n}{n} \in U(\sup(M), \varepsilon)$. Hence $\frac{a_n}{n} \to \sup(M)$. □

Four exercises show applications of Theorem 2.3.25 in extremal combinatorics.

Exercise 2.3.26 (*abba*-free words) *Let* $f(n) = l$ *be the maximum length of a word* $u = a_1a_2\dots a_l$ *over* $[n]$ *such that* (i) *for every* $i \in [l-1]$ *one has that* $a_i \ne a_{i+1}$ *and* (ii) *u does not contain the pattern abba, which means that there are no four indices* $1 \le k_1 < k_2 < k_3 < k_4 \le l$ *such that* $a_{k_1} = a_{k_4} \ne a_{k_2} = a_{k_3}$. *Show with the help of Fekete's lemma that the limit* $L \equiv \lim \frac{f(n)}{n}$ *exists*

It is not too hard to show that $f(n) = 3n - 2$, so that $L = 3$.

Exercise 2.3.27 (*abab*-free words) *The same problem for abab-free words.*

Here similarly $f(n) = 2n - 1$, so that $L = 2$.

Exercise 2.3.28 (*aabb*-free words) *The same problem for aabb-free words.*

An arithmetic progression X $(\subset \mathbb{Z})$ of length $k \in \mathbb{N}$ is any set

$$X = \{a + jd: \; j \in [k]\}, \; a \in \mathbb{Z} \;\&\; d \in \mathbb{N}\,.$$

Exercise 2.3.29 (function $r_k(n)$) *For $k, n \in \mathbb{N}$ let $r_k(n)$ be the maximum size of a set $A \subset [n]$ containing no arithmetic progression of length k. Show by Fekete's lemma that for every k the finite limit $L_k \equiv \lim_{n\to\infty} \frac{r_k(n)}{n}$ $(\in [0, 1])$ exists.*

Clearly, $L_1 = L_2 = 0$. For $k \geq 3$ the next famous theorem holds.

Theorem 2.3.30 (E. Szemerédi, 1975) *Also for every $k \geq 3$ it holds that $L_k = 0$.*

The proof is quite complicated, see [38] or [40]. *Endre Szemerédi (1940)* is a Hungarian mathematician.

Chapter 3

Arithmetic of limits. AK series

This chapter, based on the lecture

`https://kam.mff.cuni.cz/~klazar/MAI24_pred3.pdf`

of March 7, 2024, starts with Section 3.1 on relations between limits of sequences and arithmetic operations. The main result is Theorem 3.1.2 on limits of sums, products and ratios. In two supplements, Propositions 3.1.4 and 3.1.6, we consider situations not covered in the theorem, and explain indefiniteness of indefinite expressions; proofs are moved to exercises.

In Section 3.2 we give in Proposition 3.2.2 an example of a computation of the limit of a recurrent sequence. In Proposition 3.2.4 we determine the limits $\lim q^n$. Section 3.3 is devoted to relations between limits of sequences and the order of real numbers. Our Theorem 3.3.1 on this relation is stronger than the standard one. Likewise the squeeze Theorem 3.3.10. Section 3.4 introduces $\liminf$ and $\limsup$ of a real sequence. Theorem 3.4.4 shows that these quantities are always defined and Theorem 3.4.6 gives their basic properties.

In the final Section 3.5 infinite series come on the stage. We introduce them in a novel way, as AK series that properly generalize finite sums. We wrote about AK series in *Some highlights*. We return to infinite series in the next lecture and in *MA 1*$^+$.

3.1 Arithmetic of limits of sequence

• *Arithmetic of limits of sequences.* Recall that (a_n), (b_n) and (c_n) denote real sequences, that always $\varepsilon, \delta, \theta > 0$ and that $\mathbb{R}^*$ are extended reals, with elements denoted by A, B, K and L. Recall the computing with infinities in Definition 2.1.3. The next theorem is the main tool for computing limits. But first an exercise.

Exercise 3.1.1 (variants of the Δ-inequality) *For every $a, b \in \mathbb{R}$ it holds that*

$$|a+b| \geq |a| - |b| \text{ and } |a-b| \geq |a| - |b|\,.$$

Theorem 3.1.2 (AL of sequecnes) *Let $(a_n), (b_n) \subset \mathbb{R}$, $\lim a_n = K$ and $\lim b_n = L$. Then $\lim(a_n + b_n) = K + L$, $\lim a_n b_n = KL$ and $\lim \frac{a_n}{b_n} = \frac{K}{L}$, if the expression on the right side is not indeterminate.*

Proof. Sum. Let $K, L \in \mathbb{R}$ and an ε be given. For every large n we have that $|a_n - K| \leq \frac{\varepsilon}{2}$ and $|b_n - L| \leq \frac{\varepsilon}{2}$. By the Δ-inequality it holds for these n that

$$|(a_n + b_n) - (K + L)| \leq |a_n - K| + |b_n - L| \leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon\,.$$

Hence $a_n + b_n \to K + L$.

Let $K = L = \pm\infty$ and an ε be given. For every large n the numbers a_n and b_n have the same sign as K and $|a_n|, |b_n| \geq \frac{1}{2\varepsilon}$. Thus for these n the sum $a_n + b_n$ has the same sign as K and $|a_n + b_n| = |a_n| + |b_n| \geq \frac{1}{2\varepsilon} + \frac{1}{2\varepsilon} = \frac{1}{\varepsilon}$. Hence $a_n + b_n \to K + L = K = L$.

Let $K = \pm\infty$, $L \in \mathbb{R}$ and an ε be given. For every large n the number a_n has the same sign as K, $|a_n| \geq \frac{1}{\varepsilon} + |L| + 1$ and $|b_n - L| \leq 1$, thus $|b_n| \leq |L| + 1$. For these n the sum $a_n + b_n$ has the same sign as K and, by Exercise 3.1.1, $|a_n + b_n| \geq |a_n| - |b_n| \geq \frac{1}{\varepsilon} + |L| + 1 - |L| - 1 = \frac{1}{\varepsilon}$. Hence $a_n + b_n \to K + L = K$. The cases $K \in \mathbb{R}$ and $L = \pm\infty$ follow from the commutativity of addition.

Product. Let $K, L \in \mathbb{R}$ and an $\varepsilon \leq 1$ be given. For every large n one has that $|a_n - K| \leq \frac{\varepsilon}{2|L|+1}$, thus $|a_n| \leq |K| + 1$, and $|b_n - L| \leq \frac{\varepsilon}{2|K|+2}$. By the Δ-inequality it holds for these n that

$$\begin{aligned} |a_n b_n - KL| &\leq |a_n| \cdot |b_n - L| + |L| \cdot |a_n - K| \\ &\leq (|K| + 1) \cdot \tfrac{\varepsilon}{2|K|+2} + |L| \cdot \tfrac{\varepsilon}{2|L|+1} \leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon\,. \end{aligned}$$

Hence $a_n b_n \to KL$.

Let $K = \pm\infty$, $L = \pm\infty$ and an ε be given. For every large n the number a_n has the same sign as K, b_n as L and $|a_n|, |b_n| \geq \frac{1}{\sqrt{\varepsilon}}$. Thus for these n the product $a_n b_n$ has the same sign as KL and $|a_n b_n| = |a_n| \cdot |b_n| \geq \frac{1}{\sqrt{\varepsilon}} \cdot \frac{1}{\sqrt{\varepsilon}} = \frac{1}{\varepsilon}$. Hence $a_n b_n \to KL$.

Let $K = \pm\infty$, $L \in \mathbb{R} \setminus \{0\}$ ($L = 0$ yields an indefinite expression) and let an ε be given. For every large n the number a_n has the same sign as K, $|a_n| \geq \frac{2}{\varepsilon|L|}$ and $|b_n - L| \leq \frac{|L|}{2}$, thus $|b_n| \geq \frac{|L|}{2}$. So for these n the product $a_n b_n$ has the same sign as KL and $|a_n b_n| = |a_n| \cdot |b_n| \geq \frac{2}{\varepsilon|L|} \cdot \frac{|L|}{2} = \frac{1}{\varepsilon}$. Hence $a_n b_n \to KL$. The cases $K \in \mathbb{R} \setminus \{0\}$ and $L = \pm\infty$ follow from the commutativity of multiplication.

Ratio. Let $K \in \mathbb{R}$, $L \in \mathbb{R} \setminus \{0\}$ ($L = 0$ yields an indefinite expression) and an ε be given. For every large n it holds that $|a_n - K| \leq \min(\{1, \frac{\varepsilon L^2}{4(|L|+1)}\})$ and $|b_n - L| \leq \min(\{1, \frac{\varepsilon L^2}{4(|K|+1)}, \frac{|L|}{2}\})$, thus $|a_n| \leq |K| + 1$, $|b_n| \leq |L| + 1$ and $|b_n| \geq \frac{|L|}{2}$. By the Δ-inequality it holds for these n that

$$\left|\tfrac{a_n}{b_n} - \tfrac{K}{L}\right| = \left|\tfrac{a_n L - b_n K}{b_n L}\right| \leq \tfrac{|a_n| \cdot |L - b_n| + |b_n| \cdot |a_n - K|}{|b_n| \cdot |L|} \leq \tfrac{\varepsilon L^2/4 + \varepsilon L^2/4}{L^2/2} = \varepsilon\,.$$

Hence $\frac{a_n}{b_n} \to \frac{K}{L}$.

Let $K = \pm\infty$, $L \in \mathbb{R} \setminus \{0\}$ and an ε be given. For every large n the number a_n has the same sign as K, $|a_n| \geq \frac{|L|+1}{\varepsilon}$ and $|b_n - L| \leq 1$, thus $|b_n| \leq |L|+1$. So for these n the ratio $\frac{a_n}{b_n}$ has the same sign as $\frac{K}{L}$ and $\left|\frac{a_n}{b_n}\right| = \frac{|a_n|}{|b_n|} \geq \frac{|L|+1}{\varepsilon(|L|+1)} = \frac{1}{\varepsilon}$. Hence $\frac{a_n}{b_n} \to \frac{K}{L}$.

Let $K \in \mathbb{R}$, $L = \pm\infty$ and an ε be given. For every large n one has that $|a_n - K| \leq 1$, thus $|a_n| \leq |K| + 1$, and $|b_n| \geq \frac{|K|+1}{\varepsilon}$. For these n it holds that $\left|\frac{a_n}{b_n} - 0\right| = \left|\frac{a_n}{b_n}\right| = \frac{|a_n|}{|b_n|} \leq \frac{|K|+1}{(|K|+1)/\varepsilon} = \varepsilon$. Hence $\frac{a_n}{b_n} \to \frac{K}{L} = 0$. □

If $\lim a_n = K$, $\lim b_n = L$ and $\frac{K}{L}$ is not an indefinite expression, then $L \neq 0$. There are only finitely many n with $b_n = 0$ and the corresponding undefined ratios $\frac{a_n}{b_n}$ may be ignored or defined arbitrarily.

• *Supplements to* AL *of sequences.* The previous theorem is not a complete description of arithmetic of limits of sequences. Even if one the two sequences does not have a limit, their sum sequence or product sequence or ratio sequence still may have a unique limit. We present six scenarios of this kind and leave proofs for them as an exercise.

Exercise 3.1.3 *Prove the following proposition.*

Proposition 3.1.4 (supplement 1 to AL) *Let $(a_n), (b_n) \subset \mathbb{R}$. Even if (a_n) possibly does not have a limit, the following implications hold.*

1. *If (a_n) is bounded and $L \equiv \lim b_n = \pm\infty$ <u>then</u> $\lim(a_n + b_n) = L$.*
2. *If (a_n) is bounded and $\lim b_n = 0$ <u>then</u> $\lim a_n b_n = 0$.*
3. *If for every large n it holds that $a_n \geq c > 0$ and $L \equiv \lim b_n = \pm\infty$ <u>then</u> $\lim a_n b_n = L$.*
4. *If (a_n) is bounded and $\lim b_n = \pm\infty$ <u>then</u> $\lim \frac{a_n}{b_n} = 0$.*
5. *If for every large n we have that $a_n \geq c > 0$ and $b_n > 0$, and $\lim b_n = 0$ <u>then</u> $\lim \frac{a_n}{b_n} = +\infty$.*
6. *If for every large n one has that $0 < a_n \leq c$ and $L \equiv \lim b_n = \pm\infty$ <u>then</u> $\lim \frac{b_n}{a_n} = L$.*

In parts 3 and 5 we may also have $a_n \leq c < 0$, in part 6 we may have $c \leq a_n < 0$ and in part 5 we may have $b_n < 0$. It is not hard to state precisely and prove these modifications.

If the expression $K+L$ or KL or $\frac{K}{L}$ is indefinite, then the corresponding limit is not uniquely determined. We show on several examples what may happen.

Exercise 3.1.5 *Prove the following proposition.*

Proposition 3.1.6 (supplement 2 to AL) *Let $A \in \mathbb{R}^*$. Then there exist eight sequences $(a_{n,i}), (b_{n,i}) \subset \mathbb{R}$, $i \in [4]$, such that the following hold.*

1. $\lim a_{n,1} = +\infty$, $\lim b_{n,1} = -\infty$ *and* $\lim(a_{n,1} + b_{n,1}) = A$.
2. $\lim a_{n,2} = 0$, $\lim b_{n,2} = +\infty$ *and* $\lim a_{n,2} \cdot b_{n,2} = A$.
3. $\lim a_{n,3} = \lim b_{n,3} = 0$ *and* $\lim \frac{a_{n,3}}{b_{n,3}} = A$.
4. $\lim a_{n,4} = \pm\infty$, $\lim b_{n,4} = \pm\infty$ *and* $\lim \frac{a_{n,4}}{b_{n,4}} = A$.

3.2 Limits of recurrent sequences

In this section we compute as an example the limit of a recurrent sequence. We return to recurrent sequences in *MA 1*$^+$.

• *The limit of a recurrent sequence.* We begin with the well known inequality between the arithmetic and geometric mean.

Exercise 3.2.1 (AG inequality) *For every two real numbers $a, b \ge 0$ it is true that $\frac{a+b}{2} \ge \sqrt{ab}$.*

Proposition 3.2.2 (one recurrent limit) *Let $(a_n) \subset \mathbb{Q}$ be given by $a_1 \equiv 1$ and for $n \ge 2$ by $a_n \equiv \frac{a_{n-1}}{2} + \frac{1}{a_{n-1}}$. Then $\lim a_n = \sqrt{2}$.*

Proof. First we show that $a_2 \ge a_3 \ge \cdots \ge 0$, so that (a_n) converges by Corollary 2.3.4. Clearly, always $a_n > 0$ and a_n is defined for every n. For $n \ge 2$ we get by the AG inequality that

$$a_n = \frac{a_{n-1}}{2} + \frac{1}{a_{n-1}} \ge 2\sqrt{\frac{a_{n-1}}{2} \cdot \frac{1}{a_{n-1}}} = \sqrt{2}\,.$$

Then for $n \ge 3$ it holds that $a_{n-1} \ge a_n$ iff $\frac{a_{n-1}}{2} \ge \frac{1}{a_{n-1}}$ which means iff $a_{n-1} \ge \sqrt{2}$ which is true. Let $a \equiv \lim a_n \ge \sqrt{2}$. By the AL of sequences and limits of subsequences one has that

$$a = \lim a_n = \frac{\lim a_{n-1}}{2} + \frac{1}{\lim a_{n-1}} = \frac{a}{2} + \frac{1}{a}\,.$$

Hence $\frac{a}{2} = \frac{1}{a}$, $a^2 = 2$ and $a = \sqrt{2}$. □

In order that computation of this kind be correct we always have to show that the limit of the recurrent sequence exists. For example, the recurrent sequence (a_n) given by $a_1 \equiv 1$ and for $n \ge 2$ by $a_n \equiv -a_{n-1}$ does *not* have the limit $\lim a_n = 0$, although in $\mathbb{R}^*$ the equation $L = -L$ has the only solution $L = 0$. The sequence is alternating, $(a_n) = (1, -1, 1, -1, \dots)$, and has no limit.

• *Limits of geometric sequences.* A geometric sequence is the sequence of powers

$$(q^n) = (q,\, q^2,\, q^3,\, \dots),\ q \in \mathbb{R}\,.$$

Exercise 3.2.3 (one more supplement to AL) *For every* $(a_n) \subset \mathbb{R}$ *it holds that*

$$\lim a_n = 0 \iff \lim |a_n| = 0\,.$$

Proposition 3.2.4 (limits of geometric sequences) *Let* $q \in \mathbb{R}$. *Then for* $|q| < 1$ *we have that* $\lim q^n = 0$*, for* $q = 1$ *that* $\lim q^n = 1$*, for* $q > 1$ *that* $\lim q^n = +\infty$ *and for* $q \le -1$ *the limit does not exist.*

Proof. Let $|q| < 1$. By Exercise 3.2.3 we can assume that $q \in [0, 1)$. Then (q^n) weakly decreases and is bounded from below. By Corollary 2.3.10 it has the limit $L \equiv \lim q^n \in [0, +\infty)$. Since $q^n = q \cdot q^{n-1}$, we have the equation $L = q \cdot L$. Thus $L = \frac{0}{1-q} = 0$. For $q = 1$ we have the constant sequence $(1, 1, \dots)$. Let $q > 1$. By the first part of this proposition and by part 5 of Proposition 3.1.4 one has that $\lim q^n = \lim \frac{1}{(1/q)^n} = \frac{1}{0^+} = +\infty$. For $q \le -1$ the sequence (q^n) does not have a limit because its subsequences with odd, respectively even, indices have different limits. □

3.3 Limits of sequences versus order

We consider interaction of limits of sequences with the LO $(\mathbb{R}^*, <)$. We revisit and investigate this topic further in *MA 1*$^+$.

• *One standard theorem in stronger clothes.* If we can compare terms in two sequences then we can compare their limits, and vice versa. But which terms in the sequences are being compared?

Theorem 3.3.1 (limits versus order 1) *Let* $(a_n), (b_n) \subset \mathbb{R}$ *be sequences with* $\lim a_n = K$ *and* $\lim b_n = L$. *Then the following hold.*
1. *If* $K < L$ *then for some* n_0 *for every two, not necessarily equal, indices* $m, n \ge n_0$ *it holds that* $a_m < b_n$.
2. *If for every* n_0 *there exist indices* m *and* n *such that* $m, n \ge n_0$ & $a_m \ge b_n$*, then* $K \ge L$.

Proof. 1. Let $K < L$. By Exercise 2.1.12 there is an ε such that $U(K, \varepsilon) < U(L, \varepsilon)$. By the definition of limit we have an n_0 such that if $m, n \ge n_0$ then $a_m \in U(K, \varepsilon)$ and $b_n \in U(L, \varepsilon)$. So $m, n \ge n_0 \Rightarrow a_m < b_n$.

2. Logic tells us that the implication $\varphi \Rightarrow \psi$ is equivalent to the reversal $\neg\psi \Rightarrow \neg\varphi$ (Exercise 1.2.3). The reversal of the implication in part 1 is exactly part 2. □

Version 2 of the theorem will concern limits of functions. One of the mysteries in teaching mathematical analysis is that (elsewhere) the theorem is stated unnecessarily weakly, in the form of part 1 as: if $K < L$ then there is an n_0 such that for every $n \ge n_0$ it holds that $a_n < b_n$. Or in the form of part 2 as: if $a_n \le b_n$ for every $n \ge n_0$ then $K \le L$. I was teaching these weakish forms of the theorem for many years.

Exercise 3.3.2 *Explain why part* 1 *of Theorem 3.3.1 which allows distinct indices m and n is stronger than the form with $m = n$.*

Corollary 3.3.3 (limits preserve $\le$) *Suppose that $(a_n), (b_n) \subset \mathbb{R}$ are sequences such that $\lim a_n = K$, $\lim b_n = L$ and that for every n_0 there exist indices m and n satisfying $m, n \ge n_0$ & $a_m \le b_n$. Then $K \le L$.*

Proof. This an equivalent restatement of Theorem 3.3.1, basically part 2. □

In general strict inequalities are not preserved in limits as they may turn in equalities. This is another reason why non-strict equalities are safer than the strict ones.

Exercise 3.3.4 *Find convergent sequences (a_n) and (b_n) such that for every m and n it holds that $a_m < b_n$ but at the same time $\lim a_n = \lim b_n$.*

Next proposition further strengthens Theorem 3.3.1.

Exercise 3.3.5 *Prove the next strengthening and state it in the form of part 2.*

Proposition 3.3.6 (strengthening Thm 3.3.1) *If $\lim a_n = K$, $\lim b_n = L$ and $K < L$, then there exists an n_0 and real numbers $a < b$ such that for every $m, n \ge n_0$ we have that $a_m \le a < b \le b_n$.*

• *Intervals.* For $a, b \in \mathbb{R}$ we denote the closed interval with the endpoints a and b as $I(a, b)$:

$$I(a, b) \equiv [a, b] \text{ if } a \le b \text{ and } I(a, b) \equiv [b, a] \text{ if } a \ge b\,.$$

A set $M \subset \mathbb{R}$ is convex if for every $a, b \in M$ the whole $I(a, b) \subset M$. For example, every neighborhood $U(A, \varepsilon)$ is convex.

Proposition 3.3.7 (on intervals) *Convex sets of real numbers are exactly the sets $\emptyset$, the singletons $\{a\}$ for $a \in \mathbb{R}$, the whole $\mathbb{R}$ and for reals $a < b$ the intervals (a, b), $(-\infty, a)$, $(a, +\infty)$, $(a, b]$, $[a, b)$, $[a, b]$, $(-\infty, a]$ & $[a, +\infty)$.*

Proof. The transitivity of $<$ shows that all stated sets are convex. We show that there are no other real convex sets. Let $X \subset \mathbb{R}$ be a convex set different from $\emptyset$, $\mathbb{R}$ and $\{a\}$ and let $a \in \mathbb{R} \setminus X$. Convexity of X implies that $a \in H(X)$ (upper bounds of X) or $a \in D(X)$ (lower bounds of X). We discuss only the former case as the latter can be reduced to it by reversing inequalities.

So let $H(X) \ne \emptyset$. We set $b \equiv \sup(X)$. Let $D(X) = \emptyset$. If $b \in X$ then $X = (-\infty, b]$. If $b \notin X$ then $X = (-\infty, b)$. Let $D(X) \ne \emptyset$. Then we set $c \equiv \inf(X)$, clearly $c < b$. If $b \notin X$ and $c \notin X$ then $X = (c, b)$. If $b \notin X$ and $c \in X$ then $X = [c, b)$. If $b \in X$ and $c \notin X$ then $X = (c, b]$. Finally if $b \in X$ and $c \in X$ then $X = [c, b]$. □

Exercise 3.3.8 *Are there nonempty finite intervals?*

Definition 3.3.9 (nontrivial intervals) *Nontrivial intervals are the nonempty and non-singleton intervals.*

• *The squeeze theorem.* Czech textbooks of mathematical analysis call the following theorem the "two cops theorem", the idea being that two cops lead between them the suspect to the common limit.

Theorem 3.3.10 (squeeze theorem 1) *If* $\lim a_n = \lim b_n = a$*,* $(c_n) \subset \mathbb{R}$ *and for every large* n *it holds that* $c_n \in I(a_n, b_n)$*, then also* $\lim c_n = a$*.*

Proof. Let the sequences (a_n), (b_n) and (c_n) be as stated and let an ε be given. Then for every large n it holds that $a_n, b_n \in U(a, \varepsilon)$. By the convexity of $U(a, \varepsilon)$ it holds for every large n that $c_n \in I(a_n, b_n) \subset U(a, \varepsilon)$. Hence $c_n \to a$. □

Exercise 3.3.11 *For infinite limits one cop suffices: if* $\lim a_n = -\infty$ *and for every large* n *one has that* $b_n \le a_n$*, then* $\lim b_n = -\infty$*. Similarly for* $+\infty$*.*

3.4 Limes inferior and limes superior

These Latin terms mean "the lowest limit" and "the highest limit", respectively.

• *Limit points of sequences.* These are limits of subsequences.

Definition 3.4.1 (limit point) *An element* $A \in \mathbb{R}^*$ *is a limit point of* (a_n) *if* $A = \lim b_n$ *for some* $(b_n) \preceq (a_n)$*. We denote the set of limit points of* (a_n) *by* $L(a_n)$ *(*$\subset \mathbb{R}^*$*).*

For example, $(a_n) \equiv (n - 1 + (-1)^n n + \frac{1}{n})$ has $L(a_n) = \{-1, +\infty\}$.

Exercise 3.4.2 *Every real sequence has at least one limit point.*

• *Limes inferior and limes superior of a sequence.* We already revealed that these are, respectively, the smallest and the largest limit point of it.

Definition 3.4.3 (liminf and limsup) *Let* (a_n) *be a real sequence. We define* $\liminf a_n \equiv \min(L(a_n))$ *and* $\limsup a_n \equiv \max(L(a_n))$*. The minimum and maximum are taken in the* LO $(\mathbb{R}^*, <)$*.*

We show that these minima and maxima always exist.

Theorem 3.4.4 (liminf and limsup exist) *For every* $(a_n) \subset \mathbb{R}$*,* $L(a_n) \neq \emptyset$ *and the set* $L(a_n)$ *has in the* LO $(\mathbb{R}^*, <)$ *both minimum and maximum.*

Proof. Let $(a_n) \subset \mathbb{R}$. Then $L(a_n) \neq \emptyset$ by Exercise 3.4.2. We show that $\max(L(a_n))$ exists, the minimum is treated similarly. Let $A \equiv \sup(L(a_n))$, taken in the LO $(\mathbb{R}^*, <)$ (by Proposition 2.1.6 this supremum exists). We show that $A \in L(a_n)$. If $A = -\infty$ then $L(a_n) = \{-\infty\}$ and we are done, $A \in L(a_n)$.

Let $A > -\infty$. Then there is a real sequence $(b_n) \subset L(a_n)$ such that $\lim b_n = A$. For $A < +\infty$ and for $A = +\infty \notin L(a_n)$ it follows from the definition of supremum. For $A = +\infty \in L(a_n)$ we are done. Since every number b_n is the limit of a subsequence of (a_n), it is easy to construct a subsequence (a_{m_n}) such that for every n we have $a_{m_n} \in U(b_n, 1/n)$. Then $\lim a_{m_n} = \lim b_n = A$ and $A \in L(a_n)$. □

It is clear that if $\lim a_n$ exists then $L(a_n) = \{\lim a_n\}$. We obtain some more properties of liminfs and limsups.

Proposition 3.4.5 ($\liminf \stackrel{?}{=} \limsup$) *Always* $\liminf a_n \le \limsup a_n$. *The equality holds iff the limit* $\lim a_n$ *exists. Then* $\liminf a_n = \limsup a_n = \lim a_n$.

Proof. Let $(a_n) \subset \mathbb{R}$. The inequality is obvious as $\liminf a_n = \min(L(a_n))$ and $\limsup a_n = \max(L(a_n))$. If it is an equality then $L(a_n)$ is a singleton and (a_n) does not have two subsequences with different limits. Then by part 2 of Theorem 2.2.5 the sequence (a_n) has a limit which equals to $\liminf a_n$ and $\limsup a_n$. If $\liminf a_n \ne \limsup a_n$, the sequence (a_n) has two subsequences with different limits and $\lim a_n$ does not exist. □

Theorem 3.4.6 (properties of liminfs and limsups) *Let* $(a_n) \subset \mathbb{R}$, $A \equiv \liminf a_n$ *and* $B \equiv \limsup a_n$. *Then the following hold.*
1. *If* $A = -\infty$ *then for every* $c < 0$ *it holds for infinitely many* n *that* $a_n \le c$. *If* $A = +\infty$ *then* $\lim a_n = +\infty$.
2. *If* $A \in \mathbb{R}$ *then for every* ε *it holds for infinitely many* n *that* $a_n \le A + \varepsilon$, *and for every* $n \ge n_0$ *that* $a_n \ge A - \varepsilon$.
3. *If* $B = +\infty$ *then for every* $c > 0$ *it holds for infinitely many* n *that* $a_n \ge c$. *If* $B = -\infty$ *then* $\lim a_n = -\infty$.
4. *If* $B \in \mathbb{R}$ *then for every* ε *it holds for infinitely many* n *that* $a_n \ge B - \varepsilon$, *and for every* $n \ge n_0$ *that* $a_n \le B + \varepsilon$.

Proof. We prove parts 1 and 2. Proofs for 3 and 4 are the exercise below.

1. Let $A = -\infty$. Then for some $(b_n) \preceq (a_n)$ it holds that $\lim b_n = -\infty$ and the claim holds by the definition of the limit $-\infty$. Let $A = +\infty$. Then $L(a_n) = \{+\infty\}$ and by Proposition 3.4.5 we have that $\lim a_n = +\infty$.

2. Let $A \in \mathbb{R}$ and an ε be given. Since there is a $(b_n) \preceq (a_n)$ with $\lim b_n = A$, we have for infinitely many n that $a_n \le A + \varepsilon$. If we had $a_n < A - \varepsilon$ for infinitely many n, a $(c_n) \preceq (a_n)$ would exist with $\lim c_n$ less than $A - \varepsilon$. This is impossible because $A = \min(H(a_n))$. Hence for every $n \ge n_0$ it holds that $a_n \ge A - \varepsilon$. □

As an example of use of liminfs and limsups we mention a number-theoretic estimate. One can show that the function $\tau(n)$ that counts divisors of n, for example $\tau(6) = |\{1, 2, 3, 6\}| = 4$, is such that

$$\limsup \frac{\log(\tau(n))}{(\log 2)(\log n)/(\log\log n)} = 1 \;\text{ while }\; \liminf \tau(n) = 2\,.$$

Exercise 3.4.7 *Prove the latter bound.*

The corresponding sequence begin as

$$(\tau(n)) = (1,\, 2,\, 2,\, 3,\, 2,\, 4,\, 2,\, 4,\, 3,\, 4,\, 2,\, 6,\, 4,\, \dots)\,.$$

Exercise 3.4.8 *Prove parts 3 and 4 of the last theorem.*

Exercise 3.4.9 *Find a sequence* (a_n) *such that* $L(a_n) = \mathbb{R}^*$.

Exercise 3.4.10 *Is there a sequence* (a_n) *such that* $L(a_n) = [-1, 0) \cup (0, 1]$*?*

Exercise 3.4.11 *Find* $\liminf a_n$ *and* $\limsup a_n$ *for* $a_n \equiv n(1 + (-1)^n)$.

3.5 AK series

Important applications of limits are rigorous treatments of infinite sums.

• AK *series*. Our goal is to extend, in a commutative and associative way, finite sums $\sum_{x\in X} r_x$ with finite index sets X and $r_x \in \mathbb{R}$ — we take them for granted — to sums with countable index sets. The paradoxes in Section 1.1 show that straightforward infinite addition is neither commutative nor associative. AK series are a remedy for this and are interesting on their own.

Definition 3.5.1 (AK series) *Any map* $r\colon X \to \mathbb{R}$ *defined on an at most countable set* X *with the property that for some* $c \ge 0$ *for every finite set* $Y \subset X$ *it holds that* $\sum_{x\in Y} |r(x)| \le c$ *is called an* <u>*AK series*</u> *(absolutely convergent series). We write it as* $\sum_{x\in X} r_x$*, with* $r_x \equiv r(x)$*.*

We use the acronym AK because AC is already taken by the axiom of choice. In the definition, "absolutely" refers not so much to the absolute value as to independence of sums of AK series on the order of summation. Let

$$\mathfrak{S} \equiv \{r :\; r\colon X \to \mathbb{R} \text{ is an AK series}\}$$

(see Exercise 1.2.2 and Definition 1.2.14) be the (proper) <u>class of AK series</u>. Clearly, any finite sum $\sum_{x\in X} r_x$ with finite index set X is an AK series. If $\sum_{x\in X} r_x \in \mathfrak{S}$ and $Y \subset X$ then $\sum_{x\in Y} r_x \in \mathfrak{S}$ as well.

Exercise 3.5.2 *Prove it.*

We say that $\sum_{x\in Y} r_x$ is a <u>subseries</u> of $\sum_{x\in X} r_x$.

Let $R = \sum_{x\in X} r_x \in \mathfrak{S}$. If X is finite we define the <u>sum</u> $S(R)$ $(\in \mathbb{R})$ of R to be simply the sum of of the real numbers r_x in this finite list. For countable X the sum $S(X)$ arises in the following theorem.

Theorem 3.5.3 (sum of AK series) *Let $R = \sum_{x\in X} r_x \in \mathfrak{S}$ with infinite X. Then for any bijection $f\colon \mathbb{N} \to X$ there exists a unique finite limit*

$$S(R) \equiv \lim_{n\to\infty} \sum_{i=1}^{n} r_{f(i)} \quad (\in \mathbb{R})$$

that is independent of f. $S(R)$ is the sum of R.

Proof. Let $f, g\colon \mathbb{N} \to X$ be bijections and an ε be given. We set

$$c \equiv \sup(\{\textstyle\sum_{x\in Y} |r_x| :\ Y \subset X \text{ and is finite}\})\,.$$

We take a finite set $Y \subset X$ such that $c - \varepsilon \le \sum_{x\in Y} |r_x| \le c$. Then for every finite set $Z \subset X \setminus Y$ one has that $\sum_{x\in Z} |r_x| \le \varepsilon$. We take an n_0 such that $f[\,[n_0]\,], g[\,[n_0]\,] \supset Y$. Then for every $m, n \ge n_0$ it holds that

$$\big|\textstyle\sum_{i=1}^{m} r_{f(i)} - \sum_{i=1}^{n} r_{g(i)}\big| \le \sum_{x\in Z_m} |r_x| + \sum_{x\in W_n} |r_x| \le \varepsilon + \varepsilon = 2\varepsilon\,,$$

because Z_m and W_n are some finite subsets of $X \setminus Y$. The choice $g = f$ shows that the sequence $\big(\sum_{i=1}^{n} r_{f(i)} :\ n \in \mathbb{N}\big)$ is Cauchy. By Theorem 2.3.20 it has a finite limit. The choice $g \ne f$ shows that the limit is independent of the bijection f. □

The map $S\colon \mathfrak{S} \to \mathbb{R}$, which is a class, sends any AK series to its sum.

Proposition 3.5.4 (approximating sums) *Let $R = \sum_{x\in X} r_x \in \mathfrak{S}$. Then for every ε there is a finite subset $Y = Y(\varepsilon, R)$ of X such that for every finite set Z with $Y \subset Z \subset X$ it holds that $|\sum_{x\in Z} r_x - S(R)| \le \varepsilon$.*

Proof. For finite X it is trivial, $Y \equiv X$. For infinite X we take any bijection $f\colon \mathbb{N} \to X$, and then an n_0 such that for $n \ge n_0$ it holds that $|\sum_{i=1}^{n} r_{f(i)} - S(R)| \le \frac{\varepsilon}{2}$ and that for every finite set $M \subset \mathbb{N} \setminus [n_0]$ it holds that $\sum_{i\in M} |r_{f(i)}| \le \frac{\varepsilon}{2}$. We set $Y \equiv f[\,[n_0]\,]$. Let Z with $Y \subset Z \subset X$ be a finite set. Then

$$|\textstyle\sum_{x\in Z} r_x - S(R)| \le |\sum_{x\in Y} r_x - S(R)| + \sum_{x\in Z\setminus Y} |r_x| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

□

• AK *series correctly extend finite sums.* We show that sums of countable AK series are commutative and associative. Thus AK series form a correct extension of finite sums. Their commutativity is an immediate corollary of Theorem 3.5.3.

Corollary 3.5.5 (commutativity of $S(\cdot)$) *Let $R = \sum_{x\in X} r_x \in \mathfrak{S}$ with infinite X. Then for every bijection $f\colon X \to X$ it holds that $R' \equiv \sum_{x\in X} r_{f(x)} \in \mathfrak{S}$ and $S(R') = S(R)$.*

We prove the associativity.

Theorem 3.5.6 (associativity of $S(\cdot)$) *Let $R = \sum_{x\in X} r_x \in \mathfrak{S}$ and Y be a partition of X. For $Z \in Y$ let $r_Z \equiv S(R_Z)$ where $R_Z \equiv \sum_{x\in Z} r_x$. Then $R' \equiv \sum_{Z\in Y} r_Z \in \mathfrak{S}$ and $S(R') = S(R)$.*

Proof. Let R and Y be as stated. First we show that $R' \in \mathfrak{S}$. Let

$$c \equiv \sup(\{\textstyle\sum_{x\in Z} |r_x| :\ Z \subset X \text{ and is finite}\})$$

and let $Y' = \{Z_1, \dots, Z_n\} \subset Y$ be a finite set. For any $i \in [n]$ we apply Proposition 3.5.4 and take finite sets $Z_i' \equiv Y(2^{-i}, R_{Z_i})$ $(\subset Z_i)$. We set $Z_0 \equiv Z_1' \cup \dots \cup Z_n'$. Then

$$\textstyle\sum_{Z\in Y'} |r_Z| \le \sum_{i=1}^n |r_{Z_i} - \sum_{x\in Z_0} r_x| + |\sum_{x\in Z_0} r_x| \le 1 + c\,.$$

Hence $\sum_{Z\in Y} r_Z \in \mathfrak{S}$.

Let an ε be given. We show that $|S(R)-S(R')| \le \varepsilon$. We use Proposition 3.5.4 and take finite sets $X' \equiv Y(\frac{\varepsilon}{3}, R)$ $(\subset X)$, $\{Z_1, \dots, Z_n\} \equiv Y(\frac{\varepsilon}{3}, R')$ $(\subset Y,\, n \in \mathbb{N})$ and $Z_i' \equiv Y(2^{-i}\frac{\varepsilon}{3}, R_{Z_i})$ $(\subset Z_i)$, $i \in [n]$. Let $X_0 \equiv X' \cup Z_1' \cup \dots \cup Z_n'$ and $Z_n'' \equiv (X' \setminus \bigcup_{i=1}^n Z_i') \cup Z_n'$. Then $|S(R) - S(R')|$ is at most

$$\textstyle|S(R) - \sum_{x\in X_0} r_x| + \sum_{i=1}^{n-1} |\sum_{x\in Z_i'} r_x - r(Z_i)| + |\sum_{x\in Z_n''} r_x - r(Z_n)| +$$
$$\textstyle+ |\sum_{i=1}^n r(Z_i) - S(R')| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,.$$

So $|S(R) - S(R')| \le \varepsilon$. This holds for any ε and therefore $S(R) = S(R')$. □

Exercise 3.5.7 *Why do we define the set Z_n'' in the above way?*

Exercise 3.5.8 *Where was associativity of finite sums used in the previous proof?*

• *Congruence of* AK *series.* $R = \sum_{x\in X} r_x$ and $R' = \sum_{x\in Y} s_x$ in $\mathfrak{S}$ are congruent, in symbols $R \sim R'$, if there is a bijection $f\colon X \to Y$ such that for every $x \in X$ we have that $r_x = s_{f(x)}$.

Exercise 3.5.9 *Show that $\sim$ is an equivalence relation on $\mathfrak{S}$. (Equivalence relation on a class is defined in the same way as on a set.)*

Exercise 3.5.10 *If $R, R' \in \mathfrak{S}$ are congruent then $S(R) = S(R')$.*

• *Scalar multiple, binary sum and product of* AK *series.* We introduce three operations on $\mathfrak{S}$ and begin with the scalar multiple. This is actually a system of unary operations on $\mathfrak{S}$, indexed by $\mathbb{R}$.

Exercise 3.5.11 *Prove the following proposition.*

Proposition 3.5.12 (scalar multiple) *For $a \in \mathbb{R}$ and $R = \sum_{x\in X} r_x \in \mathfrak{S}$ let $aR \equiv \sum_{x\in X} a r_x$. Then $aR \in \mathfrak{S}$ and $S(aR) = aS(R)$. We call aR the scalar multiple of R (by a).*

Exercise 3.5.13 *If $a \in \mathbb{R}$ and $R \sim R'$ then $aR \sim aR'$.*

The other two operations on $\mathfrak{S}$ are binary.

Theorem 3.5.14 (binary sum) *For $R = \sum_{x\in X} r_x$ and $R' = \sum_{y\in Y} s_y$ in $\mathfrak{S}$ we define $Z \equiv X \times \{0\} \cup Y \times \{1\}$ and*

$$R + R' \equiv \sum_{z\in Z} t_z\,,$$

where $t_z \equiv r_x$ if $z = (x, 0)$ and $t_z \equiv s_y$ if $z = (y, 1)$. Then $R + R' \in \mathfrak{S}$ and $S(R + R') = S(R) + S(R')$. We call $R + R'$ the binary sum of R and R'.

Proof. We show that $R + R' \in \mathfrak{S}$. Let c be a constant witnessing that $R \in \mathfrak{S}$ and $R' \in \mathfrak{S}$, and let $W \subset Z$ be a finite set. Then $W = (X' \times \{0\}) \cup (Y' \times \{1\})$, where $X' \subset X$ and $Y' \subset Y$ are finite sets, and

$$\sum_{z\in W} |t_z| = \sum_{x\in X'} |r_x| + \sum_{y\in Y'} |s_y| \le c + c = 2c\,.$$

Hence $R + R' \in \mathfrak{S}$.

We prove that $S(R+R') = S(R)+S(R')$; our argument does not distinguish finite and infinite index sets. Let $r \equiv S(R)$, $s \equiv S(R')$ and $t \equiv S(R + R')$, and let an ε be given. We show that $|t - (r + s)| \le \varepsilon$. We use Proposition 3.5.4 and take finite sets $X' \equiv Y(\frac{\varepsilon}{3}, R)$ $(\subset X)$, $Y' \equiv Y(\frac{\varepsilon}{3}, S)$ $(\subset Y)$ and $Z \equiv Y(\frac{\varepsilon}{3}, R+S)$ $(\subset X \times \{0\} \cup Y \times \{1\})$. We take finite sets X'' and Y'' such that $X' \subset X'' \subset X$, $Y' \subset Y'' \subset Y$ and $Z \subset X'' \times \{0\} \cup Y'' \times \{1\} \equiv W$. Then $|t - (r + s)|$ is at most

$$|t - \sum_{z\in W} t_z| + |\sum_{x\in X''} r_x - r| + |\sum_{y\in Y''} s_y - s| \le \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon\,.$$

This holds for every ε and $t = r + s$. □

We could call this operation also the *disjoint union* of two AK series.

Exercise 3.5.15 *If $Q \sim Q'$ and $R \sim R'$ then $Q + R \sim Q' + R'$.*

Theorem 3.5.16 (product) *For $R = \sum_{x\in X} r_x$ and $R' = \sum_{y\in Y} s_y$ in $\mathfrak{S}$ let*

$$R \cdot R' \equiv \sum_{(x,\,y)\in X\times Y} r_x s_y\,.$$

Then $R \cdot R' \in \mathfrak{S}$ and $S(R \cdot R') = S(R)S(R')$. We call $R \cdot R'$ the product of R and R'.

Proof. First we show that $R \cdot R' \in \mathfrak{S}$. We take a constant c witnessing that R and R' are AK series. Let $Z \subset X \times Y$ be a finite set. We take finite sets $X' \subset X$ and $Y' \subset Y$ such that $Z \subset X' \times Y'$. Then

$$\sum_{(x,\,y)\in Z} |r_x s_y| \le \sum_{x\in X'} |r_x| \cdot \sum_{y\in Y'} |s_y| \le c \cdot c = c^2\,.$$

Hence $R \cdot R' \in \mathfrak{S}$.

We prove that $S(R \cdot R') = S(R)S(R')$; our argument does not distinguish finite and infinite index sets. Let $r \equiv S(R)$, $s \equiv S(R')$ and $t \equiv S(R \cdot R')$, and let an $\varepsilon \le 1$ be given. We show that $|t - rs| \le \varepsilon$. We use Proposition 3.5.4 and take finite sets $X' \equiv Y(\frac{\varepsilon}{3(|s|+1)}, R)$ $(\subset X)$, $Y' \equiv Y(\frac{\varepsilon}{3(|r|+1)}, S)$ $(\subset Y)$ and $Z \equiv Y(\frac{\varepsilon}{3}, R \cdot R')$ $(\subset X \times Y)$. We take finite sets X'' and Y'' such that $X' \subset X'' \subset X$, $Y' \subset Y'' \subset Y$ and $Z \subset X'' \times Y''$. Then $|t - rs|$ is at most

$$\begin{aligned}&|t - \textstyle\sum_{(x,\,y)\in X''\times Y''} r_x s_y| + |\sum_{x\in X''} r_x \cdot \sum_{y\in Y''} s_y - rs| \le \\ &\le \tfrac{\varepsilon}{3} + |(r+\delta)(s+\theta) - rs| \ \text{ where } |\delta| \le \tfrac{\varepsilon}{3(|s|+1)} \ \text{ and } |\theta| \le \tfrac{\varepsilon}{3(|r|+1)}\,.\end{aligned}$$

Hence $|t - rs| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. This holds for every ε and $t = rs$. □

Exercise 3.5.17 *If $Q \sim Q'$ and $R \sim R'$ then $Q \cdot R \sim Q' \cdot R'$.*

• *A semiring of factorized* AK *series.* We show that AK series, when factorized by $\sim$, form a semiring with respect to binary sum and product. We define $\mathfrak{T} \equiv \mathfrak{S}/\sim$ and call this class the factorized AK series. By a semiring we mean the ring structure on a set or a class with the existence of additive inverses dropped. Let $0_{\mathfrak{T}} \equiv \emptyset$ be the empty AK series and $1_{\mathfrak{T}} \equiv \left[\sum_{x\in\{1\}} 1\right]_\sim$ be the AK series that have just a single summand 1.

Theorem 3.5.18 (semiring $\mathfrak{T}_{\mathrm{SR}}$) *The structure*

$$\mathfrak{T}_{\mathrm{SR}} \equiv \langle \mathfrak{T},\, 0_{\mathfrak{T}},\, 1_{\mathfrak{T}},\, +,\, \cdot \rangle$$

is a semiring. In more details, $+$ and $\cdot$ are commutative and associative operations on $\mathfrak{T}$, the element $0_{\mathfrak{T}}$ and $1_{\mathfrak{T}}$ in $\mathfrak{T}$ is neutral to $+$ and $\cdot$, respectively, and $\cdot$ is distributive to $+$.

Proof. Exercises 3.5.15 and 3.5.17 show that $+$ and $\cdot$ indeed operate on the class $\mathfrak{T}$. We show that $+$ is commutative. Let $R = \sum_{x\in X} r_x$ and $R' = \sum_{x\in Y} s_x$ be in $\mathfrak{S}$, and let $Z \equiv X \times \{0\} \cup Y \times \{1\}$ and $W \equiv X \times \{1\} \cup Y \times \{0\}$. We take the bijection $f\colon Z \to W$ that sends $(x, 0) \in Z$ to $(x, 1) \in W$, and $(y, 1) \in Z$ to $(y, 0) \in W$. Then we see that for $R + R' = \sum_{z\in Z} t_z$ and $R' + R = \sum_{z\in W} t'_z$ it holds for every $z \in Z$ that $t_z = t'_{f(z)}$ because $t_z = r_x = t'_{f(z)}$ if $z = (x, 0)$, and $t_z = s_y = t'_{f(z)}$ if $z = (y, 1)$. Hence $R + R' \sim R' + R$. The proof of associativity of $+$ is similar, see Exercise 3.5.19. We leave proofs of commutativity and associativity of $\cdot$ to respective Exercises 3.5.20 and 3.5.21.

Let $R = \sum_{x\in X} r_x \in \mathfrak{S}$. Since $X \times \{0\} \cup \emptyset \times \{1\} = X \times \{0\}$, the bijection sending $x \in X$ to $(x, 0) \in X \times \{0\}$ proves that $R \sim R + 0_{\mathfrak{T}}$. Similarly the bijection sending $x \in X$ to $(x, 1) \in X \times \{1\}$ proves that $R \sim R \cdot 1_{\mathfrak{T}}$. We finally show that $\cdot$ is distributive to $+$. Let $R = \sum_{x\in X} r_x$, $R' = \sum_{x\in Y} s_x$ and $R'' = \sum_{x\in Z} t_x$ be in $\mathfrak{S}$, and let $W \equiv X \times (Y \times \{0\} \cup Z \times \{1\})$ and $W' \equiv (X \times Y) \times \{0\} \cup (X \times Z) \times \{1\}$. We take the bijection $f\colon W \to W'$ that sends $(x, (y, 0)) \in W$ to $((x, y), 0) \in W'$, and $(x, (z, 1)) \in W$ to $((x, z), 1) \in W'$. Then we see that for $R \cdot (R' + R'') = \sum_{w\in W} u_w$ and $R \cdot R' + R \cdot R'' = \sum_{w\in W'} u'_w$ it holds for every $w \in W$ that

$u_w = u'_{f(w)}$ because $u_w = r_x s_y = u'_{f(w)}$ if $w = (x, (y, 0))$, and $u_w = r_x t_z = u'_{f(w)}$ if $w = (x, (z, 1))$. We get that $R \cdot (R' + R'') \sim R \cdot R' + R \cdot R''$. □

Exercise 3.5.19 *Give the bijection proving that* $R + (R' + R'') \sim (R + R') + R''$.

Exercise 3.5.20 *Give the bijection proving that* $R \cdot R' \sim R' \cdot R$.

Exercise 3.5.21 *Give the bijection proving that* $R \cdot (R' \cdot R'') \sim (R \cdot R') \cdot R''$.

In conclusion we describe interaction of scalar multiples with $+$ and $\cdot$.

Proposition 3.5.22 (on scalar multiples) *For every* $a, b \in \mathbb{R}$ *and* $T = [R]_\sim$, $T' = [R']_\sim$ *in* $\mathfrak{T}$ *it holds that* $a(T+T') = aT + aT'$, $(ab)T = a(bT)$, $(ab)(T \cdot T') = aT \cdot bT'$ *and* $1T = T$.

Proof. Let $R = \sum_{x \in X} r_x$ and $R' = \sum_{x \in Y} s_x$. The congruence $a(R + R') \sim aR + aR'$ follows from the identity bijection from the set $X \times \{0\} \cup Y \times \{1\}$ to itself. Similarly identical bijections work in the other three identities. □

Exercise 3.5.23 *What about* $(a + b)R = aR + bR$*?*

• AK *series versus classical series.* For classical series $\sum_{n=1}^{\infty} a_n$, which are sequences $(a_n) \subset \mathbb{R}$, the sum is (defined as) the limit $\lim(a_1 + \cdots + a_n)$. This is a simple definition but it does not work as needed — we saw in the first chapter that the limit may depend on the order of summands. Absolutely convergent series fix it but in the classical approach are introduced only afterwards; we decided to begin with them. Most applications of infinite series use absolutely convergent series. Another disadvantage of the classical approach to infinite series is its fixing a single index set, $\mathbb{N}$ or $\mathbb{N}_0$. This is too restrictive and indeed classical series often do not obey it and use other index sets like $\mathbb{Z}$ or $\mathbb{N} \times \mathbb{N}$. In the next chapter we apply Theorems 3.5.6 and 3.5.16 in Theorem 4.3.4 in the proof of the identity $\mathrm{e}^{x+y} = \mathrm{e}^x \cdot \mathrm{e}^y$, where $\mathrm{e}^x = \sum_{n \in \mathbb{N}_0} \frac{x^n}{n!}$. For the convenience of the reader and to be in sync with the standard syllabus we begin the next chapter with an introduction to classical infinite series.

Chapter 4

Infinite series. Elementary functions

In contrast to the last section where we introduced AK series, in the first section of this chapter, which is based on the lecture

`https://kam.mff.cuni.cz/~klazar/MAI24_pred4.pdf`

of March 14, 2024, we approach infinite series in the standard, classical way. In Proposition 4.1.10 we show that the harmonic series diverges and in Theorem 4.1.22 we deduce the formula for sum of geometric series. Riemann's Theorem 4.1.17 describes a family of series with the property that their sums may be arbitrarily changed by reordering summands. In Section 4.2 we generalize limits of real sequences to limits of functions. We prove Theorem 4.2.13 on Heine's definition of limits of functions. This theorem reduces limits of functions to limits of sequences.

In Section 4.3 Basic Elementary Functions appear: constants, $\exp x$, $\log x$, a^b, $\cos x$, $\sin x$, $\tan x$, $\cot x$ and inverses of the last four. In Theorem 4.3.4 we prove that $\exp x \cdot \exp y = \exp(x+y)$. Compared to the lecture, Sections 4.4 and 4.5 are new. We introduce Really Basic Elementary Functions. Definitions 4.4.5 and 4.4.14 precisely describe Elementary Functions as functions obtained from Basic Elementary Functions by repeated addition, multiplication, division and composition. In Definitions 4.5.1 and 4.5.6 we introduce in a novel way polynomials and rational functions.

4.1 Classical infinite series

In Section 3.5 we introduced AK series. Now we give a more standard introduction to the theory of infinite series.

• *Series in general.* An infinite series is a sequence $(a_n) \subset \mathbb{R}$. We denote it by $\sum a_n$, $\sum_{n=1}^{\infty} a_n$ or $a_1 + a_2 + \cdots$. The numbers a_n are summands of the series.

The sum of the series is the limit

$$\lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n) \;\; (\in \mathbb{R}^*)\,.$$

We denote the sum again by $\sum a_n$, $\sum_{n=1}^{\infty} a_n$ or $a_1 + a_2 + \cdots$. Terms in the sequence $(s_n) \equiv (a_1 + \cdots + a_n)$ are partial sums s_n of the series. So the sum is the limit $\lim s_n$. A series with finite sum converges, else it diverges.

Exercise 4.1.1 *Both convergence and divergence of a series is a robust property of sequences.*

The sum itself is, however, sensitive to changes of summands.

Exercise 4.1.2 *In any convergent series any change of any single summand changes the sum.*

Exercise 4.1.3 *Every series $a_1+a_2+\dots$ such that $a_n \ge 0$ for every $n \ge n_0$ has a sum and this sum is not $-\infty$. Likewise every series with almost all summands non-positive has a sum and this sum is not $+\infty$.*

Exercise 4.1.4 $\sum_{n=1}^{\infty} 1 = +\infty$.

A reordering of a series $\sum a_n$ is any series $\sum b_n$ such that there is a bijection $f\colon \mathbb{N} \to \mathbb{N}$ for which $b_n = a_{f(n)}$ for every n.

Proposition 4.1.5 (commutativity of sums) *Suppose that $\sum a_n$ has only finitely many negative summands or only finitely many positive summands. Then all reorderings of $\sum a_n$ have the same sum.*

Proof. Let $\sum a_n$ have finitely many positive summands, with indices $I \subset \mathbb{N}$. Let $f, g\colon \mathbb{N} \to \mathbb{N}$ be bijections and let $s_n \equiv \sum_{i=1}^{n} a_{f(i)}$ and $t_n \equiv \sum_{i=1}^{n} a_{g(i)}$. We take an m such that $f[\,[m]\,], g[\,[m]\,] \supset I$. Then sequences (s_n) and (t_n) weakly decrease starting from the index m, and by Corollary 2.3.4 both have a limit. It is not hard to see that for every $n_1 \ge m$ there exist $n_2, n_3 \ge m$ such that $t_{n_2} \le s_{n_1}$ and $s_{n_3} \le t_{n_1}$. Thus $\lim s_n = \lim t_n$. In the other case that $\sum a_n$ has finitely many negative summands we argue similarly. □

The necessary convergence condition of a series $\sum a_n$ is that $\lim a_n = 0$.

Proposition 4.1.6 (NCC of a series) *If $\sum a_n$ converges then $\lim a_n = 0$.*

Proof. Suppose that $s \equiv \lim s_n = \lim(a_1 + \cdots + a_n)$ is finite. Then $\lim a_n = \lim(s_n - s_{n-1}) = \lim s_n - \lim s_{n-1} = s - s = 0$. □

Thus if $\lim a_n$ does not exist or is not 0 then $\sum a_n$ diverges. For sums $\pm\infty$ NCC does not hold.

Exercise 4.1.7 *Explain the equalities in the conclusion of the last proof.*

• The *harmonic series*. It is the series $\sum \frac{1}{n}$. Although $\lim \frac{1}{n} = 0$, we show that this series diverges and has the sum $\sum \frac{1}{n} = +\infty$.

Exercise 4.1.8 *If (a_n) weakly increases and has a subsequence with the limit $+\infty$, then $\lim a_n = +\infty$.*

Exercise 4.1.9 *If the series $\sum a_n$ and $\sum b_n$ satisfy for every $n \ge n_0$ that $a_n \ge b_n$ and if the sum $\sum b_n = +\infty$, then the sum $\sum a_n = +\infty$.*

Proposition 4.1.10 (summing harmonic series) *The sum $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = +\infty$.*

Proof. We take the series $\sum b_n \equiv \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \cdots$. In general $b_{2^k} = b_{2^k+1} = \cdots = b_{2^{k+1}-1} = \frac{1}{2^{k+1}}$. For every n we have that $\frac{1}{n} \ge b_n$. Partial sums (s_n) of $\sum b_n$ increase and for $k \in \mathbb{N}_0$ it holds that $s_{2^{k+1}-1} = \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \cdots + 2^k \cdot \frac{1}{2^{k+1}} = \frac{k+1}{2}$. By Exercise 4.1.8 the sum $\sum b_n = \lim s_n = +\infty$. By Exercise 4.1.9 the sum $\sum \frac{1}{n} = +\infty$ as well. $\square$

By Proposition 4.1.5 every reordering of the harmonic series sums to $+\infty$. The partial sums $(h_n) \equiv \left(\sum_{i=1}^{n} \frac{1}{i}\right) = (1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \dots)$ $(\subset \mathbb{Q})$ are so called harmonic numbers. That $h_n \to +\infty$ was proven already in 1350 by the French medieval philosopher *Nicolas Oresme (1320 to 1325–1382)*.

Theorem 4.1.11 (asymptotics of h_n) *For $n \in \mathbb{N}$ we have that*

$$h_n = \log n + \gamma + O(1/n)\,.$$

Here $\gamma = 0.57721\dots$ is so called Euler's constant.

By $O(1/n)$ we denote the term a_n of a sequence (a_n) satisfying for some constant $c \ge 0$ for every n that $|a_n| \le c \cdot \frac{1}{n}$. We prove this theorem in lecture 14 with the help of integrals. Asymptotic notation, including the symbol $O(\cdot)$, will be introduced in Section 5.5.

Exercise 4.1.12 *Prove that $h_n \in \mathbb{N} \iff n = 1$. Hint: $m = (2l-1)2^k$.*

Exercise 4.1.13 (unsolved) *Euler's constant γ is an irrational number.*

• *Riemannian series.* In the first lecture we met the series $1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n} + \dots$ with the sum 0. By reordering its summands we changed the sum to a positive one. In the forthcoming theorem we prove that this and similar series can be reordered to have any sum. But we begin with less general changes of sums. Let $\sum_{n=1}^{\infty} a_n$ be a series. By $k_1 < k_2 < \dots$ we denote the indices n such that $a_n \ge 0$, and similarly $z_1 < z_2 < \dots$ are indices n such that $a_n < 0$. If the number of indices k_n is infinite, by Proposition 4.1.5 all reorderings of the series $\sum a_{k_n}$ have the same sum. The same holds for the eventual series $\sum a_{z_n}$.

Proposition 4.1.14 (reordering to $\pm\infty$) *A series $\sum a_n$ can be reordered to have sum $-\infty$ $\iff$ the sum $\sum a_{z_n} = -\infty$. Similarly, $\sum a_n$ has a reordering with the sum $+\infty$ $\iff$ the sum $\sum a_{k_n} = +\infty$.*

Proof. We prove the first equivalence and leave the second one for an exercise. Let $\sum a_{z_n} = -\infty$ (so there are infinitely many indices z_n). We may assume that there are infinitely many indices k_n. If their number is finite, by Proposition 4.1.5 every reordering of $\sum a_n$ sums to $-\infty$. We define a bijection $f\colon \mathbb{N} \to \mathbb{N}$ such that the sum $\sum a_{f(n)} = -\infty$. It is the "limit" of certain injective sequences $P_k = (m_{n,k}) \subset \mathbb{N}$, $k \in \mathbb{N}_0$, whose terms are indices k_n and z_n. We begin with the sequence $P_0 \equiv (z_n)$. We take an initial segment U_1 of P_0 such that $\sum_{i\in U_1} a_i \le -1 - a_{k_1}$. We insert in P_0 after U_1 the index k_1 and get the sequence P_1. We take an initial segment U_2 in P_1 such that $|U_2| > |U_1|$ and $\sum_{i\in U_2} a_i \le -2 - a_{k_2}$. We insert in P_1 after U_2 the index k_2 and get the sequence P_2. And so on. Since $|U_1| < |U_2| < \ldots$, the sequences P_0, P_1, P_2, $\ldots$ converge, in the obvious sense, to the sought for bijection f.

Suppose that it is not true that $\sum a_{z_n} = -\infty$. Then either the number of indices z_n is finite or the sum $\sum a_{z_n} \in \mathbb{R}$. In the former case by Proposition 4.1.5 all reorderings of $\sum a_n$ have the same sum different from $-\infty$. In the latter case there exists a c such that for every n it holds that $\sum_{i=1}^{n} a_{z_i} \ge c$. It follows that no reordering of $\sum a_n$ has the sum $-\infty$. □

Exercise 4.1.15 *Prove the second equivalence of the previous proposition by reducing it to the first one.*

Proposition 4.1.16 (reordering to no sum) *A series $\sum a_n$ has an reordering without sum $\iff$ the sum $\sum a_{z_n} = -\infty$ and the sum $\sum a_{k_n} = +\infty$.*

Proof. Let the sum $\sum a_{z_n} = -\infty$ and the sum $\sum a_{k_n} = +\infty$. We take a (finite) initial segment U_1 of (z_n) such that $\sum_{i\in U_1} a_i \le -1$. We take an initial segment V_1 of (k_n) such that $\sum_{i\in V_1} a_i \ge 2$. We take an initial segment U_2 of the sequence $(z_n) \setminus U_1$ such that $\sum_{i\in U_2} a_i \le -2$. We take an initial segment V_2 of the sequence $(k_n) \setminus V_1$ such that $\sum_{i\in V_2} a_i \ge 2$. And so on. The sequence

$$U_1 V_1 U_2 V_2 \ldots \subset \mathbb{N}$$

is a bijection f from $\mathbb{N}$ to $\mathbb{N}$ and the series $\sum a_{f(n)}$ does not have sum because both $\sum_{i=1}^{n} a_{f(i)} \le -1$ and $\sum_{i=1}^{n} a_{f(i)} \ge 1$ holds for infinitely many n.

Let for example the sum $\sum a_{z_n} \in \mathbb{R}$; if $\sum a_{k_n} \in \mathbb{R}$, we argue similarly. If $\sum a_{k_n} = +\infty$ then it follows that every reordering of $\sum a_n$ has the sum $+\infty$. If $\sum a_{k_n} \in \mathbb{R}$, the series $\sum a_n$ is an AK series and by Theorem 3.5.3 all its reorderings have the same finite sum. □

A series $\sum a_n$ is <u>Riemannian</u> if $\lim a_n = 0$, the sum $\sum a_{k_n} = +\infty$ and the sum $\sum a_{z_n} = -\infty$.

Theorem 4.1.17 (on Riemannian series) *Let $\sum a_n$ be a Riemannian series. Then for every $A \in \mathbb{R}^*$ the series $\sum a_n$ has a reordering with the sum A. The series can be also reordered so that it has no sum.*

Proof. The cases $A = \pm\infty$ are proven in Proposition 4.1.14. The case of no sum is proven in Proposition 4.1.16. Let $A = b \in \mathbb{R}$. We construct an injective sequence $(p_n) \subset \mathbb{N}$ that is a bijection from $\mathbb{N}$ to $\mathbb{N}$ and is such that the sum $\sum_{n=1}^{\infty} a_{p_n} = b$. In our construction we mark (for $n = 1, 2, \dots$) terms of the sequences (z_n) and (k_n) as *used*. We start with the terms in both sequences marked as unused. We set $p_1 \equiv k_1$ and mark the term k_1 as used. Suppose that $p_1, p_2, \dots, p_n$ are defined and let $s_n \equiv \sum_{i=1}^{n} a_{p_i}$. If $s_n \le b$ then p_{n+1} is the first unused term of (k_n) which we mark as used. If $s_n > b$ then p_{n+1} is the first unused term of (z_n) which we mark as used.

It follows that for infinitely many n both $s_n \le b$ and $s_n > b$ hold. So we exhaust both (z_n) and (k_n), and (p_n) is a bijection from $\mathbb{N}$ to $\mathbb{N}$. It holds for every $n \ge n_0$ that

$$s_n \in [b + a_{z_{i_n}},\, b + a_{k_{j_n}}]\,,$$

where $\lim i_n = \lim j_n = +\infty$. As $\lim a_n = 0$, we get that $\lim s_n = b$. □

The theorem is in essence due to the German mathematician *Bernhard Riemann (1826–1866)* whose main discovery is the link between the complex *zeta function* $\zeta(s) = \sum \frac{1}{n^s}$ and the distribution of prime numbers in $\mathbb{N}$.

• *Abscon series.* We considered much more general series of this kind in the previous Section 3.5. In this section we look standardly on a particular subset of the class $\mathfrak{S}$.

Definition 4.1.18 (abscon series) *$\sum a_n$ is an absolutely convergent series (an abskon series) if the sum $\sum |a_n| < +\infty$.*

It is clear that every abscon series is an AK series.

Theorem 4.1.19 (infinite triangle inequality) *Let $\sum a_n$ be an abscon series. Then $\sum a_n$ converges and its sum satisfies the inequality $|\sum a_n| \le \sum |a_n|$.*

Proof. Let $\sum a_n$ be an abscon series with partial sums (s_n) and let $\sum |a_n|$ have partial sums (t_n). The series $\sum a_n$ converges by Theorem 3.5.3, but we give a direct proof. By the assumption and Theorem 2.3.20 the sequence (t_n) is Cauchy. Hence for a given ε for every two large indices $m \le n$ it holds that

$$|t_n - t_m| = \big||a_{m+1}| + |a_{m+2}| + \cdots + |a_n|\big| = |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| \le \varepsilon$$

(for $m = n$ these sums are zero). By the Δ-inequality we have for the same indices $m \le n$ that

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n| \le |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| \le \varepsilon\,.$$

Hence (s_n) is Cauchy. By Theorem 2.3.20 the sequence (s_n) converges. Hence the series $\sum a_n$ converges.

By the Δ-inequality it holds for every n that $|s_n| \le t_n$, equivalently $-t_n \le s_n \le t_n$. Sending $n \to \infty$ gives by Theorem 3.3.1 that the sums satisfy inequalities $-\sum |a_n| \le \sum a_n \le \sum |a_n|$. Hence $\left|\sum a_n\right| \le \sum |a_n|$. □

Exercise 4.1.20 *Every reordering of an abscon series is an abscon series.*

Proposition 4.1.21 (on abscon series) $\sum a_n$ *is an abscon series* $\iff$ *all reorderings of it have the same finite sum.*

Proof. Implication $\Rightarrow$ follows from Theorem 3.5.3. If $\sum a_n$ is not an abscon series then the sum $\sum a_{z_n} = -\infty$ or the sum $\sum a_{k_n} = +\infty$, and by Proposition 4.1.14 some reordering does not have finite sum. □

• *Geometric series.* These are series of the form $\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + \cdots + q^n + \cdots$ with $q \in \mathbb{R}$. We call the number q the quotient of the (geometric) series.

Theorem 4.1.22 (sum of geometric series) *The sum* $\sum_{n=0}^{\infty} q^n$ *equals* $\frac{1}{1-q}$ *if* $-1 < q < 1$, $+\infty$ *if* $q \ge 1$ *and does not exist if* $q \le -1$.

Proof. For every $q \in \mathbb{R} \setminus \{1\}$ and $n \in \mathbb{N}$ we have the identity

$$s_n \equiv 1 + q + q^2 + \cdots + q^{n-1} = \tfrac{1-q^n}{1-q} = \tfrac{1}{1-q} + \tfrac{q^n}{q-1}\,.$$

So for $q < -1$ we have by the AL of sequences that $\lim s_{2n-1} = +\infty$ and $\lim s_{2n} = -\infty$. Hence $\lim s_n$ does not exist. For $q = -1$ similarly $s_{2n-1} = 1$ and $s_{2n} = 0$, the sum again does not exist. For $-1 < q < 1$ we have that $\lim q^n = 0$. Thus by the AL of sequences, $\lim s_n = \frac{1}{1-q}$. For $q = 1$ one has that $s_n = n$ and the sum $\lim s_n = +\infty$. For $q > 1$ it holds that $\lim q^n = +\infty$ and by the AL of sequences, $\lim s_n = +\infty$. □

As an application of this formula we express a periodic decimal expansion as a fraction:

$$27.272727\cdots = 27(1 + 10^{-2} + 10^{-4} + \dots) = 27 \cdot \tfrac{1}{1-10^{-2}} = \tfrac{27\cdot 100}{99} = \tfrac{300}{11}\,.$$

Exercise 4.1.23 *Let* $q \in (-1, 1)$ *and* $m \in \mathbb{Z}$. *Then the sum* $\sum_{n \ge m} q^n = \frac{q^m}{1-q}$ *(we ignore* 0^m *for* $m < 0$*).*

Exercise 4.1.24 *Which geometric series are abskon?*

• *The zeta series* $\zeta(s)$. In the definition we use real exponentiation a^b which we soon introduce in Section 4.3.

Definition 4.1.25 (series $\zeta(s)$) *For* $s \in \mathbb{R}$ *the zeta series is* $\zeta(s) \equiv \sum \frac{1}{n^s}$.

We determine the convergence of $\zeta(s)$ by the Cauchy condensation criterion.

Theorem 4.1.26 (CCC) *Let $a_1 \geq a_2 \geq \cdots \geq 0$ be real numbers. Then the series $\sum a_n$ converges $\iff$ the series $R \equiv \sum 2^n \cdot a_{2^n}$ converges.*

Proof. Suppose that R has the sum $+\infty$. Hence also the series $\frac{1}{2}R = \sum 2^{n-1} \cdot a_{2^n}$ has sum $+\infty$. We have the inequalities $a_2 \geq a_2$, $a_3 + a_4 \geq 2a_4$, $a_5 + \cdots + a_8 \geq 4a_8$, $\dots$, $\sum_{j=2^{k-1}+1}^{2^k} a_j \geq 2^{k-1} a_{2^k}$, $\dots$. Summing them we get that $\sum a_n = +\infty$.

Suppose that R converges. We have the inequalities $a_2 + a_3 \leq 2a_2$, $a_4 + \cdots + a_7 \leq 4a_4$, $\dots$, $\sum_{j=2^k}^{2^{k+1}-1} a_j \leq 2^k a_{2^k}$, $\dots$. Summing them we get that $\sum a_n$ converges. □

The proof of convergence of $\zeta(s)$ for $s > 1$ is a nice application of CCC.

Theorem 4.1.27 (convergence of $\zeta(s)$) *For $s \leq 1$ the sum $\zeta(s) = +\infty$. For $s > 1$ the zeta series converges.*

Proof. To prove the former claim is Exercise 4.1.28. Let $s > 1$. The series R in CCC for $\zeta(s)$ is

$$\sum \tfrac{2^n}{(2^n)^s} = \sum \tfrac{1}{(2^{s-1})^n}\,.$$

Since $0 < \frac{1}{2^{s-1}} < 1$, by Theorem 4.1.22 this geometric series converges. So by Theorem 4.1.26 the series $\zeta(s)$ converges. □

In *MA 1*$^+$ we show that $\zeta(2) = \frac{\pi^2}{6}$. This is due to the Swiss mathematician *Leonhard Euler (1707–1783)*. In *MA 1*$^+$ we also show that the sum $\zeta(3)$ is an irrational number; this was proved by the French mathematician *Roger Apéry (1916–1994)* in 1979.

Exercise 4.1.28 *Prove that for $s \leq 1$ the sum $\zeta(s) = +\infty$.*

Exercise 4.1.29 *For which real s does the series $\sum_{n\geq 2} \frac{1}{n(\log n)^s}$ converge?*

4.2 Limits of functions

We extend the notion of the limit of a real sequence (a_n), which is a function $a\colon \mathbb{N} \to \mathbb{R}$, to any function $f\colon M \to \mathbb{R}$ with arbitrary $M \subset \mathbb{R}$.

• *Deleted neighborhoods and limit points.* Recall ε-neighborhoods $U(A, \varepsilon)$. The deleted ε-neighborhood of $A \in \mathbb{R}^*$ is

$$P(A,\, \varepsilon) \equiv U(A,\, \varepsilon) \setminus \{A\}\,.$$

Let $M \subset \mathbb{R}$. An element $L \in \mathbb{R}^*$ is a limit point of M if

$$\forall\, \varepsilon \left(P(L,\, \varepsilon) \cap M \neq \emptyset\right).$$

The set of limit points of $M \subset \mathbb{R}$ is denoted by $L(M)$ ($\subset \mathbb{R}^*$).

Exercise 4.2.1 *Prove the following proposition.*

Proposition 4.2.2 (on limit points) *Let $M \subset \mathbb{R}$ and $A \in \mathbb{R}^*$. The next four claims are mutually equivalent.*
1. $A \in L(M)$.
2. *There is a sequence $(a_n) \subset M \setminus \{A\}$ such that $\lim a_n = A$.*
3. *There is an injective sequence $(a_n) \subset M$ such that $\lim a_n = A$.*
4. *For every $n \in \mathbb{N}$ it holds that $P(A, \frac{1}{n}) \cap M \neq \emptyset$.*

Exercise 4.2.3 *If $M \subset \mathbb{R}$ is finite then $L(M) = \emptyset$.*

Exercise 4.2.4 *If $M \subset \mathbb{R}$ is infinite then $L(M) \neq \emptyset$.*

Exercise 4.2.5 *If $b \in L(M)$ then also $b \in L(M \setminus \{b\})$.*

• *Real functions and their limits.* We introduce the following important notation for functions.

Definition 4.2.6 (functions) *For $M \subset \mathbb{R}$ we set*

$$\underline{\mathcal{F}(M)} \equiv \{(M, \mathbb{R}, f) : \ f\colon M \to \mathbb{R} \ \text{ and } \ M \subset \mathbb{R}\}\,.$$

Let $\underline{\mathcal{R}} \equiv \bigcup_{M \subset \mathbb{R}} \mathcal{F}(M)$. For $f \in \mathcal{F}(M)$ we define $\underline{Z(f)} \equiv \{b \in M : \ f(b) = 0\}$ $(\subset M)$. Recall that the definition domain of any $f\colon X \to Y$ is $M(f) = X$.

Thus for $M \subset \mathbb{R}$ we denote by $\mathcal{F}(M)$ the set of functions with the definition domain M and range $\mathbb{R}$, and $\mathcal{R}$ is the set of all such functions for all M. By $Z(f)$ we denote the set of zeros of f.

Definition 4.2.7 (limit of a function) *Suppose that $f \in \mathcal{R}$, $A \in L(M(f))$ and that $L \in \mathbb{R}^*$. If for every ε there is a δ such that*

$$f[P(A, \delta)] \subset U(L, \varepsilon)\,, \tag{$*$}$$

we write $\lim_{x \to A} f(x) = L$ and say that the function $\underline{f \text{ has in } A \text{ the limit } L}$.

Due to our definition of the image of a set by a function it suffices to write just $f[P(A, \delta)]$; we do not have to write $f[P(A, \delta) \cap M(f)]$. The limit does not depend on the value $f(A)$ and the function f even need not be defined in A. If $A = \pm\infty$ then it in fact cannot be defined in A. For a sequence $(a_n) \subset \mathbb{R}$, which is a function $a\colon \mathbb{N} \to \mathbb{R}$, it clearly holds that $\lim_{x \to +\infty} a(x) = \lim a_n$.

Exercise 4.2.8 *Besides $+\infty$, what other limit points does the set $\mathbb{N}$ $(\subset \mathbb{R})$ have?*

For $A = a \in \mathbb{R}$ & $L = b \in \mathbb{R}$ we can write the relation that $\lim_{x\to a} f(x) = b$ as

$$\forall\,\varepsilon\,\exists\,\delta\,\big(x \in M(f) \wedge 0 < |x-a| \le \delta \Rightarrow |f(x) - b| \le \varepsilon\big)$$

— recall that we like more non-strict inequalities than strict ones. It should be stressed that if $f \in \mathcal{F}(M)$ and $A \notin L(M)$ then $\lim_{x\to A} f(x)$ is not defined. Then for some δ one has that $P(A,\delta) \cap M = \emptyset$ and $f[P(A,\delta)] = \emptyset$. Then inclusion $(*)$ in Definition 4.2.7 would hold for every L and every ε. The existence of the limit $\lim_{x\to A} f(x)$ always means that $A \in L(M(f))$.

Proposition 4.2.9 (locality of limits) *If $f, g \in \mathcal{R}$, $A \in \mathbb{R}^*$ and there is a θ such that $f = g$ on $P(A,\theta)$ then $\lim_{x\to A} f(x) = \lim_{x\to A} g(x)$, if one side is defined.*

Proof. This is immediate from Definition 4.2.7 because we can take the δ in it such that $\delta \le \theta$. Then $P(A,\delta) \subset P(A,\theta)$ and $f[P(A,\delta)] = g[P(A,\delta)]$. □

Here "locality" means "1ocalness", in the sense of the "principle of locality" [31] in physics. Later we will see that also continuity and derivatives are local.

Proposition 4.2.10 (uniqueness of limits) *Limits of functions are unique, if $\lim_{x\to K} f(x) = L$ and $\lim_{x\to K} f(x) = L'$ then $L = L'$.*

Proof. For every ε there is a δ such that the nonempty (!) set $f[P(K,\delta)]$ is contained both in $U(L,\varepsilon)$ and $U(L',\varepsilon)$. Thus $\forall\varepsilon\big(U(L,\varepsilon) \cap U(L',\varepsilon) \ne \emptyset\big)$. Exercise 2.1.12 gives that $L = L'$. □

We show that limits of restrictions are equal to limits of original functions.

Proposition 4.2.11 (limits of restrictions) *Let $f \in \mathcal{F}(M)$, X be any set, $A \in L(X \cap M)$ and let $\lim_{x\to A} f(x) = L$. Then $\lim_{x\to A}(f \mid X)(x) = L$.*

Proof. Let an ε be given. Then there is a δ such that $f[P(A,\delta)] \subset U(L,\varepsilon)$. The inclusion $P(A,\delta) \cap (X \cap M) \subset P(A,\delta) \cap M$ and the definition of restriction give the inclusions

$$(f \mid X)[P(A,\,\delta)] \subset f[P(A,\,\delta)] \subset U(L,\,\varepsilon)\,.$$

Hence $\lim_{x\to A}(f \mid X)(x) = L$. □

This is the first result of several forthcoming ones which concern interaction of limits of functions with an operation on $\mathcal{R}$. Later we consider interaction with composition and with the arithmetic operations of addition, multiplication and division. Interaction with inverting functions will be investigated in *MA 1*$^+$.

Exercise 4.2.12 *Give an example of a function $f \in \mathcal{F}(M)$ and a set X such that $\lim_{x\to A} f(x)$ does not exist but $\lim_{x\to A}(f \mid X)(x)$ exists (in particular we have $A \in L(X \cap M)$).*

• *Heine's definition of the limit of a function.* We noted that limits of sequences are particular cases of limits of functions. The German mathematician *Eduard Heine (1821–1881)* showed how to go in the other way and reduce limits of functions to limits of sequences.

Theorem 4.2.13 (Heine's definition of LF) *Let $f \in \mathcal{F}(M)$ and $K \in L(M)$. Then $\lim_{x \to K} f(x) = L \iff$ for every sequence $(a_n) \subset M \setminus \{K\}$ with $\lim a_n = K$ we have that $\lim f(a_n) = L$.*

Proof. The implication $\Rightarrow$. Suppose that $\lim_{x \to K} f(x) = L$, $(a_n) \subset M \setminus \{K\}$ and has the limit K, and that an ε is given. Then there is a δ such that for every $x \in P(K, \delta) \cap M$ we have $f(x) \in U(L, \varepsilon)$. For this δ there is an n_0 such that for every $n \geq n_0$ we have $a_n \in P(K, \delta) \cap M$. Hence $n \geq n_0 \Rightarrow f(a_n) \in U(L, \varepsilon)$ and $f(a_n) \to L$.

The reversal $\neg \Rightarrow \neg$. Suppose that it is not true that $\lim_{x \to K} f(x) = L$. Then there is an ε such that for every δ there is a $b = b(\delta) \in P(K, \delta) \cap M$ with $f(b) \notin U(L, \varepsilon)$. For every $n \in \mathbb{N}$ we set $\delta = \frac{1}{n}$ and choose a point $b_n \equiv b(\frac{1}{n})$ in $P(K, \frac{1}{n}) \cap M$ such that $f(b_n) \notin U(L, \varepsilon)$. Then $(b_n) \subset M \setminus \{K\}$ and $\lim b_n = K$, but $\neg\big(\lim f(b_n) = L\big)$. The right side of the equivalence does not hold. □

In the proof of the reversal $\neg \Rightarrow \neg$ we used the axiom of choice; we explain it in *MA 1*$^+$.

Exercise 4.2.14 *How did we use it?*

• *A few limits of functions.* We compute one limit. Using the identities $x - y = \frac{x^2 - y^2}{x+y}$ a $\frac{x}{y} = \frac{1}{y/x}$ we see that $\lim_{x \to +\infty} \big(\sqrt{x + \sqrt{x}} - \sqrt{x}\big) = \lim_{x \to +\infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x}} + \sqrt{x}}$ $= \lim_{x \to +\infty} \frac{1}{\sqrt{1 + 1/\sqrt{x}} + 1} = \frac{1}{\sqrt{1 + 1/(+\infty)} + 1} = \frac{1}{1+1} = \frac{1}{2}$.

Exercise 4.2.15 *Compute the following limits.*
1. $\lim_{x \to -\infty} \frac{x}{\sqrt{1 + x^2 - 1}}$,
2. $\lim_{x \to +\infty} \frac{1}{\sqrt{1 + x} - \sqrt{x}}$,
3. $\lim_{x \to 0} \frac{1}{x}$ *and*
4. $\lim_{x \to -\infty} \frac{1}{x}$.

4.3 Basic Elementary Functions

We introduce five important subsets of $\mathcal{R}$: Basic Elementary Functions (BEF), Elementary Functions (EF), Really Basic Elementary Functions (RBEF), Polynomials (POL) and Rational Functions (RAC).

Definition 4.3.1 (BEF) *Basic Elementary Functions, abbreviated BEF, are the constant functions (constants) $k_c(x)$ for $c \in \mathbb{R}$ and the functions $\exp x$, $\log x$, a^x for $a > 0$, x^b for $b \in \mathbb{R}$, 0^x, x^m for $m \in \mathbb{Z}$, $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\arcsin x$, $\arccos x$, $\arctan x$ and $\operatorname{arccot} x$.*

Now we define them. Note that x^b and x^m mean functions of different types, depending on whether the exponent is real or integral.

• *Constant functions or constants* are the functions $k_c\colon \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$, with the only value $k_c(x) = c$. Instead of $k_c(x)$, we often write just k_c or even only c.

Exercise 4.3.2 *How many constants $k_c(x)$ are there?*

• *The exponential function* $\exp x = \exp(x) = \mathrm{e}^x\colon \mathbb{R} \to \mathbb{R}$ is for $x \in \mathbb{R}$ given by the sum $\exp x \equiv \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (here $0^0 \equiv 1$).

Exercise 4.3.3 *For every $x \in \mathbb{R}$ the series* $\exp x$ *is an abscon series, hence an* AK *series.*

Using AK series we prove the exponential identity.

Theorem 4.3.4 (exponential identity) *For every real numbers x and y it holds that* $\exp(x + y) = \exp(x) \cdot \exp(y)$.

Proof. Let $x, y \in \mathbb{R}$. Since $\exp x$ is an AK series, using Theorems 3.5.16 and 3.5.6 we get that the product of sums $\exp x \cdot \exp y$ equals

$$\sum_{m\in\mathbb{N}_0} \frac{x^m}{m!} \cdot \sum_{n\in\mathbb{N}_0} \frac{y^n}{n!} = \sum_{(m,\,n)\in\mathbb{N}_0^2} \frac{x^m}{m!} \cdot \frac{y^n}{n!} = \sum_{k=0}^{\infty} \sum_{\substack{m,\,n\in\mathbb{N}_0 \\ m+n=k}} \frac{x^m}{m!} \cdot \frac{y^n}{n!}\,.$$

Due to an algebraic rearrangement and Exercise 2.2.9 the last sum equals

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{m=0}^{k} \binom{k}{m} x^m y^{k-m} = \sum_{k=0}^{\infty} \frac{1}{k!}(x + y)^k = \exp(x + y)\,.$$

□

Usually this identity is proved by means of a theorem on classical series due to the Polish mathematician *Franz (Franciszek) Mertens (1840–1927)*, which we state here without proof.

Theorem 4.3.5 (F. Mertens) *Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be series with respective sums a & b and let*

$$c_n \equiv \sum_{i=0}^{n} a_i b_{n-i},\quad n \in \mathbb{N}_0\,.$$

Suppose that at least one of the two series is abscon. Then $\sum_{n=0}^{\infty} c_n$ converges and has the sum ab.

We list some more properties of $\exp x$.

Exercise 4.3.6 *Prove parts 1–3 in the following proposition.*

Proposition 4.3.7 (properties of e^x) *The exponential function has the following properties.*

1. $\exp 0 = 1$, *and for every* $x \in \mathbb{R}$ *it holds that* $\exp x > 0$ *and* $\exp(-x) = \frac{1}{\exp x}$.

2. *For every real* $x < y$ *it holds that* $\exp x < \exp y$.

3. *We have the limits* $\lim_{x \to -\infty} \exp x = 0$ *and* $\lim_{x \to +\infty} \exp x = +\infty$.

4. *The function* $\exp$ *is a bijection from* $\mathbb{R}$ *to* $(0, +\infty)$.

Part 4 will be proven later in Corollary 6.3.3.

• *Euler's number* e is the sum $\mathrm{e} \equiv \exp 1 = \sum_{n \ge 0} \frac{1}{n!} = 2 + \frac{1}{2!} + \frac{1}{3!} + \cdots = 2.71828\cdots$.

Exercise 4.3.8 *Show that* e *is irrational. Hint: multiply the equality* $\sum_{j \ge 0} \frac{1}{j!} = \frac{n}{m}$ *by* $m!$.

• *(Natural)* *logarithm* $\log\colon (0, +\infty) \to \mathbb{R}$. It is the inverse to the exponential function, $\log \equiv \exp^{-1}$. We obtain its properties by inverting the properties of the exponential.

Exercise 4.3.9 *Prove the following proposition.*

Proposition 4.3.10 (properties of $\log x$) *Logarithm has the following properties.*

1. $\log 1 = 0$, *for every real* $x, y > 0$ *it holds that* $\log(xy) = \log x + \log y$ *and if* $x < y$ *then* $\log x < \log y$.

2. *We have the limits* $\lim_{x \to 0} \log x = -\infty$ *and* $\lim_{x \to +\infty} \log x = +\infty$.

3. *Logarithm is a bijection from* $(0, +\infty)$ *to* $\mathbb{R}$.

• *Real exponentiation* a^b. Here a is the base and b is the exponent. It looks like a single bivariate function, but for our purposes we introduce two (by inclusion) incomparable families of univariate functions. The first family uses the relation $a^b \equiv \exp(b \log a)$ and its extension by $\lim_{x \to -\infty} \exp x = 0$. The second family is based on iterated multiplication.

Definition 4.3.11 (a^b analytically) *We define three systems of functions.*
1. *For any* $a > 0$ *we have the function* $a^x \equiv \exp(x \log a)$*; it is in* $\mathcal{F}(\mathbb{R})$.
2. *For any* $b > 0$ *we have the function* $x^b \equiv \exp(b \log x) \cup \{(0, 0)\}$*; it is in* $\mathcal{F}([0, +\infty))$. *For any* $b \le 0$ *we have the function* $x^b \equiv \exp(b \log x)$*; it is in* $\mathcal{F}((0, +\infty))$.
3. *The function* $0^x \equiv k_0(x) \mid (0, +\infty)$.

This definition leaves 0^0 undefined and always $a^b \ge 0$. Odd roots, that is $\sqrt[3]{x} = x^{1/3}$, $\sqrt[5]{x} = x^{1/5}$, etc. are sometimes defined for every $x \in \mathbb{R}$. Then, for example, $\sqrt[3]{-8} = -2$. We do not allow this. Note the different definition domains in part 2 in the cases $b > 0$ and $b \le 0$.

Definition 4.3.12 (a^b algebraically) *For $m \in \mathbb{Z}$ we define functions x^m. For any $m > 0$ we have the function $x^m \equiv x \cdot x \cdot \ldots \cdot x$, with m factors x; it is in $\mathcal{F}(\mathbb{R})$. We set $x^0 \equiv k_1(x)$; it is in $\in \mathcal{F}(\mathbb{R})$. For any $m < 0$ we have the function $x^m \equiv \frac{1}{x^{-m}} = k_1(x)/x^{-m}$; it is in $\mathcal{F}(\mathbb{R} \setminus \{0\})$.*

In this definition $0^0 = 1$ and a^b may be negative.

Exercise 4.3.13 *Definitions 4.3.11 and 4.3.12 coincide on the intersection of their domains of validity.*

Exercise 4.3.14 *Show that for every $x \in \mathbb{R}$ one has that $\mathrm{e}^x = \exp x$. On the left side we have real exponentiation with the base $\mathrm{e} = 2.71\ldots$ and exponent x. On the right side we have the value of the exponential function in x.*

• *Exponential identities.* We discuss some well known, but also some not so well known, identities for the real exponentiation.

Theorem 4.3.15 (three basic exponential identities) *Let $a, b > 0$ & x, y be real numbers. Then*

$$(a \cdot b)^x = a^x \cdot b^x, \quad a^x \cdot a^y = a^{x+y} \quad \& \quad (a^x)^y = a^{x \cdot y} .$$

Proof. $(ab)^x$ is $\exp(x \log(ab)) = \exp(x \log a + x \log b) = \exp(x \log a) \exp(x \log b) = a^x b^x$. $a^x a^y$ is $\exp(x \log a) \exp(y \log a) = \exp(x \log a + y \log a) = \exp((x + y) \log a) = a^{x+y}$. $(a^x)^y$ is $\exp(y \log(\exp(x \log a))) = \exp(yx \log a) = a^{xy}$. □

But $\big((-1)^2\big)^{\frac{1}{2}} = 1^{\frac{1}{2}} = 1 \neq -1 = (-1)^1 = (-1)^{2 \cdot \frac{1}{2}}$; we used both definitions of real exponentiation.

The Polish-American mathematician *Alfred Tarski (1901–1983)*, who was the second greatest mathematical logician of the 20th century, conjectured that every identity for real exponentiation, like $x^y \cdot (x^y)^y = x^{y+y^2}$, can be derived from the three previous basic identities (and other basic properties of addition, multiplication and exponentiation). In 1981 the British mathematician *Alex Wilkie (1948)* refuted Tarski's conjecture; he proved that identities like

$$\begin{aligned}&\big((1+x)^y + (1+x+x^2)^y\big)^x \cdot \big((1+x^3)^x + (1+x^2+x^4)^x\big)^y \\ &= \big((1+x)^x + (1+x+x^2)^x\big)^y \cdot \big((1+x^3)^y + (1+x^2+x^4)^y\big)^x\end{aligned}$$

cannot be derived from the three basic identities. Identities of this kind are now called Wilkie's identities. In *MA 1*$^+$ we return to this topic.

Exercise 4.3.16 *Prove that for every real $x, y > 0$ the stated Wilkie's identity holds. Hint: $(1+x) \cdot (1+x^2+x^4) = (1+x^3) \cdot (1+x+x^2)$.*

Exercise 4.3.17 *Show that 0^0 is an indefinite expression: for every $A \in \mathbb{R}^*$ with $A \geq 0$ there exist sequences $(a_n) \subset (0, +\infty)$ and $(b_n) \subset \mathbb{R}$ such that $\lim a_n = \lim b_n = 0$ and $\lim (a_n)^{b_n} = A$. Could A be negative?*

It is often useful to define 0^0 as 1.

• *Cosine and sine*. Functions $\cos, \sin\colon \mathbb{R} \to \mathbb{R}$ originated in geometry, but the most concise definition is via sums of series. Let $t \in \mathbb{R}$. Then $\cos t$ is the sum $\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{(2n)!}$ (here $0^0 = 1$), and $\sin t$ the sum $\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{(2n+1)!}$. So $\cos t = 1 - \frac{t^2}{2} + \frac{t^4}{24} - \cdots$ a $\sin t = t - \frac{t^3}{6} + \frac{t^5}{120} - \cdots$.

Exercise 4.3.18 *For every t both $\cos t$ and $\sin t$ is an abscon series, hence also an* AK *series.*

The planar set $S \equiv \{(x, y) \in \mathbb{R}^2 :\ x^2 + y^2 = 1\}$ is the unit circle. It has radius 1 and center $(0, 0)$. The next theorem, which we do not prove here, exemplifies the main geometric property of cosine and sine.

Theorem 4.3.19 (runner, $\cos x$ and $\sin x$) *Let $t \in \mathbb{R}$. A runner starts in the point $(1, 0)$ of the track S and runs on S with the unit speed; for $t > 0$ she runs counter-clockwisely, and for $t \le 0$ clockwisely. Then in the time $|t|$ the runner is in the point $(\cos t, \sin t)$ of S.*

A rigorous geometric definition of the functions $\cos x$ and $\sin x$ is not an easy undertaking and we move it, as well as the proof of the theorem, to *MA 1^+*.

The number π can be defined in two ways. Firstly, $\pi = 3.14159\dots$ is twice the minimum $x > 0$ such that $\cos x = 0$. Secondly, 2π is the circumference of S, that is, the time when the runner passes for the second time through the point $(1, 0)$. The second definition is informal because we define the length of a circular arc only in *MA 1^+*. Then we prove the equivalence of both definitions. Thus the next basic properties of sine and cosine are given here only conditionally, assuming Theorem 4.3.19.

Exercise 4.3.20 *Deduce from Theorem 4.3.19 the next proposition.*

Proposition 4.3.21 (properties of $\sin x$ and $\cos x$) *These properties are as follows.*

1. *Both $\sin x$ and $\cos x$ is a 2π-periodic function: $\cos(t + 2\pi) = \cos t$ and $\sin(t + 2\pi) = \sin t$.*
2. *On $[0, \frac{\pi}{2}]$ sine increases from 0 to 1.*
3. $\forall\, t \in [0, \pi]\,\big(\sin(t) = \sin(\pi - t)\big)$ *and* $\forall\, t \in [0, 2\pi]\,\big(\sin(t) = -\sin(2\pi - t)\big)$.
4. *For every $t \in \mathbb{R}$ we have that $\cos t = \sin(t + \frac{\pi}{2})$ and $\cos^2 t + \sin^2 t = 1$.*
5. *The summation formulae hold: for every $s, t \in \mathbb{R}$,*

$$\begin{aligned} \sin(s \pm t) &= \sin s \cdot \cos t \pm \cos s \cdot \sin t \text{ and} \\ \cos(s \pm t) &= \cos s \cdot \cos t \mp \sin s \cdot \sin t\ . \end{aligned}$$

Exercise 4.3.22 (Euler's formula) $\forall\, t\,\big(\exp(it) = \cos t + i \sin t\big)$, $i \equiv \sqrt{-1}$.

• *The functions* <u>*tangent*</u> *and* <u>*cotangent*</u> in $\mathcal{R}$ are defined by $\tan t \equiv \frac{\sin t}{\cos t}$ and $\cot t \equiv \frac{\cos t}{\sin t}$.

Exercise 4.3.23 $M(\tan) = \mathbb{R} \setminus \{\frac{(2m-1)\pi}{2} : m \in \mathbb{Z}\}$ *and* $M(\cot) = \mathbb{R} \setminus \{m\pi : m \in \mathbb{Z}\}$.

• *The function* <u>*arcsine*</u> *(inverse sine)* $\arcsin x\colon [-1,1] \to \mathbb{R}$ *and* <u>*arccosine*</u> *(inverse cosine)* $\arccos x\colon [-1,1] \to \mathbb{R}$ is congruent to the inverse of the restriction of $\sin x$ and $\cos x$ to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $[0, \pi]$, respectively.

Exercise 4.3.24 *Thus they are (congruent to) the bijections*

$$\arcsin\colon [-1,\, 1] \to [-\tfrac{\pi}{2},\, \tfrac{\pi}{2}] \;\; \textit{and} \;\; \arccos\colon [-1,\, 1] \to [0,\, \pi]\,.$$

For simplicity of notation we denote these pairs of congruent functions by the same symbols.

• *The function* <u>*arctangent*</u> *(inverse tangent)* $\arctan\colon \mathbb{R} \to \mathbb{R}$ *and* <u>*arccotangent*</u> *(inverse cotangent)* $\operatorname{arccot} x\colon \mathbb{R} \to \mathbb{R}$ is the inverse of the restriction of $\tan x$ and $\cot x$ to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $(0, \pi)$, respectively.

Exercise 4.3.25 *Thus they are (congruent to) the bijections*

$$\arctan\colon \mathbb{R} \to (-\tfrac{\pi}{2},\, \tfrac{\pi}{2}) \;\; \textit{and} \;\; \operatorname{arccot}\colon \mathbb{R} \to (0,\, \pi)\,.$$

We again denote two congruent functions by the same symbol.

4.4 Elementary Functions

In this section we define the set of Elementary Functions.

• *Six operations on functions.* We introduce and remind six operations on the set of functions $\mathcal{R}$. Recall that $\mathcal{R}$ is the set of functions f of the type $f\colon M \to \mathbb{R}$ where $M \subset \mathbb{R}$.

Definition 4.4.1 (six operations on $\mathcal{R}$) *Let* $f, g \in \mathcal{R}$.

1. <u>*Sum*</u> $f + g\colon M(f) \cap M(g) \to \mathbb{R}$ *has values* $(f+g)(x) \equiv f(x) + g(x)$.
2. <u>*Product*</u> $fg = f \cdot g\colon M(f) \cap M(g) \to \mathbb{R}$ *has values* $(fg)(x) \equiv f(x)g(x)$.
3. <u>*Ratio*</u> *(<u>division</u>)* $f/g\colon M(f) \cap M(g) \setminus Z(g) \to \mathbb{R}$, *where* $Z(g) = \{x \in M(g) : g(x) = 0\}$, *has values* $(f/g)(x) \equiv f(x)/g(x)$.
4. *Recall that for any set* X <u>*restriction*</u> $f \,|\, X\colon M(f) \cap X \to \mathbb{R}$ *has values* $(f \,|\, X)(x) \equiv f(x)$.
5. *Recall that* <u>*composition*</u> $f(g) = f \circ g\colon M(f(g)) \to \mathbb{R}$, *where* $M(f(g)) \equiv \{x \in M(g) : g(x) \in M(f)\}$, *has values* $f(g)(x) \equiv f(g(x))$.

6. *Finally recall that if f is injective then* inverse $f^{-1}\colon f[M(f)] \to \mathbb{R}$ *has values* $f^{-1}(y) \equiv x$ *iff* $f(x) = y$.

The operations in parts 1, 2, 3 and 5 are binary; in parts 4 and 6 they are unary. In part 4 we have a set system of unary operations indexed by subsets of $\mathbb{R}$. The unary operation in part 6 is partial because it is not defined on non-injective functions. The ratio operation in part 3 is always defined; in this functional arithmetic there is nothing like the forbidden division by zero. In Chapter 7 we introduce another, seventh, operation on $\mathcal{R}$, the unary operation of derivative $f \mapsto f'$.

Exercise 4.4.2 *Prove the following proposition.*

Proposition 4.4.3 (monoids of functions) *Let $\mathcal{R}$ be as in Definition 4.2.6, $0_{\mathcal{R}} \equiv k_0(x)$ and $1_{\mathcal{R}} \equiv k_1(x)$. The structures*

$$\mathcal{R}_{\mathrm{amo}} \equiv \langle \mathcal{R},\, 0_{\mathcal{R}},\, + \rangle \ \text{ and } \ \mathcal{R}_{\mathrm{mmo}} = \langle \mathcal{R},\, 1_{\mathcal{R}},\, \cdot \rangle$$

are commutative monoids, *that is, the above operations $+$ and $\cdot$ are commutative and associative, and have neutral elements $0_{\mathcal{R}}$ and $1_{\mathcal{R}}$. They are not groups because inverses in general do not exist (but see Exercise 4.4.9). Operation $\cdot$ is distributive to $+$, always*

$$f \cdot (g + h) = (f \cdot g) + (f \cdot h)\,.$$

Let $f, g \in \mathcal{R}$. Their difference is the function $f - g\colon M(f) \cap M(g) \to \mathbb{R}$ with values $(f - g)(x) \equiv f(x) - g(x)$.

Exercise 4.4.4 *For every $f, g \in \mathcal{R}$ it holds that $f - g = f + (k_{-1} \cdot g)$.*

• *"How elementary, dear Watson!"*[1] We define the set of so called Elementary Functions; sometimes it is confounded with the previous set BEF.

Definition 4.4.5 (EF 1) *A function $f \in \mathcal{R}$ is* elementary $\iff$ *there exist $n \in \mathbb{N}$ and functions $f_i \in \mathcal{R}$, $i \in [n]$, such that $f_n = f$ and for every $i \in [n]$ one has that $f_i \in \mathrm{BEF}$ or there exist indices $j, k \in [i-1]$ such that $f_i = f_j + f_k$ or $f_i = f_j \cdot f_k$ or $f_i = f_j / f_k$ or $f_i = f_j(f_k)$. The set of elementary functions is denoted as* EF *and is called the* Elementary Functions.

Clearly, every function f_i in the generating word $f_1, f_2, \dots, f_n$ (of f_n) is elementary too. Said less formally and less precisely, we get EF from BEF by repeated applications of addition, multiplication, division, and composition. For example, the identity function $\mathrm{id}(x) \equiv x$ for every $x \in \mathbb{R}$ is elementary because $\mathrm{id}(x) = \log(\exp x)$.

[1] By [13] the only "elementary" statement in the work of A. C. Doyle on Sherlock Holmes is found in the story *The Crooked Man* and reads: " 'Excellent!' I [Watson] cried. 'Elementary,' said he." All other "elementary" exclamations of S. Holmes originate probably in movie and TV adaptations. The Czech version "Jak prosté, milý Watsone!", if not "elementary", even rhymes.

Exercise 4.4.6 *The absolute value* $|x|$ *(*$\in \mathcal{F}(\mathbb{R})$*) is elementary.*

Exercise 4.4.7 $k_1(x)/k_0(x) = ?$

Exercise 4.4.8 *Is the empty function* $\emptyset$ *elementary?*

Exercise 4.4.9 *For every* $f \in \mathrm{EF}$ *there is a unique* $g \in \mathrm{EF}$ *such that* $M(g) = M(f)$ *and* $f + g = k_0 \,|\, M(f)$*.*

We remind the types of real intervals given in Proposition 3.3.7:

$$\mathcal{I} \equiv \{\emptyset, \{a\}, \mathbb{R}, (a, b), (-\infty, a), (a, +\infty), (a, b], [a, b), [a, b], (-\infty, a], [a, +\infty)\}\,,$$

for any $a, b \in \mathbb{R}$ with $a < b$.

Proposition 4.4.10 (interval restrictions) *For every* $f \in \mathrm{EF}$ *and every interval* $I \in \mathcal{I}$ *it holds that* $f \,|\, I \in \mathrm{EF}$*.*

Proof. It suffices to show that for every interval $I \in \mathcal{I}$ the function $g \equiv k_0 \,|\, I$ is elementary. Then $f + g$ gives the required restriction. For $I = \emptyset$ let $g \equiv k_1/k_0$. For $I = \{a\}$ let $g \equiv \sqrt{a-x} + \sqrt{x-a}$. For $I = (-\infty, b]$ let $g \equiv \sqrt{b-x} - \sqrt{b-x}$. For $I = \mathbb{R}$ let $g \equiv k_0$. For $I = (a, b)$ let $g \equiv \log(x-a) + \log(b-x) - \log(x-a) - \log(b-x)$. Of course, x is $\mathrm{id}(x)$, a is $k_a(x)$ and b is $k_b(x)$. For other intervals in $\mathcal{I}$ we combine these square roots and logarithms in similar ways. □

Exercise 4.4.11 *Find* $f, g \in \mathrm{EF}$ *such that* $M(f) = \mathbb{Z}$ *and* $M(g) = \mathbb{R} \setminus (\{0\} \cup \{\frac{1}{n} :\ n \in \mathbb{Z} \setminus \{0\}\})$*.*

BEF contains many redundant functions that can be expressed from other functions in BEF: $\cos x = \sin(x + \pi/2)$, $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$, $\arccos x = \frac{\pi}{2} + \arcsin x$, $\arctan x = \arcsin\big(x/\sqrt{1+x^2}\big)$ and $\mathrm{arccot}\, x = \frac{\pi}{2} - \arctan x$. For real exponentiation we have the following reduction.

Exercise 4.4.12 *Show that of the functions in Definitions 4.3.11 and 4.3.12 it suffices to keep just the functions* x^b*,* $b \in (0, +\infty) \setminus \mathbb{N}$*, which lie in* $\mathcal{F}([0, +\infty))$*.*

We remove from BEF these redundant functions and get the next set of functions.

Definition 4.4.13 (RBEF) *Really Basic Elementary Functions are: the constants* $k_c(x)$ *for* $c \in \mathbb{R}$*,* $\exp x$*,* $\log x$*,* x^b *for* $b > 0$ *and* $b \notin \mathbb{N}$*,* $\sin x$ *and* $\arcsin x$*.*

In *MA 1*$^+$ we show that functions x^b for $b \in (0, +\infty) \setminus \mathbb{N}$ cannot be expressed in terms of other functions in RBEF. Now we can give a more compact definition of EF.

Definition 4.4.14 (EF 2) *In Definition 4.4.5 the set of functions* BEF *may be replaced with the smaller set* RBEF*.*

For example, the function $\sqrt{x} \equiv x^{1/2}$, which is in $\mathcal{F}([0,+\infty))$, is in RBEF. Note that $\exp(\frac{1}{2}\log x) \neq \sqrt{x}$ because $M(\exp(\frac{1}{2}\log x)) = (0,+\infty)$. We return to EF in *MA 1*$^+$ where we prove that some antiderivatives are non-elementary.

The elementary function $\frac{|x|}{x}\colon \mathbb{R}\setminus\{0\} \to \{-1,1\}$ is -1 for $x < 0$ and 1 for $x > 0$. It resembles the function signum $\mathrm{sgn}\colon \mathbb{R} \to \{-1,0,1\}$ given by $\mathrm{sgn}\,x = -1$ for $x < 0$, $\mathrm{sgn}\,0 = 0$ and $\mathrm{sgn}\,x = 1$ for $x > 0$.

Proposition 4.4.15 (sgn $\notin$ EF) *However,* sgn $\notin$ EF.

Proof. Every elementary function is continuous, see Definition 6.1.1 and Theorem 6.6.16, but $\mathrm{sgn}\,x$ is not continuous. □

Maybe we should reconsider the definition of EF so that $\mathrm{sgn}\,x$ is elementary after all. We return to this in *MA 1*$^+$.

Exercise 4.4.16 *Give examples of functions in* EF *which are not differentiable in some points of their definition domain.*

4.5 Polynomials and Rational Functions

We define these subsets of $\mathcal{R}$ by restricting generation of Elementary Functions in Definition 4.4.5.

• *Polynomials.* We propose somewhat unorthodox, at least for the textbook of mathematical analysis, approach to polynomials.

Definition 4.5.1 (POL) *A function $f \in \mathcal{R}$ is a polynomial $\iff$ there exist $n \in \mathbb{N}$ and functions $f_i \in \mathcal{R}$, $i \in [n]$, such that $f_n = f$ and for every $i \in [n]$ one has that $f_i \in \{k_c(x):\ c \in \mathbb{R}\} \cup \{\mathrm{id}(x)\}$ or there exist indices $j,k \in [i-1]$ such that $f_i = f_j + f_k$ or $f_i = f_j \cdot f_k$. The set of polynomials is denoted as* POL *and is called the Polynomials.*

It is clear that every function f_i, $i \in [n]$, is a polynomial. In our approach polynomials arise from identity and constants by repeated addition and multiplication. We show that every polynomial has the well known canonical form. For $f \in \mathcal{R}$ and $n \in \mathbb{N}_0$ we define the power f^n for $n = 0$ as $f - f + k_1$ $(= k_1 \mid M(f))$ and for $n > 0$ as $f \cdot f \cdot \ldots \cdot f$ with n factors f.

Proposition 4.5.2 (canonical form) *Every polynomial p has $M(p) = \mathbb{R}$ and either $p = k_0$ and is the zero polynomial or p has the unique canonical form*

$$p = \textstyle\sum_{j=0}^{n} k_{a_j} \cdot \mathrm{id}_{\mathbb{R}}^{j} \quad (= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)$$

with $n \in \mathbb{N}_0$, $a_j \in \mathbb{R}$ & $a_n \neq 0$.

Proof. It is immediate that $M(p) = \mathbb{R}$ for every $p \in \mathrm{POL}$. Let p be a polynomial. By Definition 4.5.1, $p = f_n$ has the generating word $f_1, \dots, f_n$. We use induction on n. If $n = 1$ then either p is the zero polynomial or it has the canonical form $p = k_c \cdot \mathrm{id}_{\mathbb{R}}^0$ with $c \neq 0$ or $p = k_0 \cdot \mathrm{id}_{\mathbb{R}}^0 + k_1 \cdot \mathrm{id}_{\mathbb{R}}^1$. Let $n > 1$. If $p = f_n$ is a constant or the identity, we are in the previous case. Let $p = f_j + f_k$ or $p = f_j \cdot f_k$ with $1 \le j, k < n$. Then either f_j is the zero polynomial or, by induction, has the canonical form. The same holds for f_k. If f_j or f_k is the zero polynomial, we see that p is the zero polynomial or has the canonical form. If both f_j and f_k has the canonical form, we see with the help of Proposition 4.4.3 that p is the zero polynomial or has the canonical form.

Uniqueness of canonical forms follows from the next exercise. Suppose that p has two distinct canonical forms. By Proposition 4.4.3 we see that the difference $p - p$ has also canonical form. By Exercise 4.5.3 the polynomial $p - p$ has only finitely many zeros. This is a contradiction because $p - p = k_0$. □

The degree $\deg p$ of a nonzero polynomial p is the index $n \in \mathbb{N}_0$ in its canonical form. It is useful to set $\deg k_0 \equiv -\infty$.

Exercise 4.5.3 *Show that every nonzero polynomial p has finite set $Z(p) = \{b \in \mathbb{R} :\ p(b) = 0\}$ of zeros.*

An integral domain is a ring in which product of two nonzero elements is never zero.

Exercise 4.5.4 *Prove the following proposition.*

Proposition 4.5.5 (POL is ID) *The structure $\mathbb{R}[x] \equiv \langle \mathrm{POL}, k_0, k_1, +, \cdot \rangle$ is an integral domain.*

• *Rational functions.* Their definition is very similar to the definition of polynomials, we only add the operation of division. But the result is that we get in a considerably more complicated situation.

Definition 4.5.6 (RAC) *A function $f \in \mathcal{R}$ is rational $\iff$ there exist $n \in \mathbb{N}$ and functions $f_i \in \mathcal{R}$, $i \in [n]$, such that $f_n = f$ and for every $i \in [n]$ one has that $f_i \in \{k_c(x) :\ c \in \mathbb{R}\} \cup \{\mathrm{id}(x)\}$ or there exist indices $j, k \in [i-1]$ such that $f_i = f_j + f_k$ or $f_i = f_j \cdot f_k$ or $f_i = f_j / f_k$. The set of rational functions is denoted as RAC and is called the Rational Functions.*

Again every function f_i, $i \in [n]$, in the generating word is rational. Our rational functions arise from constants and the identity by repeated addition, multiplication *and division.* For instance, the function $\frac{1}{x} = k_1/\mathrm{id}$ ($\in \mathcal{F}(\mathbb{R} \setminus \{0\})$) is rational. Every polynomial is rational, POL ⊂ RAC. The empty function $\emptyset$ is not a polynomial but is rational because, for example, $\emptyset = k_1/k_0$. For obtaining canonical forms of functions in RAC we need three lemmas confirming on computations with ratios in $\mathcal{R}$.

Lemma 4.5.7 (adding two ratios) *For every four functions* f_1*,* g_1*,* f_2 *and* g_2 *in* $\mathcal{R}$ *it holds that*

$$F \equiv \frac{f_1}{g_1} + \frac{f_2}{g_2} = \frac{f_1 g_2 + f_2 g_1}{g_1 g_2} \equiv G\,.$$

Proof. The definition domain $M(F) = M(f_1/g_1) \cap M(f_2/g_2)$ equals to

$$\big((M(f_1) \cap M(g_1)) \setminus Z(g_1)\big) \cap \big((M(f_2) \cap M(g_2)) \setminus Z(g_2)\big)\,.$$

The definition domain $M(G) = (M(f_1g_2 + f_2g_1) \cap M(g_1g_2)) \setminus Z(g_1g_2)$ equals to

$$\big((M(f_1) \cap M(g_2)) \cap (M(f_2) \cap M(g_1)) \cap (M(g_1) \cap M(g_2))\big) \setminus (Z(g_1) \cup Z(g_2))\,.$$

The equality $Z(g_1g_2) = Z(g_1) \cup Z(g_2)$ follows from the fact that $\mathbb{R}$, like any field, is an integral domain. $M(F) = M(G)$ because it is easy to see that both displayed sets are equal to the set

$$\big(M(f_1) \cap M(f_2) \cap M(g_1) \cap M(g_2)\big) \setminus \big(Z(g_1) \cup Z(g_2)\big)\,.$$

Due to the arithmetic in $\mathbb{R}$ we know that $F(x) = G(x)$ for every $x \in M(F) = M(G)$. Hence $F = G$. $\square$

The proof of the second lemma is similar and we omit it.

Lemma 4.5.8 (multiplying two ratios) *For every four functions* f_1*,* g_1*,* f_2 *and* g_2 *in* $\mathcal{R}$ *it holds that*

$$\frac{f_1}{g_1} \cdot \frac{f_2}{g_2} = \frac{f_1 f_2}{g_1 g_2}\,.$$

We prove the third lemma; the next exercise shows that it is a bit tricky.

Exercise 4.5.9 *It is not true that for every four functions* f_1*,* g_1*,* f_2 *and* g_2 *in* $\mathcal{R}$ *it holds that*

$$\frac{f_1/g_1}{f_2/g_2} = \frac{f_1 g_2}{f_2 g_1}$$

Lemma 4.5.10 (ratio of two ratios) *For every four functions* f_1*,* g_1*,* f_2 *and* g_2 *in* $\mathcal{R}$ *it holds that*

$$F \equiv \frac{f_1/g_1}{f_2/g_2} = \frac{f_1 g_2^2}{f_2 g_1 g_2} \equiv G\,.$$

Proof. Then $M(F) = (M(f_1/g_1) \cap M(f_2/g_2)) \setminus Z(f_2/g_2)$ equals to

$$\big((M(f_1) \cap M(g_1) \setminus Z(g_1)) \cap (M(f_2) \cap M(g_2) \setminus Z(g_2))\big) \setminus \big(Z(f_2) \setminus Z(g_2)\big)\,.$$

Then $M(G) = (M(f_1g_2^2) \cap M(f_2g_1g_2)) \setminus Z(f_2g_1g_2))$ equals to

$$\big(M(f_1) \cap M(f_2) \cap M(g_1) \cap M(g_2)\big) \setminus \big(Z(f_2) \cup Z(g_1) \cup Z(g_2)\big)\,.$$

Thus $M(F) = M(G)$ because both displayed sets are clearly equal to

$$\big(M(f_1) \cap M(f_2) \cap M(g_1) \cap M(g_2)\big) \setminus \big(Z(g_1) \cup Z(f_2) \cup Z(g_2)\big)\,.$$

Due to the arithmetic in $\mathbb{R}$ we know that $F(x) = G(x)$ for every $x \in M(F) = M(G)$. Hence $F = G$. □

Proposition 4.5.11 (canonical form) *Every nonempty rational function r has $M(r) = \mathbb{R} \setminus Z$, where $Z \subset \mathbb{R}$ is a finite set, and there exist two polynomials p and q such that $q \neq k_0$, $Z = Z(q)$ and $r = p/q$.*

Proof. Let $r \in$ RAC. By Definition 4.5.6, $r = f_n$ has the generating word $f_1, \ldots, f_n$. We proceed by induction on n. If $n = 1$ then r is a constant or the identity and has the canonical form k_c/k_1 or id/k_1. Let $n > 1$. If $r = f_n$ is a constant or the identity, we are in the previous case. Suppose that j and k with $1 \leq j, k < n$ are such that $r = f_j + f_k$ or $r = f_j \cdot f_k$ or $r = f_j/f_k$. Then $f_j = \emptyset$ or by induction f_j has a canonical form, and the same holds for f_k. If $f_j = \emptyset$ or $f_k = \emptyset$ then also $r = \emptyset$.

It remains to deal with the case of canonical forms $f_j = p/q$ and $f_k = p'/q'$, where $M(f_j) = \mathbb{R} \setminus Z(q)$ and $M(f_k) = \mathbb{R} \setminus Z(q')$. We consider the three stated cases for r. In the first case we have by Lemma 4.5.7 that

$$r = f_j + f_k = \tfrac{pq' + p'q}{qq'}\,.$$

The numerator and the denominator are polynomials and $qq' \neq k_0$, due to Proposition 4.5.5. Also, $M(r)$ is

$$M(f_j) \cap M(f_k) = (\mathbb{R} \setminus Z(q)) \cap (\mathbb{R} \setminus Z(q')) = \mathbb{R} \setminus (Z(q) \cup Z(q')) = \mathbb{R} \setminus Z(qq')\,.$$

Hence we have a canonical form for r. In the second case we have by by Lemma 4.5.8 that

$$r = f_j f_k = \tfrac{pp'}{qq'}$$

and like in the first case we see that this is a canonical form for r. Finally in the third case we have by Lemma 4.5.10 that

$$r = f_j/f_k = \tfrac{p(q')^2}{qq'p'}\,.$$

The numerator and the denominator are polynomials. If $p' = k_0$ then $r = \emptyset$, else $qq'p' \neq k_0$. Also, $M(r)$ is

$$\begin{aligned} &M(f_j) \cap M(f_k) \setminus Z(f_k) = (\mathbb{R} \setminus Z(q)) \cap (\mathbb{R} \setminus Z(q')) \setminus (Z(p') \setminus Z(q')) \\ &= \mathbb{R} \setminus (Z(q) \cup Z(q') \cup Z(p')) = \mathbb{R} \setminus Z(qq'p')\,. \end{aligned}$$

We again have a canonical form for r. □

Unlike for polynomials, canonical forms of rational functions are not unique. For instance, both $\frac{0+1x}{0+1x}$ and $\frac{0+0x+1x^2}{0+0x+1x^2}$ is a canonical form of the rational function $x/x = \mathrm{id}/\mathrm{id}$ $(= k_1 \,|\, \mathbb{R} \setminus \{0\})$.

We introduce the congruence $\sim$ on $\mathrm{RAC} \setminus \{\emptyset\}$ by setting

$$r \sim s \overset{\mathrm{def}}{\iff} r \,|\, M(s) = s \,|\, M(r)\ .$$

Exercise 4.5.12 *Show that $\sim$ is an equivalence relation.*

For example, $k_1 \sim x/x \sim (x \cdot (x-1))/(x \cdot (x-1))$.

Exercise 4.5.13 *Prove the next proposition.*

Proposition 4.5.14 (the field $\mathbb{R}(x)$) *The structure*

$$\mathbb{R}(x) \equiv \langle (\mathrm{RAC} \setminus \{\emptyset\})/\!\sim,\ [k_0]_\sim,\ [k_1]_\sim,\ +,\ \cdot \rangle$$

is a field, the field of rational functions.

Unlike in algebra, elements of our $\mathbb{R}(x)$ are really functions, more precisely equivalence blocks of them.

Chapter 5

Limits of functions. Asymptotic notation

This chapter reflects my lecture

`https://kam.mff.cuni.cz/~klazar/MAI24_pred5.pdf`

given on March 21, 2024. We begin with Section 5.1 on one-sided limits of functions. Propositions 5.1.7 and 5.1.13 describe relations between ordinary and one-sided limits. In Section 5.2 we introduce pointwise continuity of functions. In Proposition 5.2.5 we characterize it by limits, and in Exercise 5.2.6 by Heine's definition. Section 5.3 contains Theorem 5.3.1 on limits of monotone functions, Theorem 5.3.3 on arithmetic of limits of functions, Theorem 5.3.7 on relations of limits of functions and the order $(\mathbb{R}^*, <)$ and the squeeze Theorem 5.3.11. In Section 5.4 we present Theorem 5.4.1 on limits of composite functions in a form stronger than it is common. In the final Section 5.5 we explain asymptotic symbols O, $\ll$, $\gg$, Ω, Θ, $\asymp$, o, ω and $\sim$. We discuss asymptotic expansions of functions and give three examples of them.

5.1 One-sided limits

The complement of a point a to the real axis is $\mathbb{R} \setminus \{a\} = (-\infty, a) \cup (a, +\infty)$ and comprises two separated intervals. In the plane $\mathbb{R}^2$ we can go around a point but in the real line $\mathbb{R}$ this is impossible. So we consider left-sided and right-sided limits of functions. We begin with one-sided neighborhoods.

• *One-sided neighborhoods and one-sided limit points.* Left, respectively right, ε-neighborhood of a point $b \in \mathbb{R}$ is

$$U^-(b,\, \varepsilon) \equiv (b - \varepsilon,\, b], \text{ respectively } U^+(b,\, \varepsilon) \equiv [b,\, b + \varepsilon)\,.$$

Left, respectively right, deleted ε-neighborhood of a point b is

$$P^-(b,\, \varepsilon) \equiv (b - \varepsilon,\, b), \text{ respectively } P^+(b,\, \varepsilon) \equiv (b,\, b + \varepsilon)\,.$$

A point $b \in \mathbb{R}$ is a left, respectively right, limit point of $M \subset \mathbb{R}$ if

$$\forall\, \varepsilon \left(P^-(b,\, \varepsilon) \cap M \neq \emptyset\right), \text{ respectively } \forall\, \varepsilon \left(P^+(b,\, \varepsilon) \cap M \neq \emptyset\right).$$

The set of these points is denoted by $L^-(M)$, respectively $L^+(M)$ ($\subset \mathbb{R}$). A point $b \in \mathbb{R}$ is a two-sided limit point of $M \subset \mathbb{R}$ if

$$\forall\, \varepsilon \left(P^-(b,\, \varepsilon) \cap M \neq \emptyset \,\wedge\, P^+(b,\, \varepsilon) \cap M \neq \emptyset\right).$$

The set of these points is denoted by $L^{\mathrm{TS}}(M)$ ($\subset \mathbb{R}$). Two-sided limit points play key role in the criterion of local extremes. We do not define one-sided neighborhoods for infinities, nor $\pm\infty$ can be a one-sided limit point of a set.

Exercise 5.1.1 *$b \in L^-(M)$, respectively $b \in L^+(M)$ $\iff$ $\exists\ (a_n) \subset (-\infty, b) \cap M$, respectively $\exists\ (a_n) \subset (b, +\infty) \cap M$, such that $\lim a_n = b$.*

Exercise 5.1.2 *Let $M \subset \mathbb{R}$ and $b \in \mathbb{R}$. Prove the following.*
1. *$b \in L^-(M) \Rightarrow b \in L(M)$.*
2. *$b \in L^+(M) \Rightarrow b \in L(M)$.*
3. *$b \in L(M) \Rightarrow b \in L^-(M)$ or $b \in L^+(M)$.*
4. *It can be that $b \in L(M)$ but $b \notin L^-(M)$ or $b \notin L^+(M)$.*

By Exercise 4.2.3 no finite set has a limit point, the less one-sided limit point. By Exercise 4.2.4 every infinite real set has a limit point. This is not true for one-sided limit points.

Exercise 5.1.3 *Give an example of an infinite subset of $\mathbb{R}$ that has no one-sided limit points. Hint: $\pm\infty$ is never a one-sided limit point.*

Exercise 5.1.4 *Every infinite and bounded real set has a one-sided limit point.*

- *One-sided limits of functions.* We refine limits of functions by one-sided limits.

Definition 5.1.5 (one-sided limit) *Let $f \in \mathcal{F}(M)$, $b \in L^-(M)$ and L be in $\mathbb{R}^*$. If for every ε there is a δ such that $f[P^-(b,\delta)] \subset U(L,\varepsilon)$, we write $\lim_{x\to b^-} f(x) = L$ and say that the function f has in b the left-sided limit L. By replacing the sign $^-$ with the sign $^+$ we get the right-sided limit in b, denoted by $\lim_{x\to b^+} f(x) = L$.*

Like for the ordinary limit, the one-sided limit in b is not defined if b is not the respective one-sided limit point. Again if $\lim_{x\to b^\pm} f(x)$ exists, it means in particular that b is the respective one-sided limit point.

Exercise 5.1.6 *Prove the following proposition.*

Proposition 5.1.7 (on one-sided limits) *The following hold.*
1. *$\lim_{x\to a} f(x) = L \Rightarrow \lim_{x\to a^-} f(x) = L$ or $\lim_{x\to a^+} f(x) = L$.*
2. *$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L \Rightarrow \lim_{x\to a} f(x) = L$.*
3. *$\lim_{x\to a^-} f(x) = K$, $\lim_{x\to a^+} f(x) = L$ and $K \neq L \Rightarrow \neg\,\exists\, \lim_{x\to a} f(x)$.*

For instance, the limit $\lim_{x\to 0}\operatorname{sgn} x$ does not exist because $\lim_{x\to 0^-}\operatorname{sgn} x = -1$ and $\lim_{x\to 0^+}\operatorname{sgn} x = 1$.

Exercise 5.1.8 *Prove the following proposition.*

Proposition 5.1.9 (uniqueness of $\lim_{x\to b^{\pm}} f(x)$) *If $\lim_{x\to b^{\pm}} f(x) = K$ and $\lim_{x\to b^{\pm}} f(x) = L$ <u>then</u> $K = L$ (equal signs).*

Exercise 5.1.10 *Prove the following proposition.*

For $b \in \mathbb{R}$ let $I^-(b) \equiv (-\infty, b)$ and $I^+(b) \equiv (b, +\infty)$.

Proposition 5.1.11 (Heine's definition of $\lim_{x\to b^{\pm}} f(x)$) *Suppose that f is in $\mathcal{F}(M)$ and that b is in $L^{\pm}(M)$. <u>Then</u> $\lim_{x\to b^{\pm}} f(x) = L \iff$ for every sequence $(a_n) \subset M \cap I^{\pm}(b)$ with $\lim a_n = b$ one has that $\lim f(a_n) = L$ (equal signs).*

Sometimes ordinary limits are unnecessarily replaced with one-sided limits. For example, we can sometimes read that $\lim_{x\to 0^+}\log x = -\infty$. By our definitions we can write simply that $\lim_{x\to 0}\log x = -\infty$. Here

$$\lim_{x\to 0}\log x = \lim_{x\to 0^+}\log x = -\infty$$

but $\lim_{x\to 0^-}\log x$ is not defined because $0 \notin L^-(M(\log x)) = L^-((0, +\infty))$. In conclusion we mention one more relation between ordinary and one-sided limits. We use it in the proof of Corollary 5.4.5.

Exercise 5.1.12 *Prove the following proposition.*

Proposition 5.1.13 (using restriction) *Let $f \in \mathcal{F}(M)$ and $b \in L^{\pm}(M)$. <u>Then</u> $\lim_{x\to b^{\pm}} f(x) = L \iff \lim_{x\to b}(f \mid I^{\pm}(b))(x) = L$ (equal signs).*

5.2 Point-wise continuity

• *Continuity of a function in a point of the definition domain.* We arrive at an important definition.

Definition 5.2.1 (pointwise continuity) *Let $f \in \mathcal{F}(M)$ and $b \in M$. We say that f is <u>continuous in b</u> if for every ε there is a δ such that $f[U(b,\delta)] \subset U(f(b),\varepsilon)$. Else we say that f is <u>discontinuous in b</u>.*

Exercise 5.2.2 *So when is a function f discontinuous in b ($\in M(f)$)?*

For example $\operatorname{sgn} x$ is discontinuous in $x = 0$, but it is continuous in every other point. If $b \notin M(f)$ then the function f is in b *neither continuous nor discontinuous*. Comparing the above definition with that of $\lim_{x\to b} f(x) = L$ we see that L is replaced with $f(b)$, and the deleted neighborhood $P(b, \delta)$ with the ordinary neighborhood $U(b, \delta)$.

Proposition 5.2.3 (locality of continuity) *If $f, g \in \mathcal{R}$, $b \in M(f) \cap M(g)$ and there is a θ such that $f = g$ on $U(b, \theta)$ then f is continuous in $b \iff g$ is continuous in b.*

Proof. This is immediate from Definition 5.2.1 because we can take the δ in it such that $\delta \le \theta$. Then $U(b, \delta) \subset U(b, \theta)$ and $f[U(b, \delta)] = g[U(b, \delta)]$. □

Exercise 5.2.4 *A function f is continuous in $b \in M(f)$ iff for every ε there exists a δ such that $x \in M(f) \wedge |x - b| \le \delta \Rightarrow |f(x) - f(b)| \le \varepsilon$.*

Continuity of f in b is *not* equivalent with $\lim_{x\to b} f(x) = f(b)$. This only holds in limit points of M.

Proposition 5.2.5 (on pointwise continuity) *For any function $f \in \mathcal{F}(M)$ and any point $b \in M \cap L(M)$ the claims 1, 2 and 3 simultaneously hold, or simultaneously do not hold.*

1. *The function f is continuous in b.*
2. *The limit $\lim_{x\to b} f(x) = f(b)$.*
3. *For every sequence $(a_n) \subset M$ with $\lim a_n = b$ it holds that $\lim f(a_n) = f(b)$.*

Proof. The implication $1 \Rightarrow 2$. Let f be continuous in b by Definition 5.2.1 and an ε be given. Thus there is a δ such that $f[U(b, \delta)] \subset U(f(b), \varepsilon)$. Then $b \in L(M(f))$ and also $f[P(b, \delta)] \subset U(f(b), \varepsilon)$. Hence $\lim_{x\to b} f(x) = f(b)$.

The implication $2 \Rightarrow 3$. Suppose that $\lim_{x\to b} f(x) = f(b)$, that $(a_n) \subset M$ has $\lim a_n = b$ and that an ε is given. Thus there is a δ such that

$$f[P(b,\, \delta)] \subset U(f(b),\, \varepsilon)\ . \qquad (*)$$

We take an n_0 such that $n \ge n_0 \Rightarrow a_n \in U(b, \delta)$. Then also $n \ge n_0 \Rightarrow f(a_n) \in U(f(b), \varepsilon)$: for $a_n \ne b$ we use the inclusion $(*)$, and for $a_n = b$ it holds that $f(a_n) = f(b) \in U(f(b), \varepsilon)$. Hence $\lim f(a_n) = f(b)$.

The implication $3 \Rightarrow 1$. We prove its reversal $\neg 1 \Rightarrow \neg 3$. Suppose that f is not continuous in b. Then there is an ε such that $\forall\, \delta\, \exists\, a = a(\delta) \in U(b, \delta) \cap M$ with $f(a) \notin U(f(b), \varepsilon)$. For every n we *chose* such $a_n = a(1/n)$ and get the sequence $(a_n) \subset M$ such that $\lim a_n = b$, but for every n it holds that $f(a_n) \notin U(f(b), \varepsilon)$. Hence $(f(a_n))$ does not converge to $f(b)$ and part 3 does not hold. □

We proved the last implication again with the help of the axiom of choice. Part 3 describes Heine's definition of pointwise continuity. The next exercise makes it more precise.

Exercise 5.2.6 *In Proposition 5.2.5 in the equivalence* $1 \iff 3$ *it is possible to omit the assumption that* $b \in L(M)$*. Thus* f *is continuous in a point* $b \in M(f)$ $\iff$ *for every sequence* $(a_n) \subset M$ *with* $\lim a_n = b$ *one has that* $\lim f(a_n) = f(b)$*.*

The right side of this equivalence is sometimes taken as the definition of pointwise continuity.

• *Isolated points.* Let $M \subset \mathbb{R}$. The set $M \setminus L(M)$ consists of so called isolated points of M.

Exercise 5.2.7 *A point* $b \in M$ *is an isolated point of* $M \subset \mathbb{R}$ *iff for some* ε *it holds that* $U(b, \varepsilon) \cap M = \{b\}$*.*

Exercise 5.2.8 *Let* $b \in M \subset \mathbb{R}$*. Then* b *is either a limit point of* M *or an isolated point of* M*.*

Proposition 5.2.9 (continuity in isolated points) *Every function* $f \in \mathcal{R}$ *is continuous in every isolated point of* $M(f)$*.*

Proof. Let $f \in \mathcal{F}(M)$ and $b \in M$ be an isolated point. By Exercise 5.2.7 there is a δ such that $U(b, \delta) \cap M = \{b\}$. For this δ the inclusion $f[U(b, \delta)] = f[\{b\}] = \{f(b)\} \subset U(f(b), \varepsilon)$ holds for every ε. Hence f is continuous in b by Definition 5.2.1. □

Thus every sequence $(a_n) \subset \mathbb{R}$, understood as a function $a \in \mathcal{F}(\mathbb{N})$, is continuous in every point n of its definition domain $\mathbb{N}$.

Exercise 5.2.10 *A function* $f \in \mathcal{F}(M)$ *is not continuous in* $b \in M \iff$ *there is a sequence* $(a_n) \subset M$ *such that* $\lim a_n = b$ *and* $\lim f(a_n) = A \neq f(b)$*.*

• *One-sided continuity.* A function f is left-continuous in $b \in M(f)$ if for every ε there is a δ such that $f[U^-(b, \delta)] \subset U(f(b), \varepsilon)$. By replacing the sign $^-$ with the sign $^+$ we get the right-continuity.

Exercise 5.2.11 *A function is continuous at a point iff it is both left- and right-continuous at the point.*

• *Riemann's function.* It is the function $r \in \mathcal{F}(\mathbb{R})$ with values $r(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$ and $r(\frac{m}{n}) = \frac{1}{n}$ if the fraction $\frac{m}{n}$ is in lowest terms.

Proposition 5.2.12 (on Riemann's function) *Riemann's function is continuous exactly at irrational numbers.*

Proof. Let $x = \frac{m}{n}$ be a fraction in lowest terms and $\varepsilon \le \frac{1}{n}$. For every δ there is an irrational $\alpha \in U(x, \delta)$. But $r(\alpha) = 0 \notin U(r(x), \varepsilon) = U(\frac{1}{n}, \varepsilon)$, so that r is discontinuous in x. Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and an $\varepsilon \in (0, 1)$ be given. We set

$$M \equiv \{|x - \tfrac{m}{n}| : \tfrac{m}{n} \in \mathbb{Q} \cap U(x, 1) \wedge \tfrac{1}{n} \ge \varepsilon\} \text{ and } \delta \equiv \min(M) .$$

This δ exists and is positive because by Exercise 5.2.13 the set $M \neq \emptyset$ and is a finite set of positive real numbers. For this δ we have that $y \in U(x, \delta) \Rightarrow r(y) \in U(r(x), \varepsilon) = U(0, \varepsilon)$ — for every $y \in U(x, \delta)$ we have that $r(y) = 0$ or $r(y) = \frac{1}{n} < \varepsilon$. Hence r is continuous in x. □

Exercise 5.2.13 *Why is M a nonempty finite set of positive real numbers?*

5.3 Limits and order, arithmetic of limits

We extend these kind of results obtained earlier for limits of sequences to limits of functions.

• *Limits of monotone functions.* Let $f \in \mathcal{F}(M)$ and X be any set. A function f weakly increases, respectively weakly decreases, on X if for every $x \leq y$ in $X \cap M$ we have that $f(x) \leq f(y)$, respectively that $f(x) \geq f(y)$. Weakly increasing or weakly decreasing f is monotone on X. Note that X need not be a subset of M.

Theorem 5.3.1 (limits of monotone functions) *Let $f \in \mathcal{F}(M)$. The following hold.*
1. *If $b \in L^-(M)$ and there is a θ such that f weakly increases on $P^-(b, \theta)$ then*

$$\lim_{x \to b^-} f(x) = \sup(f[P^-(b,\, \theta)])\,.$$

2. *If $+\infty \in L(M)$ and there is a θ such that f weakly increases on $U(+\infty, \theta)$ then*

$$\lim_{x \to +\infty} f(x) = \sup(f[U(+\infty,\, \theta)])\,.$$

Suprema are taken in the LO $(\mathbb{R}^*, <)$.

Proof. 1. Let f, M, b and θ be as stated and an ε be given. We set $A \equiv \sup(f[P^-(b, \theta)])$ and take any $a \in U(A, \varepsilon)$ with $a < A$. By the definition of supremum there is a $c \in P^-(b, \theta) \cap M$ such that $a < f(c) \leq A$. We set $\delta \equiv b - c$. For every $d \in M$ with $c < d < b$ it holds that $a < f(c) \leq f(d) \leq A$. Hence, by Exercise 2.1.11, $f(d) \in U(A, \varepsilon)$. Thus $f[P^-(b, \delta)] \subset U(A, \varepsilon)$ and $\lim_{x \to b^-} f(x) = A$.

2. Let f, M and θ be as stated and an ε be given. $A \equiv \sup(f[U(+\infty, \theta)])$ and we take any $a \in U(A, \varepsilon)$ with $a < A$. By the definition of supremum there is a $c \in U(+\infty, \theta) \cap M$ such that $a < f(c) \leq A$. We set $\delta \equiv \frac{1}{c}$. For every $d \in M$ with $c < d$ it holds that $a < f(c) \leq f(d) \leq A$. Using Exercise 2.1.11 we get that $f(d) \in U(A, \varepsilon)$. Hence $f[U(+\infty, \delta)] \subset U(A, \varepsilon)$ and $\lim_{x \to +\infty} f(x) = A$. □

The theorem is not valid for ordinary limits: the function $\mathrm{sgn}\colon \mathbb{R} \to \{-1, 0, 1\}$ weakly increases on $\mathbb{R}$ but $\lim_{x \to 0} \mathrm{sgn}\, x$ does not exist. We find ordinary limits of monotone functions by reducing them via Proposition 5.1.7 to one-sided limits. These we compute by means of the previous theorem and the next exercise.

Exercise 5.3.2 *Describe further variants of the theorem: for locally weakly decreasing functions and/or the right-sided limit in b, respectively in $-\infty$.*

• *Arithmetic of limits of functions.* We extend arithmetic of limits (AL) from sequences to functions. In proofs we use Heine's definition of limits of functions.

Theorem 5.3.3 (AL of functions) *Let $f, g \in \mathcal{R}$, $A \in L(M(f) \cap M(g))$, $\lim_{x\to A} f(x) = K$ and $\lim_{x\to A} g(x) = L$. Then $\lim_{x\to A}(f+g)(x) = K + L$, $\lim_{x\to A}(fg)(x) = KL$ and $\lim_{x\to A}(f/g)(x) = K/L$, if the expression on the right side is not indefinite.*

Proof. We only consider ratio, the proofs for sum and product are similar and easier. We assume that K/L is not an indefinite expression. Then $L \neq 0$ and $A \in L(M(f/g))$ (Exercise 5.3.4). Let $(a_n) \subset M(f/g) \setminus \{A\}$ be any sequence with $\lim a_n = A$. The implication $\Rightarrow$ in Heine's definition of limits of functions gives that $\lim f(a_n) = K$ and $\lim g(a_n) = L$. Using Theorem 3.1.2 we get that $\lim \frac{f(a_n)}{g(a_n)} = \frac{\lim f(a_n)}{\lim g(a_n)} = \frac{K}{L}$. Since for every sequence (a_n) as above the sequence $\left(\frac{f(a_n)}{g(a_n)}\right) = ((f/g)(a_n))$ has this limit, the implication $\Leftarrow$ in Heine's definition of limits of functions gives that also $\lim_{x\to A}(f/g)(x) = K/L$. □

Exercise 5.3.4 *Why for $L \neq 0$ is $A \in L(M(f/g))$?*

Using Proposition 5.1.13 we get easily variants of the previous theorem for one-sided limits.

Exercise 5.3.5 *Deduce from the theorem the next corollary.*

Corollary 5.3.6 (the limit of $\frac{1}{g}$ 1) *If $g \in \mathcal{R}$ and $\lim_{x\to A} g(x) = B \neq 0$ then $\lim_{x\to A}(k_1/g)(x) = \lim_{x\to A} \frac{1}{g(x)} = \frac{1}{B}$.*

• *Limits of functions and the* LO $(\mathbb{R}^*, <)$. Recall that for $M, N \subset \mathbb{R}$ the notation $M < N$ means that for every $a \in M$ and $b \in N$ it holds that $a < b$. Also recall that for any function f and any set X,

$$f[X] = f[X \cap M(f)] = \{f(x) :\ x \in X \cap M(f)\}\,.$$

In the next theorem and proposition we have $f, g \in \mathcal{R}$.

Theorem 5.3.7 (limits versus order 2) *Suppose that $\lim_{x\to A} f(x) = K$ and $\lim_{x\to B} g(x) = L$ (possibly $A \neq B$). Then the following hold.*
1. *If $K < L$ then there is a δ such that $f[P(A,\delta)] < g[P(B,\delta)]$.*
2. *If for every $\delta > 0$ there exist an $x \in P(A,\delta) \cap M(f)$ and a $y \in P(B,\delta) \cap M(g)$ such that $f(x) \geq g(y)$, then $K \geq L$.*

Proof. 1. Since $K < L$,by Exercise 2.1.12 there is an ε such that $U(K,\varepsilon) < U(L,\varepsilon)$. Then by the assumption there is a δ such that $f[P(A,\delta)] \subset U(K,\varepsilon)$ and $g[P(B,\delta)] \subset U(L,\varepsilon)$. Hence $f[P(A,\delta)] < g[P(B,\delta)]$.

2. Part 2 is the reversal of the implication in part 1. $\square$

One can strengthen the theorem in the same way as Proposition 3.3.6 strengthens Theorem 3.3.1.

Exercise 5.3.8 *Prove the following proposition.*

Proposition 5.3.9 (strengthening Theorem 5.3.7) *Let* $\lim_{x\to A} f(x) = K$ *and* $\lim_{x\to B} g(x) = L$ *(possibly* $A \neq B$*).* *Then the following hold.*
1. *If* $K < L$ *then there exist a* δ *and two numbers* a, b*, such that* $f[P(A,\delta)] < \{a\} < \{b\} < g[P(B,\delta)]$.
2. *If for every* δ *and every two real numbers* $a < b$ *there is an* $x \in P(A,\delta)\cap M(f)$ *and a* $y \in P(B,\delta) \cap M(g)$ *such that* $f(x) \geq a$ *or* $g(y) \leq b$*, then* $K \geq L$.

Exercise 5.3.10 *State versions of Theorem 5.3.7 and Proposition 5.3.9 for one-sided limits and prove them.*

As we know, $I(a,b)$ denotes the real interval $\{x \in \mathbb{R} : \min(\{a,b\}) \leq x \leq \max(\{a,b\})\}$.

Theorem 5.3.11 (squeeze theorem 2) *Suppose that* $f, g, h \in \mathcal{F}(M)$*, that* $\lim_{x\to K} f(x) = \lim_{x\to K} g(x) = L$ *and that there is a* θ *such that for every* $x \in P(K,\theta) \cap M$ *it holds that* $h(x) \in I\big(f(x), g(x)\big)$. *Then* $\lim_{x\to K} h(x) = L$.

Proof. Let f, g, h, M, K, L and θ be as stated and an ε be given. We take a $\delta \leq \theta$ such that for every $x \in P(K,\delta) \cap M$ the values $f(x)$ and $g(x)$ lie in $U(L,\varepsilon)$. For these x it holds that $h(x) \in I\big(f(x), g(x)\big) \subset U(L,\varepsilon)$ because $U(L,\varepsilon)$ is a convex set. Hence $h[P(K,\delta)] \subset U(L,\varepsilon)$ and $\lim_{x\to K} h(x) = L$. $\square$

5.4 Limits of composite functions

Composing functions has no analogue for sequences, which makes the next theorem a relative novelty. It is usually stated as an implication but we present it as an equivalence.

• *A theorem on limits of composite functions.* In the following $f, g \in \mathcal{R}$. Recall that $f(g)\colon M(f(g)) \to \mathbb{R}$ has the definition domain $M(f(g)) = \{x \in M(g) : g(x) \in M(f)\}$. It is a subset, possibly proper, of $M(g)$.

Theorem 5.4.1 (limits of CF) *Let* $\lim_{x\to A} g(x) = K$, $\lim_{x\to K} f(x) = L$ *and* $A \in L(M(f(g)))$. *Then* $\lim_{x\to A} f(g)(x) = L \iff$ *condition* 1 *or condition* 2 *holds.*

1. *The implication* $K \in M(f) \Rightarrow f(K) = L$ *holds.*
2. *There is a* θ *such that* $K \notin g[P(A, \theta)]$.
If neither condition 1 *nor condition* 2 *holds then* $\lim_{x\to A} f(g)(x)$ *does not exist or equals* $f(K)$ *but* $f(K) \neq L$.

Proof. Let A, g, K, f and L be as stated and an ε be given. By the assumption there is a δ' such that (a) $f[P(K, \delta')] \subset U(L, \varepsilon)]$, and a δ such that (b) $g[P(A, \delta)] \subset U(K, \delta')$. Suppose that condition 1 holds. Then inclusion (a) strengthens to $f[U(K, \delta')] \subset U(L, \varepsilon)$ and

$$f(g)[P(A,\, \delta)] = f[\, g[P(A,\, \delta)]\,] \subset f[U(K,\, \delta')] \subset U(L,\, \varepsilon)\,.$$

Hence $\lim_{x\to A} f(g)(x) = L$. Suppose that condition 2 holds. We can take the previous δ such that in addition $\delta \leq \theta$, where θ is as in condition 2. Then inclusion (b) strengthens to $g[P(A, \delta)] \subset P(K, \delta')$ and

$$f(g)[P(A,\, \delta)] = f[\, g[P(A,\, \delta)]\,] \subset f[P(K,\, \delta')] \subset U(L,\, \varepsilon)\,.$$

Hence again $\lim_{x\to A} f(g(x)) = L$.

Suppose that neither condition 1 nor condition 2 holds. The former means that $K \in M(f)$ but $f(K) \neq L$. The latter means that for every n there is an $a_n \in P(A, 1/n) \cap M(g)$ such that $g(a_n) = K$. Then $(a_n) \subset M(f(g)) \setminus \{A\}$, $\lim\, a_n = A$ and

$$\lim\, f(g)(a_n) = \lim\, f(g(a_n)) = \lim\, f(K) = f(K)\ \ (\neq L)\,.$$

By Heine's definition of limits of functions the limit $\lim_{x\to A} f(g(x))$ either does not exist or equals to $f(K)$, which is not L. □

Condition 1 is satisfied whenever $K \notin M(f)$, for example if $K = \pm\infty$. Similarly condition 2 is satisfied if the function g is injective. We get the following corollary.

Corollary 5.4.2 (of the theorem) *Let* $\lim_{x\to A} g(x) = K$, $\lim_{x\to K} f(x) = L$ *and* $A \in L(M(f(g)))$. *If* $K = \pm\infty$ *or* g *is injective* *then* $\lim_{x\to A} f(g)(x) = L$.

Exercise 5.4.3 *Prove the last theorem with the help of Heine's definition of limits of functions.*

• *Using Theorem 5.4.1.* We give several applications of the theorem and the corollary. One can often encounter the next two equivalences of limits of functions.

Corollary 5.4.4 (shifting the argument to 0**)** *Suppose that* $f \in \mathcal{F}(M)$ *and* $b \in \mathbb{R}$. *Then* $\lim_{x\to b} f(x) = L \iff \lim_{x\to 0} f(x + b)(x) = L$.

Proof. Let f and b be as stated. The implication $\Rightarrow$: let $\lim_{x\to b} f(x) = L$. Thus $b \in L(M)$. We take the outer function f, the injective inner function $g(x) \equiv x + b$, $A \equiv 0$ and $K \equiv b$. It holds that $M(f(g)) = X \equiv \{x - b :\ x \in M\}$ and $0 \in L(X)$. Also, $\lim_{x\to 0} g(x) = b$ and $\lim_{x\to b} f(x) = L$. Corollary 5.4.2 gives that $\lim_{x\to 0} f(x+b)(x) = \lim_{x\to 0} f(g)(x) = L$.

The implication $\Leftarrow$: let $\lim_{x\to 0} f(x+b)(x) = L$. Thus $0 \in L(X)$. We take the outer function $g(x) \equiv f(x+b)$, the injective inner function $h(x) \equiv x - b$, $A \equiv b$ and $K \equiv 0$. It holds that $M(g(h)) = M(f) = M$ and $b \in L(M)$. Clearly, $\lim_{x\to b} h(x) = 0$, $\lim_{x\to 0} g(x) = L$ and $g(h) = f((x-b)+b) = f$. Corollary 5.4.2 gives that $\lim_{x\to b} f(x) = \lim_{x\to b} g(h)(x) = L$. $\Box$

Corollary 5.4.5 **($\to 0^{\pm} \iff \to \pm\infty$)** *Let $f \in \mathcal{R}$. Then $\lim_{x\to\pm\infty} f(x) = L$ $\iff$ $\lim_{x\to 0^{\pm}} f(\frac{1}{x})(x) = L$ (equal signs). Note that the last limit is one-sided.*

Proof. We confine to the sign $+$, the case of $-$ is similar. The implication $\Rightarrow$: let $\lim_{x\to+\infty} f(x) = L$, so that $+\infty \in L(M(f))$. We take the outer function f, the injective inner function $g \equiv \frac{1}{x} \,|\, (0, +\infty)$, $A \equiv 0$ and $K \equiv +\infty$. Clearly, $0 \in L(M(f(g)))$ and $\lim_{x\to 0} g(x) = +\infty$. By Corollary 5.4.2 we have that $L = \lim_{x\to 0} f(g)(x) = \lim_{x\to 0^+} f(\frac{1}{x})(x)$.

The implication $\Leftarrow$: let $\lim_{x\to 0^+} f(\frac{1}{x})(x) = L$, so that $0 \in L^+(f(\frac{1}{x}))$. For g as above we take the outer function $F \equiv f(g)$, the injective inner function g, $A \equiv +\infty$ and $K \equiv 0$. Then $A \in L(M(F(g)))$ and $\lim_{x\to 0} F(x) = \lim_{x\to 0^+} f(\frac{1}{x})(x) = L$. Clearly, $g(g) = x \,|\, (0, +\infty)$. Corollary 5.4.2 gives that $L = \lim_{x\to+\infty} F(g)(x)$. Exercise 1.3.12 implies that $f(g)(g) = f(g(g)) = f(x \,|\, (0, +\infty))$. Thus the last limit equals $\lim_{x\to+\infty} f(x)$. $\Box$

In computing limits we use that $\lim_{x\to b} f(x) = L \iff \lim_{x\to 0} f(x+b)(x) = L$ and that $\lim_{x\to\pm\infty} f(x) = L \iff \lim_{x\to 0^{\pm}} f(\frac{1}{x})(x) = L$ without thinking. Above we formally justified it.

Exercise 5.4.6 *Deduce Corollary 5.3.6 from Theorem 5.4.1. We repeat this corollary here for the convenience of the reader.*

Corollary 5.4.7 (the limit of $\frac{1}{g}$ 2) *If $g \in \mathcal{R}$ and $\lim_{x\to A} g(x) = B \neq 0$ then $\lim_{x\to A} \frac{1}{g(x)} = \frac{1}{B}$.*

5.5 Asymptotic notation

Books on computational complexity and algorithms intersect with books on analysis in definitions of asymptotic notation, see for instance Exercise 5.5.5. In this section we present our definitions. What does the adjective "asymptotic" really mean? We reveal it in a moment in Definition 5.5.3.

• *Asymptotic relations.* Let X and Y be sets. The symmetric difference $X \Delta Y \equiv (X \setminus Y) \cup (Y \setminus X)$. Next f and g are functions in $\mathcal{R}$, the set of functions f with the definition domain $M(f) \subset \mathbb{R}$ and range $\mathbb{R}$.

Definition 5.5.1 (almost equality) *Functions f and g are almost equal, in symbols $f \doteq g$, if $M(f)\Delta M(g)$ and $\{x \in M(f) \cap M(g) :\ f(x) \neq g(x)\}$ are finite sets.*

Exercise 5.5.2 *The relation $\doteq$ on the set $\mathcal{R}$ is an equivalence relation.*

Definition 5.5.3 (asymptotic relations) *$\mathcal{A} \subset \mathcal{R} \times \mathcal{R}$ is an asymptotic relation (on $\mathcal{R}$) if for any functions f, g, f_0 and g_0 in $\mathcal{R}$ such that $f \doteq f_0$ and $g \doteq g_0$ the equivalence $f\mathcal{A}g \iff f_0\mathcal{A}g_0$ holds.*

Asymptoticity of relations between functions is a concept similar to robustness of properties of sequences in Definition 2.1.18.

• *Asymptotic symbols O, $\ll$ and other.* These are not defined by limits. We say that a function $f \in \mathcal{R}$ is bounded on N ($\subset \mathbb{R}$) if there is a constant $c \geq 0$ such that for every $x \in M(f) \cap N$ one has that $|f(x)| \leq c$.

Definition 5.5.4 (O and $\ll$) *Let $f, g \in \mathcal{R}$ and let $N \subset \mathbb{R}$. We write that $f = O(g)$ (on N) and say that on N the function f is big O of g if the function $\frac{f}{g} \mid N$ is bounded, which means that there is a constant $c \geq 0$ such that for every $x \in M(f/g) \cap N$ one has that $\left|\frac{f(x)}{g(x)}\right| \leq c$. Notation $f(x) \ll g(x)$ (on N) means the same.*

For example, $20x^2 + 100x - 1 = O(x^2)$ (on $[1, +\infty)$). The notation $f = g + O(h)$ (on N) is in the error form and means that $f - g = O(h)$ (on N). Notation like $\log x = O_\varepsilon(x^\varepsilon)$ (on $[1, +\infty)$) means that the constant c in Definition 5.5.4 is a function of ε. The notation that $f \gg g$ (on N) and that $f = \Omega(g)$ (on N) mean that $g \ll f$ (on N). The notation that $f = \Theta(g)$ (on N) and that $f \asymp g$ (on N) both mean that simultaneously $f \ll g$ (on N) and $g \ll f$ (on N).

Exercise 5.5.5 *In* [25, p. 19] *we find this definition of big O: "In particular, for $f, g\colon \mathbf{N} \to \mathbf{N}$, $g = O(f)$ means that $g(n) \leq cf(n) + c$ for some constant $c \geq 1$ and all n". Is this big O equivalent with our big O in Definition 5.5.4?*

Exercise 5.5.6 *Answer next questions.*
1. *Is $x^2 = O(x^3)$ (on $\mathbb{R} \setminus (-1, 1)$)?*
2. *Is $x^2 = O(x^3)$ (on $\mathbb{R}$)?*
3. *Is $x^3 = O(x^2)$ (on $\mathbb{R}$)?*
4. *Is $x^3 = O(x^2)$ (on $(-20, 20)$)?*
5. *Is $\log x = O(x^{1/3})$ (on $(0, +\infty)$)?*
6. *Is $\log x = O(x^{1/3})$ (on $(1, +\infty)$)?*

Proposition 5.5.7 (O is asymptotic) *For every set $N \subset \mathbb{R}$ the $f = O(g)$ (on N) relation on $\mathcal{R}$ is asymptotic.*

Proof. Suppose that $N \subset \mathbb{R}$ and that f, g, f_0 and g_0 are functions in $\mathcal{R}$ such that $f \doteq f_0$, $g \doteq g_0$ and $f = O(g)$ (on N). We show that then $f_0 = O(g_0)$ (on N) as well. By the assumption there is a constant $c \geq 0$ such that for every $x \in M(f/g) \cap N$ it holds that $\left|\frac{f(x)}{g(x)}\right| \leq c$. It follows from the definition of $\doteq$ that the set $X \equiv \frac{f_0}{g_0}[N] \setminus \frac{f}{g}[N]$ is finite (Exercise 5.5.8). For $X \neq \emptyset$ we set $d \equiv \max(\{|x| :\ x \in X\})$, and $d \equiv 1$ if $X = \emptyset$. Then for every $x \in M(f_0/g_0) \cap N$ we have that $\left|\frac{f_0(x)}{g_0(x)}\right| \leq \max(\{c, d\})$, as needed. □

Exercise 5.5.8 *Why is the set X finite?*

• *Asymptotic symbols o, ω and $\sim$.* These are defined by limits.

Definition 5.5.9 (o and ω) *Let $f, g \in \mathcal{R}$ and let $A \in L(M(f/g))$. We write $f(x) = o(g(x))$ $(x \to A)$ and say that for $x \to A$ the function f is little o of g if* $\lim_{x \to A} \frac{f(x)}{g(x)} = 0$. *Notation $f(x) = \omega(g(x))$ $(x \to A)$ means the same.*

Like before notation $f = g + o(h)$ $(x \to A)$ means that $f - g = o(h)$ $(x \to A)$.

Definition 5.5.10 ($\sim$) *Let $f, g \in \mathcal{R}$ and let $A \in L(M(f/g))$. We write that $f(x) \sim g(x)$ $(x \to A)$ and say that for $x \to A$ the function f is asymptotic to g if* $\lim_{x \to A} \frac{f(x)}{g(x)} = 1$.

For example, $x^2 \sim (x-3)^2$ $(x \to +\infty)$.

Exercise 5.5.11 *Answer next questions.*
1. *Is* $x^2 = o(x^3)$ $(x \to +\infty)$*?*
2. *Is* $x^3 = o(x^2)$ $(x \to 0)$*?*
3. *Is* $x^2 = o(x^3)$ $(x \to 0)$*?*
4. *Is* $(x+1)^3 \sim x^3$ $(x \to 1)$*?*
5. *Is* $(x+1)^3 \sim x^3$ $(x \to +\infty)$*?*
6. *Is* $\mathrm{e}^{-1/x^2} = o(x^{20})$ $(x \to 0)$*?*

Proposition 5.5.12 (o and $\sim$ are asymptotic) *For every element $A \in \mathbb{R}^*$ the $f(x) = o(g(x))$ $(x \to A)$ relation and the $f(x) \sim g(x)$ $(x \to A)$ relation on $\mathcal{R}$ are asymptotic.*

Proof. Suppose that $A \in \mathbb{R}^*$ and that f, g, f_0 and g_0 are functions in $\mathcal{R}$ such that $f \doteq f_0$, $g \doteq g_0$ and $f = o(g)$ $(x \to A)$. We show that then $f_0 = o(g_0)$ $(x \to A)$ as well. For the asymptotic symbol $\sim$ the argument is the same. By the assumption, $A \in L(f/g)$ and $\lim_{x \to A} \frac{f(x)}{g(x)} = 0$. By Exercise 5.5.8 the set $\frac{f_0}{g_0}[\mathbb{R}] \Delta \frac{f}{g}[\mathbb{R}]$ is finite. It follows that $A \in L(f_0/g_0)$ and that $\lim_{x \to A} \frac{f_0(x)}{g_0(x)} = \lim_{x \to A} \frac{f(x)}{g(x)} = 0$. □

• *Properties of asymptotic symbols.* We do not have time and space to treat this theoretically and practically important topic in a more systematic way; we confine to one proposition and one exercise.

Proposition 5.5.13 (o yields O) *If functions $f, g \in \mathcal{R}$ are such that $f(x) = o(g(x))$ $(x \to A)$ <u>then</u> there is a θ such that $f = O(g)$ (on $P(A, \theta)$).*

Proof. Let f, g and A be as stated in the hypothesis of the implication. Since $\lim_{x\to A} \frac{f(x)}{g(x)} = 0$, for the given $\varepsilon = 1$ there is a θ such that for every $x \in M(f/g) \cap P(A, \theta)$ we have that

$$\left|\tfrac{f(x)}{g(x)}\right| = \left|\tfrac{f(x)}{g(x)} - 0\right| \le 1\,.$$

Hence $f = O(g)$ (on $P(A, \theta)$), with the upper bounding constant $c = 1$. □

Exercise 5.5.14 *Prove the next proposition.*

Proposition 5.5.15 (properties of O, o and $\sim$) *Let $f(x)$, $g(x)$ and $h(x)$ be in $\mathcal{R}$, $N \subset \mathbb{R}$ and $A \in \mathbb{R}^*$. <u>Then</u> the following hold.*
1. *If $f = O(h)$ (on N) and $g = O(h)$ (on N) then $f + g = O(h)$ (on N).*
2. *If $f = O(h)$ (on N) and g is bounded on N then $fg = O(h)$ (on N).*
3. *If $f = O(h)$ (on N) and $\frac{1}{g}$ is bounded on N then $f/g = O(h)$ (on N).*
4. *If $f = o(h)$ $(x \to A)$, $g = o(h)$ $(x \to A)$ and $A \in L(\frac{f+g}{h})$ then $f + g = o(h)$ $(x \to A)$.*
5. *If $f = o(h)$ $(x \to A)$, g is bounded on a $P(A, \theta)$ and $A \in L(\frac{fg}{h})$ then $fg = o(h)$ $(x \to A)$.*
6. *If $f = o(h)$ $(x \to A)$, $\frac{1}{g}$ is bounded on a $P(A, \theta)$ and $A \in L(\frac{f}{gh})$ then $f/g = o(h)$ $(x \to A)$.*
7. *If $f(x) \sim h(x)$ $(x \to A)$, $g(x) = o(h(x))$ $(x \to A)$ and $A \in L(\frac{f+g}{h})$ then $f(x) + g(x) \sim h(x)$ $(x \to A)$.*
8. *If $f(x) \sim h(x)$ $(x \to A)$, $\lim_{x\to A} g(x) = 1$ and $A \in L(\frac{fg}{h})$ then $f(x)g(x) \sim h(x)$ $(x \to A)$.*
9. *If $f(x) \sim h(x)$ $(x \to A)$, $\lim_{x\to A} g(x) = 1$ and $A \in L(\frac{f}{gh})$ then $f(x)/g(x) \sim h(x)$ $(x \to A)$.*

The notation o, O and $\sim$ originated with the German mathematicians *Paul Bachmann (1837–1920)* and *Edmund Landau (1877–1938).* Asymptotic symbols $\ll$, $\gg$ and $\asymp$ are due to the Russian-Soviet mathematician *Ivan M. Vinogradov (1891–1983).* We do not say that their asymptotic symbols are identical with ours in Definitions 5.5.4, 5.5.9 and 5.5.10.

• *Famous asymptotics.* For real x we define $\pi(x)$, the value of the *prime number counting function*, to be the number of primes $p \in \mathbb{N}$ such that $p \le x$. For example, $\pi(\sqrt{20}) = |\{2, 3\}| = 2$. In 1896 the French mathematician *Jacques Hadamard (1865–1963)* and, in parallel with him, the Belgian mathematician *Charles Jean de la Vallée Poussin (1866–1962)* proved the famous <u>Prime Number Theorem</u> which says that

$$\pi(x) \sim \frac{x}{\log x} \quad (x \to +\infty)\,.$$

In Section 2.3 we introduced for $k, n \in \mathbb{N}$ the number $r_k(n) \in \mathbb{N}_0$ as the size of the largest set $X \subset [n]$ not containing an arithmetic progression with length k. In 1975 E. Szemerédi proved the by now famous theorem, which we mentioned in Section 2.3, that for every k it holds that

$$r_k(n) = o(n) \quad (n \to +\infty)\,.$$

For $x \in \mathbb{R}$ we define $D(x) \equiv |\{(m, n) \in \mathbb{N}^2 :\ mn \le x\}|$.

Exercise 5.5.16 *Show that $D(x) = \sum_{n \le x} \tau(n)$ where $\tau(n)$ denotes the number of divisors of n; for example $\tau(28) = |\{1, 2, 4, 7, 14, 28\}| = 6$.*

The (Dirichlet) divisor problem is the problem to estimate the error in asymptotics of $D(x)$. In 1849 the German mathematician *Peter L. Dirichlet (1805–1859)* proved that

$$D(x) = x \log x + (2\gamma - 1)x + O(\sqrt{x}) \quad (\text{on } [1, +\infty))$$

where γ is Euler's constant. In 1903 the Russian-Ukrainian mathematician *Georgij F. Voronoj (1868–1908)* improved it to

$$D(x) = x \log x + (2\gamma - 1)x + O\big(x^{1/3} \log x\big) \quad (\text{on } [2, +\infty))\,.$$

Exercise 5.5.17 *Why not on $[1, +\infty)$ as before?*

The 20th century saw a series of further improvements in the divisor problem. The current record holder is the British mathematician *Martin N. Huxley (1944)* who proved in 2003 that for every ε one has that

$$D(x) = x \log x + (2\gamma - 1)x + O_\varepsilon\big(x^{131/416+\varepsilon}\big) \quad (\text{on } [1, +\infty))\,.$$

For $n \in \mathbb{N}$ and an algorithm (Turing machine) T for multiplying integers we define $T(n)$ as the smallest $k \in \mathbb{N}$ such that T multiplies any two n-digit numbers in at most k steps. The elementary school algorithm T_{es} works in $T_{\text{es}}(n) = O(n^2)$ (on $\mathbb{N}$) steps. In 1960 the Soviet-Russian mathematician *Anatolij A. Karacuba (1937–2008)* invented an algorithm T_{K} working in $T_{\text{K}}(n) = O\big(n^{\log_2 3}\big) = O\big(n^{1.585\dots}\big)$ (on $\mathbb{N}$) steps. In 2021 the Australian computer scientist *David Harvey* with the Dutch computer scientist *Joris van der Hoeven (1971)* discovered an algorithm T_{HH} for multiplying integers that has complexity

$$T_{\text{HH}}(n) = O(n \log n) \quad (\text{on } \mathbb{N})\,.$$

Exercise 5.5.18 *But for $n = 1$ the ratio $\frac{T_{\text{HH}}(n)}{n \log n}$ is not defined?*

• *Asymptotic expansions.* Compared to the above asymptotic symbols, asymptotic expansions capture asymptotic behavior of functions in greater detail.

Definition 5.5.19 (asymptotic scale) *Let* $(f_n) \subset \mathcal{R}$. *If* $A \in L(\bigcap_{n=1}^{\infty} M(f_n))$, *if there is a* θ *such that every* $f_n \neq 0$ *on* $P(A, \theta)$ *and if for every* n *it holds that* $f_{n+1}(x) = o(f_n(x))$ $(x \to A)$, *we say that* (f_n) *is an* <u>*asymptotic scale*</u> *for* $x \to A$.

For example, (x^{-n}) is an asymptotic scale for $x \to +\infty$, and (x^n) for $x \to 0$.

Definition 5.5.20 (asymptotic expansion) *Suppose that* $(a_n) \subset \mathbb{R}$, $f \in \mathcal{R}$ *and that* $(f_n) \subset \mathcal{R}$ *is an asymptotic scale for* $x \to A$. *If for every* n *it holds that*

$$f(x) = \sum_{i=1}^{n} a_i f_i(x) + o(f_{n+1}(x)) \quad (x \to A)\,,$$

we call the sequence of functions $(a_n f_n(x))$ *an* <u>*asymptotic expansion*</u> *of* $f(x)$ *for* $x \to A$ *and write this in symbols as*

$$\underline{f(x) \approx \sum_{n=1}^{\infty} a_n f_n(x) \quad (x \to A)}\,.$$

It follows from this definition that there is a $\theta = \theta_n$ such that $f = \sum_{i=1}^{n} a_i f_i + O(f_{n+1})$ (on $P(A, \theta)$). Hence we have the asymptotic approximation $\sum_{i=1}^{n} a_i f_i$ to f with an error of order f_{n+1}.

Exercise 5.5.21 *Prove it.*

The assumption that (f_n) is an asymptotic scale only ensures that as n grows, magnitudes of these errors get smaller and smaller. On the other hand, for fixed $x \in \mathbb{R}$ nothing is assumed about the convergence of the series $\sum a_n f_n(x)$ and it typically diverges. Usually it is *not* true that $f(x) = \sum_{n=1}^{\infty} a_n f_n(x)$.

The Scottish mathematician *James Stirling (1692–1770)* derived an asymptotic expansion of $\log(n!)$ for $n \in \mathbb{N}$ and $n \to +\infty$ already in 1730, we state it in a moment, but the modern theory of asymptotic expansions originates with the French mathematician *Henri Poincaré (1854–1912)* in his memoir of 1886. For expositions of the theory of asymptotic expansions see [11, 14]. We conclude this chapter with three examples of them; proofs will be given in *MA 1*$^+$.

Theorem 5.5.22 (AE of $\log(n!)$**)** *For* $n \in \mathbb{N}$ *and* $n \to +\infty$,

$$\log(n!) \approx (n + \tfrac{1}{2}) \log n - n + \tfrac{1}{2} \log(2\pi) + \sum_{k=1}^{\infty} \tfrac{B_{2k}}{2k(2k-1)} \cdot n^{1-2k}\,.$$

Theorem 5.5.23 (AE of harmonic numbers) *For* $n \in \mathbb{N}$ *and* $n \to +\infty$,

$$h_n = \sum_{i=1}^{n} \tfrac{1}{i} \approx \log n + \gamma + \tfrac{1}{2n} - \sum_{k=1}^{\infty} \tfrac{B_{2k}}{2k} \cdot n^{-2k}\,.$$

Here B_k $(\in \mathbb{Q})$, $k \in \mathbb{N}_0$, denote the <u>Bernoulli numbers</u>. They are defined by the power series expansion

$$\tfrac{x}{\exp x - 1} = \sum_{k=0}^{\infty} \tfrac{B_k}{k!} \cdot x^k\,.$$

B_k are named after their discoverer, the Swiss mathematician *Jacob Bernoulli (1655/54–1705)*, and have initial values $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_{2k+1} = 0$ for every $k \in \mathbb{N}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, $B_{10} = \frac{5}{66}$, $B_{12} = -\frac{691}{2730}$, $B_{14} = \frac{7}{6}$ and $B_{16} = -\frac{3617}{510}$.

Exercise 5.5.24 *Using the definition derive a recurrence for B_k and show that $B_{2k+1} = 0$ for every $k \in \mathbb{N}$.*

We read in [11] that the formula in Theorem 5.5.22 is in fact not the original expansion of Stirling but a similar formula due to the French mathematician *Abraham de Moivre (1667–1754)*. The asymptotic expansion of harmonic numbers is due to L. Euler.

The third asymptotic expansion is much more recent, see [27]. Recall that a graph $G = (V, E)$ is a pair of a nonempty finite set V of vertices and a set $E \subset \binom{V}{2}$ of edges, where $\binom{V}{2}$ denotes the set of two-element subsets of V. Recall that G is connected if for every partition $\{A, B\}$ of V with two blocks there is an edge $e \in E$ that intersects both blocks A and B.

Theorem 5.5.25 (AE of probability of connectedness) *For $n \in \mathbb{N}$ and $n \to +\infty$,*

$$\tfrac{1}{2^{n(n-1)/2}} \cdot |\{G = ([n],\, E) :\ G \text{ is connected}\}| \approx 1 - \textstyle\sum_{k=1}^{\infty} t_k 2^{k(k+1)/2} \cdot \binom{n}{k} 2^{-kn}$$

where $t_k \in \mathbb{N}$ is the number of irreducible tournaments of size k.

We repeat that this result is due to [27]. Since there are $2^{n(n-1)/2}$ graphs $G = ([n], E)$, the product on the left side is the probability that a random graph with the vertex set $[n]$ is connected; by the first term of the expansion, for $n \to +\infty$ this probability goes to 1. What is t_k? A tournament $T = (V, E)$ is a pair of a nonempty finite set V of vertices and an irreflexive relation $E \subset V \times V$ on V such that for every two distinct vertices $u, v \in V$ exactly one pair of (u, v) and (v, u) is in E. We say that T is irreducible if for every partition $\{A, B\}$ of V with two blocks there exist pairs $(a, b) \in A \times B$ and $(c, d) \in B \times A$ such that $(a, b), (c, d) \in E$. Then t_k is the number of irreducible tournaments $T = ([k], E)$. By [29, A054946] the sequence of numbers t_n begins as

$$(t_n) = (t_1,\, t_2,\, \dots) = (1,\, 0,\, 2,\, 24,\, 544,\, 22320,\, 1677488,\, \dots)\,.$$

The article [4] develops a calculus for computing asymptotic expansions; it can be applied to a class of problems in enumerative combinatorics.

Chapter 6

Continuous functions

In this chapter we investigate continuous functions. It is a revised version of the lecture

`https://kam.mff.cuni.cz/~klazar/MAI24_pred6.pdf`

I gave on March 28, 2024. In Section 6.1 we introduce dense and sparse sets and state Blumberg's Theorem 6.1.12. It says that every function $f\colon \mathbb{R} \to \mathbb{R}$ has a continuous restriction $f \mid M$ to a set $M \subset \mathbb{R}$ dense in $\mathbb{R}$. A proof will be given in *MA 1*$^+$. In Section 6.2 in Theorem 6.2.3 we show that the set of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ is in bijection with $\mathbb{R}$. The main result of Section 6.3 is Theorem 6.3.1 which says that continuous functions attain any intermediate value.

In Section 6.4 we introduce real compact sets. The "min-max" Theorem 6.4.1 shows that every continuous function with a compact definition domain attains both minimum and maximum. We define open and closed sets of real numbers, discuss their basic properties and in Theorem 6.4.13 we characterize compact sets. By Proposition 6.5.2 any continuous function with compact definition domain is uniformly continuous. By Theorem 6.5.6 every uniformly continuous function has a unique continuous extension to the closure of the definition domain. We leave it to the reader in Exercise 6.5.7 to prove by means of the last theorem a version of the min-max theorem for functions $f \in \mathcal{F}(M)$ with $M \subset \mathbb{Q}$.

Section 6.6 is concerned with the interplay of continuity and various operations on functions. Theorem 6.6.1 deals with arithmetic operations, and in Theorem 6.6.3 we prove continuity of functions defined as sums of power series. Theorem 6.6.9 deals with composite functions, and Theorem 6.6.11 with inverse functions — we described this theorem in detail in *Some highlights*. We conclude this chapter with Theorem 6.6.16 which says that every elementary function is continuous.

6.1 Blumberg's theorem

• *Continuous functions.* By Definition 5.2.1 a function $f \in \mathcal{F}(M)$ is continuous at a point $a \in M$ if for every ε there is a δ such that $f[U(a,\delta)] \subset U(f(a),\varepsilon)$. We will make use several times of the equivalence

$$f \text{ is continuous at } a \iff \forall\,(a_n) \subset M\,\big(\lim a_n = a \Rightarrow \lim f(a_n) = f(a)\big)\,. \quad \text{(H)}$$

This Heine's definition of point-wise continuity was proven in Exercise 5.2.6.

Definition 6.1.1 (continuous functions) *A function $f \in \mathcal{R}$ is continuous (on $M(f)$) if it is continuous at every point $b \in M(f)$. The set of continuous functions $f \in \mathcal{F}(M)$ is denoted by $\mathcal{C}(M)$. We set $\mathcal{C} \equiv \bigcup_{M \subset \mathbb{R}} \mathcal{C}(M)$. We call the functions in $\mathcal{R} \setminus \mathcal{C}$ discontinuous.*

Exercise 6.1.2 *Every function in $\mathcal{R}$ with finite definition domain is continuous.*

Exercise 6.1.3 *Every constant function $k_a \in \mathcal{F}(\mathbb{R})$, $a \in \mathbb{R}$, is continuous.*

Exercise 6.1.4 *The identity $x = \mathrm{id}_{\mathbb{R}} \in \mathcal{F}(\mathbb{R})$ is continuous.*

Proposition 6.1.5 (continuity of restriction 1) *If $f \in \mathcal{C}$ and X is a set then also $f \,|\, X \in \mathcal{C}$.*

Proof. Let f and X be as stated, and let $b \in M(f \,|\, X)$ and an ε be given. Thus $b \in M(f)$ and since $f \in \mathcal{C}$ there is a δ such that $f[U(b,\delta)] \subset U(f(b),\varepsilon)$. But $U(b,\delta) \cap M(f) \cap X \subset U(b,\delta) \cap M(f)$, so that

$$(f \,|\, X)[U(b,\,\delta)] \subset f[U(b,\,\delta)] \subset U(f(b),\,\varepsilon)$$

and $f \,|\, X$ is continuous at b. This holds for every point $b \in M(f \,|\, X)$ and $f \,|\, X \in \mathcal{C}$. □

• *Dense and sparse sets.* Suppose that $N \subset M \subset \mathbb{R}$. The set N is dense in M if for every $a \in M$ and δ it holds that $U(a,\delta) \cap N \neq \emptyset$.

Exercise 6.1.6 *N is dense in M $\iff$ for every point $a \in M$ there is a sequence $(b_n) \subset N$ such that $\lim b_n = a$.*

Exercise 6.1.7 *Show that both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.*

Let $N \subset M \subset \mathbb{R}$. The set N is sparse in M if

$$\forall\,(a,\,b) \subset M\,\exists\,c,\,d\,\big(a \le c < d \le b \wedge N \cap (c,\,d) = \emptyset\big)$$

— any nontrivial interval in M contains a nontrivial subinterval disjoint to N.

Exercise 6.1.8 *Show that* $N \equiv \{\frac{1}{n}:\ n \in \mathbb{N}\}$ *is sparse in* $M \equiv [0,1]$.

Proposition 6.1.9 (density and continuity) *Let* $f, g \in \mathcal{C}(M)$, N *be dense in* M *and* $f \,|\, N = g \,|\, N$. *Then* $f = g$.

Proof. Let $b \in M$ and $(a_n) \subset N$ have $\lim a_n = b$. By (H) we have that $f(b) = f(\lim a_n) = \lim f(a_n) = \lim g(a_n) = g(\lim a_n) = g(b)$. □

We say that $g \in \mathcal{C}$ is a kernel of $f \in \mathcal{C}$ if g is a restriction of f and $M(g)$ is dense in $M(f)$. Then we easily reconstruct f from g: for $b \in M(f)$ we take a sequence $(a_n) \subset M(g)$ with $\lim a_n = b$, and then it holds that $f(b) = \lim f(a_n)$.

Proposition 6.1.10 (compression in $\mathcal{C}$) *Every* $f \in \mathcal{C}$ *has an at most countable kernel.*

Proof. It suffices to show that every set $M \subset \mathbb{R}$ has an at most countable dense subset N. We order all finite decimal expansions in a sequence (a_n) $(\subset \mathbb{Q})$. Let $X_n \equiv \{b_n\}$ for some $b_n \in M$ with an initial segment of its decimal expansion equal to a_n if such b_n exists, and else let $X_n \equiv \emptyset$. The desired set is $N \equiv \bigcup_{n=1}^{\infty} X_n$ (Exercise 6.1.11). We again used the axiom of choice. □

Hence every function $f \in \mathcal{C}$ can be compressed in an at most countable continuous restriction. From it f can be recovered by means of the described limits.

Exercise 6.1.11 *In the previous proof, why is* N *dense in* M*?*

The following theorem was proven in 1922.

Theorem 6.1.12 (Blumberg's) *For every function* $f \in \mathcal{F}(\mathbb{R})$ *there is a set* $M \subset \mathbb{R}$ *such that* $f \,|\, M$ *is continuous and* M *is dense in* $\mathbb{R}$.

The American mathematician *Henry Blumberg (1886–1950)* was born in northern Lithuania in the town Žagarė, but the family emigrated to America already in 1891. We prove Blumberg's theorem in *MA 1*$^+$.

6.2 The number of continuous functions

We show that there exists a bijection between the sets $\mathcal{C}(\mathbb{R})$ and $\mathbb{R}$.

- *The Cantor–Bernstein theorem.* We use it in the construction of this bijection.

Theorem 6.2.1 (Cantor–Bernstein) *Let* X *and* Y *be sets. If there are injections* $f\colon X \to Y$ *and* $g\colon Y \to X$ *then there is a bijection* $h\colon X \to Y$.

We mentioned G. Cantor earlier. *Felix Bernstein (1878–1956)* was a German mathematician. We prove the theorem in *MA 1*$^+$. For example, $(m, n) \mapsto 2^m 3^n$ is an injection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ and $n \mapsto (1, n)$ is an injection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$, hence by the C.–B. theorem there exists a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. In the next exercise we define such bijection directly.

Exercise 6.2.2 *The function $s\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$, where $s(m,n)=(2m-1)\cdot 2^{n-1}$, is a bijection.*

• *So how many continuous functions $f\colon \mathbb{R}\to\mathbb{R}$ are there?* As many as real numbers.

Theorem 6.2.3 (cardinality of continuous functions) *There exists a bijection $h\colon \mathbb{R}\to\mathcal{C}(\mathbb{R})$.*

Proof. By Theorem 6.2.1 it suffices to have injections $f\colon \mathbb{R}\to\mathcal{C}(\mathbb{R})$ and $g\colon \mathcal{C}(\mathbb{R})\to\mathbb{R}$. The injection f is described in Exercise 6.2.4. We define g. We encode any function $j\in\mathcal{C}(\mathbb{R})$ in a single number $g(j)\in\mathbb{R}$. We regard real numbers as decimal expansions with the sign $+$ omitted, for example $-\pi = -3.1415\dots$ or $2022.0000\dots$. We employ two bijections

$$r\colon \mathbb{Q}\to\mathbb{N} \text{ and } s\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}\,.$$

Exercise 6.2.2 contains a formula for s. The ten digits 0, 1, ..., 9, the decimal point . and the sign $-$ are encoded in the bijection

$$c\colon \{0,\,1,\,\dots,\,9,\,.,\,-\}\equiv X\to Y\equiv\{00,\,01,\,\dots,\,09,\,10,\,11\}$$

by pairs of digits, for example as

$$c(0)\equiv 00,\ c(1)\equiv 01,\ \dots,\ c(9)\equiv 09,\ c(.)\equiv 10 \ \text{ and } \ c(-)\equiv 11\,.$$

The value $g(j)\in\mathbb{R}$ of g on $j\in C(\mathbb{R})$ has the expansion

$$g(j)=0.\,a_1\,a_2\,a_3\,\dots\,a_{2n-1}\,a_{2n}\,\dots\quad(\in[0,\,1))\,,$$

whose two-element blocks of digits $a_{2n-1}a_{2n}$ $(\in Y)$ code the values $j(\alpha)$ of j on all fractions α. By Exercise 6.1.7 and Proposition 6.1.9 these values uniquely determine j. So let $\alpha\in\mathbb{Q}$, $k\equiv r(\alpha)$ $(\in\mathbb{N})$ and let us denote the value $j(\alpha)$ as

$$j(\alpha)=b(k,\,1)\,b(k,\,2)\,\dots\,b(k,\,l)\,\dots\,,$$

where l runs in $\mathbb{N}$ and $b(k,l)\in X$. We set $n\equiv s(k,l)$ and

$$a_{2n-1}\,a_{2n}\equiv c(b(k,\,l))\,.$$

Thus g is injective because we can reconstruct j from the decimal expansion of $g(j)$. The memberships of $g(j)$ in $[0,1)$ and in $\mathbb{R}$ hint to the injection $g(j)\mapsto F(g(j))$ where $F(g(j))$ is in $[0,1)$ and $\mathbb{R}$; the function F is introduced after Definition 1.7.13. By part 2 of Theorem 1.7.16 it is an injection. □

Exercise 6.2.4 *Show that the map $a\mapsto k_a$ is an injection from $\mathbb{R}$ to $\mathcal{C}(\mathbb{R})$.*

Exercise 6.2.5 *For every nonempty set $M\subset\mathbb{R}$ there is a bijection $h\colon \mathbb{R}\to\mathcal{C}(M)$.*

6.3 Intermediate values

We show that continuous functions map intervals to intervals. In Section **??** we will see that derivatives (i.e., functions that are derivatives of other functions) have this property too.

• *Continuous functions attain intermediate values.* The image of the function sgn ($\in \mathcal{F}(\mathbb{R})$) is the set $\mathrm{sgn}[\mathbb{R}] = \{-1, 0, 1\}$. Thus although $\frac{1}{2} \in (0,1)$, for no b we have $\mathrm{sgn}(b) = \frac{1}{2}$. For continuous functions this cannot happen.

Theorem 6.3.1 (on intermediate values) *Let $a < b$ be in $\mathbb{R}$, $f \in \mathcal{C}([a,b])$ & $f(a) < c < f(b)$ or $f(a) > c > f(b)$. Then there is a $d \in (a,b)$ such that $f(d) = c$.*

Proof. Let $f(a) < c < f(b)$, the case $f(a) > c > f(b)$ is similar. We set $X \equiv \{x \in [a,b] : f(x) < c\}$ and $d \equiv \sup(X)$. Clearly $d \in [a, b]$. The continuity of f at a and at b implies that $d \in (a,b)$. We will see that $f(d) < c$ and $f(d) > c$ lead to contradictions, hence $f(d) = c$. Let $f(d) < c$. By the continuity of f at d there is a δ such that for every $x \in U(d,\delta) \cap [a,b]$ it holds that $f(x) < c$. But then X contains numbers larger than d, which is a contradiction. Let $f(d) > c$. By the continuity of f at d there is a δ such that for every $x \in U(d,\delta) \cap [a,b]$ it holds that $f(x) > c$. But then every $x < d$ and close to d lies outside X, which is also a contradiction. □

Exercise 6.3.2 *For every interval $I \subset \mathbb{R}$ and every function $f \in \mathcal{C}(I)$ the image $f[I]$ is an interval.*

Corollary 6.3.3 (the image of $\exp x$) *It holds that $\exp[\mathbb{R}] = (0, +\infty)$. Thus $\exp$ is a bijection from $\mathbb{R}$ to $(0, +\infty)$.*

Proof. Since $\exp > 0$ on $\mathbb{R}$, we have that $\exp[\mathbb{R}] \subset (0, +\infty)$. From the limits $\lim_{x\to-\infty} \exp x = 0$ and $\lim_{x\to+\infty} \exp x = +\infty$ (part 3 of Proposition 4.3.7), from the continuity of the exponential (Corollary 6.6.6) and from Theorem 6.3.1 it follows that $(0, +\infty) \subset \exp[\mathbb{R}]$. Thus $\exp[\mathbb{R}] = (0, +\infty)$. The exponential increases and hence is a bijection. □

Exercise 6.3.4 *Prove the following corollary.*

Corollary 6.3.5 (mountaineering) *A mountain climber starts her ascend at midnight, after 24 hours reaches the summit and then descends for 24 hours to the base camp. Show that there exists a moment $t_0 \in [0, 24]$ when in each of the two days she is in the same altitude.*

A function $f \in \mathcal{F}(M)$ increases, respectively decreases, on an (arbitrary) set X if for every $x < y$ in $M \cap X$ we have that $f(x) < f(y)$, respectively $f(x) > f(y)$.

Corollary 6.3.6 (continuity and injectivity) *If $I \subset \mathbb{R}$ is an interval and $f \in \mathcal{C}(I)$ is injective, then f either increases or decreases.*

Proof. If, for contrary, f neither increases nor decreases then I contains three numbers $a < b < c$ such that $f(a) < f(b) > f(c)$ or $f(a) > f(b) < f(c)$. In the first case we see by Theorem 6.3.1 that for every d such that $f(a), f(c) < d < f(b)$ there is an $x \in (a, b)$ and a $y \in (b, c)$ such that $d = f(x) = f(y)$, in contradiction with the injectivity of f. The second case leads to a similar contradiction. $\square$

Now we can easily prove Theorem 1.6.17 which is repeated here as a corollary.

Corollary 6.3.7 (Bolzano–Cauchy) *Let I be an interval, $f \in \mathcal{C}(I)$ & for some $a, b \in I$ it holds that $f(a)f(b) \le 0$. Then there exists a $c \in I$ such that $f(c) = 0$.*

Proof. If $f(a)f(b) = 0$ then $f(a) = 0$ or $f(b) = 0$. If $f(a)f(b) < 0$ then $a \ne b$ & $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$. We use Theorem 6.3.1. $\square$

6.4 Compactness

Compact sets are a basic analytical tool. We consider only real ones and describe their relations to continuous functions. A set $M \subset \mathbb{R}$ is compact if every sequence $(a_n) \subset M$ has a convergent subsequence (a_{m_n}) with the limit $\lim a_{m_n}$ in M. By the Bolzano–Weierstrass theorem and the theorem on limits and order every interval $[a, b]$ is compact. Later we give a complete description of all compact sets.

• *Minima and maxima.* We show that every continuous function with compact definition domain has always the smallest and the largest value.

Theorem 6.4.1 (min-max) *Let $f \in \mathcal{C}(M)$ where $M \ne \emptyset$ is a compact set. Then there exist points $a, b \in M$ such that for every $x \in M$ one has that $f(a) \le f(x) \le f(b)$. The point a, respectively b, is a minimum, respectively a maximum, of f.*

Proof. We show that f has a maximum, minima are treated similarly. Let $A \equiv \sup(f[M])$, in the LO $(\mathbb{R}^*, <)$. It is clear that $f[M] \ne \emptyset$ and we take a sequence $(a_n) \subset M$ with $\lim f(a_n) = A$. We take a subsequence (a_{m_n}) of (a_n) with $\lim a_{m_n} \equiv b \in M$. By (H) it holds that $f(b) = \lim f(a_{m_n}) = \lim f(a_n) = A$, in particular $A \in \mathbb{R}$. Thus for any $x \in M$ we have that $f(x) \le A = f(b)$. $\square$

Exercise 6.4.2 *The functions $f, g\colon [0, 1) \to \mathbb{R}$, $f(x) = \frac{1}{1-x}$ and $g(x) = x$, are continuous and do not have maximum.*

We extend minima and maxima by the adjective "global". Thus $f \in \mathcal{F}(M)$ has in $b \in M$ a global maximum, respectively a global minimum, if for every x in M one has that $f(x) \le f(b)$, respectively $f(x) \ge f(b)$. We say that the function f has at $b \in M$ a local maximum, respectively a local minimum if for some δ for every x in $U(b,\delta) \cap M$ it holds that $f(x) \le f(b)$, respectively $f(x) \ge f(b)$. If these inequalities hold for every $x \ne b$ as strict (as $<$, respectively $>$), we speak of a strict global maximum, etc.

• *Continuous image of a compact set is compact.* This is an important result on compact sets, especially in the general topological version (which we do not state here).

Theorem 6.4.3 (images of compacts) *If $f \in \mathcal{C}(M)$ and M is compact then $f[M]$ is a compact set.*

Proof. Let f and M be as stated and $(b_n) \subset f[M]$. Using the axiom of choice we take a sequence $(a_n) \subset M$ such that $f(a_n) = b_n$. It has a subsequence (a_{m_n}) with $\lim a_{m_n} \equiv a \in M$. By (H) we have that $\lim f(a_{m_n}) = f(a) \equiv b$, so that $(b_{m_n}) = (f(a_{m_n}))$ has the limit $b \in f[M]$. Hence $f[M]$ is compact. □

Exercise 6.4.4 *Explain the connection between Theorems 6.4.3 and 6.4.1.*

• *Open and closed sets.* A set $M \subset \mathbb{R}$ is open if for every $b \in M$ there is a δ such that $U(b,\delta) \subset M$. It is closed if its complement $\mathbb{R} \setminus M$ is open.

Exercise 6.4.5 *Let $\mathcal{A}$, respectively $\mathcal{B}$, be a nonempty set of open, respectively closed, sets. The following hold.*
1. *Sets $\emptyset$ and $\mathbb{R}$ are open and closed.*
2. *$\bigcup \mathcal{A}$ is open.*
3. *If $\mathcal{A}$ is finite then $\bigcap \mathcal{A}$ is open.*
4. *If $\mathcal{B}$ is finite then $\bigcup \mathcal{B}$ is closed.*
5. *$\bigcap \mathcal{B}$ is closed.*
6. *For every b and δ the set $U(b,\delta)$ is open.*

Exercise 6.4.6 *If $M \subset \mathbb{R}$ is open, then $M \subset L(M)$, every point $b \in M$ is a limit point of M.*

Proposition 6.4.7 (on closed sets) *A set $M \subset \mathbb{R}$ is closed $\iff$ the limit of every convergent sequence $(a_n) \subset M$ lies in M.*

Proof. $\Rightarrow$. Let $M \subset \mathbb{R}$ be closed and let $(a_n) \subset M$ have $\lim a_n = a$. If $a \in \mathbb{R} \setminus M$, then for some δ we have $U(a,\delta) \cap M = \emptyset$. This is not possible because $a_n \to a$. Hence $a \in M$.

$\neg \Rightarrow \neg$. If $M \subset \mathbb{R}$ is not closed then there is an $a \in \mathbb{R} \setminus M$ such that for every n there is an $a_n \in U(a, \frac{1}{n}) \cap M$. Thus $(a_n) \subset M$ & $\lim a_n = a \notin M$ (we again use the axiom of choice). □

For $A \subset M \subset \mathbb{R}$ we say that A is relatively closed (in M) if there is a closed set $U \subset \mathbb{R}$ such that $A = M \cap U$.

Proposition 6.4.8 (zero sets are relatively closed) *For every* $f \in \mathcal{C}$ *the set* $Z(f) = \{x \in M(f) :\ f(x) = 0\}$ *is relatively closed (in* $M(f)$*).*

Proof. Let $N \equiv M(f) \setminus Z(f)$. By the continuity of f, for every $b \in N$ there is a δ_b such that $U(b, \delta_b) \cap Z(f) = \emptyset$. Let $A \equiv \bigcup_{b \in N} U(b, \delta_b)$. By Exercise 6.4.5 the set A is open. Thus, $Z(f) = M(f) \cap (\mathbb{R} \setminus A)$ and $\mathbb{R} \setminus A$ is closed. $\square$

For $A \subset M \subset \mathbb{R}$ we say that A is relatively open (in M) if there is an open set $B \subset \mathbb{R}$ such that $A = M \cap B$.

Proposition 6.4.9 (images of open sets) *If* $f \in \mathcal{C}$ *is injective and* $M(f)$ *and* $M \subset \mathbb{R}$ *are open then* $f[M]$ *is open.*

Proof. Let $b \in f[M]$ and $a \equiv f^{-1}(b)$ $(\in M(f) \cap M)$. Since $M(f)$ and M are open, we can take an interval $I \equiv [a - \delta, a + \delta]$ such that $I \subset M(f) \cap M$. Then

$$f(a - \delta) < b < f(a + \delta) \text{ or } f(a - \delta) > b > f(a + \delta)$$

because $f(a - \delta), f(a + \delta) < b = f(a)$ and $f(a - \delta), f(a + \delta) > b = f(a)$ would lead by Theorem 6.3.1 to a contradiction with the injectivity of f. We take an $\varepsilon < \min(\{|f(a + \delta) - b|, |f(a - \delta) - b|\})$. Theorem 6.3.1 gives $U(b, \varepsilon) \subset f[I] \subset f[M]$. Hence $f[M]$ is open. $\square$

Proposition 6.4.10 (preimages of open sets) *If* $f \in \mathcal{C}$ *and* $M \subset \mathbb{R}$ *is open then* $f^{-1}[M]$ *is relatively open (in* $M(f)$*).*

Proof. For every $b \in M$ there is an ε_b such that $U(b, \varepsilon_b) \subset M$. For every $a \in f^{-1}[M]$ $(\subset M(f))$ there is a δ_a such that, with $b \equiv f(a)$, it holds that $f[U(a, \delta_a)] \subset U(b, \varepsilon_b) \subset M$. Thus, $U(a, \delta_a) \cap M(f) \subset f^{-1}[M]$. Let $B \equiv \bigcup_{a \in f^{-1}[M]} U(a, \delta_a)$. Since $f^{-1}[M] = M(f) \cap B$ and B is an open set (by parts 2 and 6 of Exercise 6.4.5), the set $f^{-1}[M]$ is relatively open in $M(f)$. $\square$

Propositions 6.4.8 and 6.4.10 are needed in the proof of Theorem 7.6.3.

The next theorem provides an idea of structure of any open set. For real numbers $a < b$ we define an open interval as an interval of the form $(-\infty, a)$, $(a, +\infty)$ or (a, b).

Theorem 6.4.11 (structure of open sets) *A set* $M \subset \mathbb{R}$ *is open* $\iff$ *there exists an at most countable system of disjoint open intervals* $\{I_j :\ j \in J\}$ *such that* $\bigcup_{j \in J} I_j = M$.

Proof. Let $M \subset \mathbb{R}$ be a nonempty open set; for $M = \emptyset$ the claim holds trivially with $J = \emptyset$. For $a \in M$ we define I_a as the inclusion-wise maximal open interval I such that $a \in I \subset M$; it is the union of all such intervals I. For any $a, b \in M$, we have $I_a = I_b$ or $I_a \cap I_b = \emptyset$. Hence, the required system of intervals is $\{I_a :\ a \in \mathbb{Q} \cap M\}$. $\square$

• *The Cantor set.* By the previous theorem the closed set $\mathbb{R} \setminus M$ is a union of the "gaps" separating the intervals I_j. For $|J| = n$ we have at most $n+1$ of them. It is hard to imagine that for the countable set J the set of gaps may be uncountable. Such closed sets are hard to visualize. An example is the Cantor set C, defined as follows.

$$C \equiv \bigcap_{n=1}^{\infty} C_n \ (\subset [0,\,1] \equiv C_0), \ \text{ for } n \geq 1 \text{ we set } \ C_n \equiv \tfrac{1}{3}C_{n-1} \cup \left(\tfrac{1}{3}C_{n-1} + \tfrac{2}{3}\right).$$

Thus C is the leftover of the interval $[0,1]$ when we delete from it the open middle third $(\frac{1}{3}, \frac{2}{3})$, then delete from the rest $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ the open middle thirds $(\frac{1}{9}, \frac{2}{9})$ a $(\frac{7}{9}, \frac{8}{9})$, and continue in this manner for infinitely many steps.

Exercise 6.4.12 *C is an uncountable closed set with zero "length".*

• *Characterization of compact sets.* A set $M \subset \mathbb{R}$ is bounded if there is a $c \geq 0$ such that $M \subset [-c, c]$. Next, we describe all compact real sets.

Theorem 6.4.13 (compact real sets) *A set $M \subset \mathbb{R}$ is compact $\iff$ M is bounded and closed.*

Proof. Let M be bounded and closed and $(a_n) \subset M$. By Theorem 2.3.15 we have a convergent subsequence (a_{m_n}) with $\lim a_{m_n} \equiv a \in \mathbb{R}$. M is closed and Proposition 6.4.7 shows that $a \in M$. Hence M is compact.

Suppose that M is not bounded. We define a sequence $(a_n) \subset M$ such that $|a_m - a_n| \geq 1$ if $m \neq n$. This is inherited by all subsequences which therefore do not converge and M is not compact. The first term a_1 is arbitrary. Suppose that $a_1, a_2, \ldots, a_n$ are defined and satisfy that $|a_i - a_j| \geq 1$ if $i \neq j$. Since M is not bounded, there is an $a_{n+1} \in M$ such that $|a_{n+1}| \geq 1 + \max(\{|a_1|, \ldots, |a_n|\})$. Then, by the Δ-inequality, for every $i \in [n]$ we have that $|a_{n+1} - a_i| \geq 1$. Doing this extension in infinitely many steps we get the required sequence (a_n).

Suppose that M is not closed. By Proposition 6.4.7 there is a sequence $(a_n) \subset M$ with limit in $\mathbb{R} \setminus M$. Every subsequence of (a_n) has the same limit and therefore does not converge in M. The set M is not compact. □

Exercise 6.4.14 *Every set $[a,b] \setminus P(c,\delta)$ is compact.*

6.5 Uniform continuity

An $f \in \mathcal{F}(M)$ is uniformly continuous if for every ε there is a δ such that for any $a, b \in M$,

$$|a - b| \leq \delta \Rightarrow |f(a) - f(b)| \leq \varepsilon\,.$$

Here we write $\leq$, these inequalities are safer than $<$. The set of uniformly continuous functions in $\mathcal{F}(M)$ is $\mathcal{UC}(M)$ and we set $\mathcal{UC} \equiv \bigcup_{M \subset \mathbb{R}} \mathcal{UC}(M)$.

Exercise 6.5.1 *$\mathcal{UC} \subset \mathcal{C}$, any uniformly continuous function is continuous.*

• *Uniform continuity and compactness.* In the other way any continuous function with compact definition domain is uniformly continuous.

Proposition 6.5.2 (compactness and $\mathcal{UC}$) *For any compact set $M \subset \mathbb{R}$ it holds that $\mathcal{C}(M) \subset \mathcal{UC}(M)$, hence $\mathcal{C}(M) = \mathcal{UC}(M)$.*

Proof. Suppose that $M \subset \mathbb{R}$ is compact and that $f \in \mathcal{F}(M)$ is not uniformly continuous. We prove that f is not continuous at a point $c \in M$. By the assumption there is an ε such that for every δ there exist points $a, b \in M$ with $|a - b| \le \delta$ and $|f(a) - f(b)| > \varepsilon$. The axiom of choice gives two sequences $(a_n), (b_n) \subset M$ such that always $|a_n - b_n| \le \frac{1}{n}$ but $|f(a_n) - f(b_n)| > \varepsilon$. Using compactness of M we pass to subsequences (Exercise 6.5.3) and get that $\lim a_n = \lim b_n = c$ for some $c \in M$. Since $|f(a_n) - f(b_n)| > \varepsilon$ for every n, it is not true that $\lim f(a_n) = \lim f(b_n) = f(c)$. By (H) f is not continuous at c. $\square$

Exercise 6.5.3 *Explain in detail the step when we pass to subsequences.*

Exercise 6.5.4 *The continuous functions $f, g\colon (0,1] \to \mathbb{R}$, $f(x) \equiv \frac{1}{x}$ and $g(x) \equiv \sin(\frac{1}{x})$, are not uniformly continuous.*

Exercise 6.5.5 *Let $M \equiv [0,1] \cap \mathbb{Q}$. Find a function $f \in \mathcal{C}(M) \setminus \mathcal{UC}(M)$.*

• *Extending uniformly continuous functions.* The theorem in this passage is quite important. For a set $M \subset \mathbb{R}$ its <u>closure</u> is the real set

$$\overline{M} \equiv \{b \in \mathbb{R} :\ \exists\,(a_n) \subset M \text{ with } \lim a_n = b\} = L(M) \cup M \setminus \{-\infty,\, +\infty\}\,.$$

Theorem 6.5.6 (extending functions in $\mathcal{UC}(M)$) *Let $M \subset \mathbb{R}$ and let f be in $\mathcal{UC}(M)$. <u>Then</u> f has a unique extension to a function $g \in \mathcal{UC}(\overline{M})$ and $g(b) = \lim f(a_n)$ for any sequence $(a_n) \subset M$ with $\lim a_n = b$.*

Proof. Let f and M be as stated and $b \in \overline{M}$. Let $(a_n), (a_n') \subset M$ have $\lim a_n = \lim a_n' = b$ and an ε be given. We take the δ provided by the uniform continuity of f. Since for every large m and n one has that $|a_m - a_n'| \le \delta$, for the same m and n it holds that $|f(a_m) - f(a_n')| \le \varepsilon$. Selecting $(a_n) = (a_n')$ we see that the sequence $(f(a_n))$ is Cauchy. By Theorem 2.3.20 it has the limit $\lim f(a_n) \equiv c$. Selecting $(a_n) \ne (a_n')$ we see that the limit c does not depend on the choice of the sequence (a_n). We get a function $g\colon \overline{M} \to \mathbb{R}$ given by $g(b) \equiv c$.

We show that (i) g extends f, (ii) g is uniformly continuous and (iii) g is the only continuous extension of f to $\overline{M}$. The claim (i) is clear, for $b \in M$ we have that $g(b) = \lim f(b) = f(b)$, due to the constant sequence $(a_n) \equiv (b, b, \dots)$. To prove claim (ii), we take for a given ε the δ provided by the uniform continuity of f. Let $b, b' \in \overline{M}$ be given, with $|b - b'| \le \frac{\delta}{2}$. Then we can take sequences $(a_n), (a_n') \subset M$ such that $\lim a_n = b$ and $\lim a_n' = b'$ and

$$|f(b) - f(b')| \le |\lim f(a_n) - \lim f(a_n')| \le \varepsilon$$

because for every large m and n it holds that $|a_m - a'_n| \le \delta$. To prove claim (iii) we take any continuous extension $h\colon \overline{M} \to \mathbb{R}$ of f and any $b \in \overline{M}$. We take any sequence $(a_n) \subset M$ with $\lim a_n = b$. Then

$$h(b) = \lim h(a_n) = \lim f(a_n) = g(b)$$

and $h = g$. The first equality is by (H) once again, the second equality is due to h extending f and the third equality is the definition of g. $\Box$

In [23] we show that with the help of this extension theorem one can build univariate real analysis without uncountable sets, that is, without use of uncountable real functions by using only functions in $\mathcal{F}(M)$ with $M \subset \mathbb{Q}$.

Exercise 6.5.7 *Prove for such functions the next version of Theorem 6.4.1.*

HMC is an acronym for *hereditarily at most countable.*

Theorem 6.5.8 (HMC min-max) *Let $M \subset \mathbb{Q}$ be a bounded set and f be in $\mathcal{UC}(M)$. Then there exist points $b, c \in \overline{M}$ such that for every $a \in M$ one has that $f(b) \le f(a) \le f(c)$. The values $f(b) \equiv g(b)$ and $f(c) \equiv g(c)$ are of the extension g of f provided by Theorem 6.5.6.*

Thus in a sense b and c are a minimum and a maximum of f, respectively, even though they may lie outside the definition domain M.

6.6 Operations on functions and continuity

We consider operations on functions in $\mathcal{R}$ that we introduced in Definition 4.4.1 and determine if they preserve continuity. We also find out if sums of power series yield a continuous function.

• *Arithmetic of continuity.* Recall the first three operations in Definition 4.4.1. They are sum, product and ratio (division). In the following $f, g \in \mathcal{R}$.

Theorem 6.6.1 (arithmetic of continuity) 1. *If f and g are continuous at $b \in M(f) \cap M(g)$ then $f + g$ and fg are continuous at b. If f and g are continuous at $b \in M(f/g)$ then also f/g is continuous at b.* 2. *If $f, g \in \mathcal{C}$ then $f + g, fg, f/g \in \mathcal{C}$.*

Proof. 1. We treat only f/g, sum and product are treated similarly and more easily. Let f, g and b be as stated & $(a_n) \subset M(f/g)$ have $\lim a_n = b$. By (H) it holds that $\lim f(a_n) = f(b)$ and $\lim g(a_n) = g(b)$. By Theorem 3.1.2 we have that

$$\lim (f/g)(a_n) = \lim \tfrac{f(a_n)}{g(a_n)} = \tfrac{\lim f(a_n)}{\lim g(a_n)} = \tfrac{f(b)}{g(b)} = (f/g)(b)\,.$$

Thus by (H) the function f/g is continuous at b.

2. This follows from the first part. $\Box$

Exercise 6.6.2 *Show that* POL $\subset \mathcal{C}$ *and* RAC $\subset \mathcal{C}$*, that is, polynomials and rational functions are continuous.*

• *Continuity of power series.* Our goal is to prove continuity of all functions introduced in Sections 4.3 and 4.4. Those that are defined by composing and inverting will be dealt with later. Continuity of the exponential, cosine and sine follows from the next theorem.

Theorem 6.6.3 (continuity of power series) *Suppose that* $(a_n) \subset \mathbb{R}$ *and* $\lim |a_n|^{1/n} = 0$. *Then for every* $x \in \mathbb{R}$ *the series* $S(x) \equiv \sum_{n=0}^{\infty} a_n x^n$ *is abscon and its sum* $S(x) \in \mathcal{C}(\mathbb{R})$.

Proof. Let the coefficients (a_n) be as stated and $x \in \mathbb{R}$. Then $0 \le |a_n|^{1/n}|x| \le \frac{1}{2}$ for $n \ge n_0$, so that $|a_n x^n| \le (\frac{1}{2})^n$ for $n \ge n_0$. The series $\sum_{n=0}^{\infty} a_n x^n$ is therefore abscon and converges. Further

$$\begin{aligned} |S(x+c) - S(x)| &= |c| \cdot \left| \textstyle\sum_{n=1}^{\infty} a_n \sum_{i=1}^{n} \binom{n}{i} c^{i-1} x^{n-i} \right| \\ &\le |c| \cdot \textstyle\sum_{n=1}^{\infty} |a_n| \cdot (|x| + |c|)^{n-1} \cdot 2^n \to 0 \text{ if } c \to 0\,. \end{aligned}$$

Thus the function S is continuous at x. □

Exercise 6.6.4 *Explain why in the previous proof the displayed equality and inequality hold.*

Exercise 6.6.5 *Prove that* $\lim (n!)^{1/n} = +\infty$.

Corollary 6.6.6 (continuity of e^x**,** $\cos x$**,** $\sin x$**)** *These functions are continuous on the common definition domain* $\mathbb{R}$.

Proof. This follows from the definitions that

$$\mathrm{e}^x = \textstyle\sum_{n=0}^{\infty} \frac{x^n}{n!},\ \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \text{ and } \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\,,$$

and from Theorem 6.6.3 and Exercise 6.6.5. □

Corollary 6.6.7 (continuity of $\tan x$ **and** $\cot x$**)** *Both functions are continuous on their definition domains.*

Proof. As we know, $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x}$. Thus continuity of these functions follows from Corollary 6.6.6 and Theorem 6.6.1. □

• *Restriction and composition.* Recall the last three operations on $\mathcal{R}$ in Definition 4.4.1. They are restriction, composition and inverse. We dealt with restriction in Proposition 6.1.5. Here we supplement it with the point-wise form. In the following f and g are in $\mathcal{R}$ and X is any set.

Proposition 6.6.8 (continuity of restriction 2) 1. *If* f *is continuous at* $b \in M(f \mid X)$ *then* $f \mid X$ *is continuous at* b. 2. *If* $f \in \mathcal{C}$ *then* $f \mid X \in \mathcal{C}$.

Proof. 1. Let f, b and X be as stated. We take any sequence $(b_n) \subset M(f \mid X) = M(f) \cap X$ with $\lim b_n = b$. By (H) we have that $\lim f(b_n) = f(b)$. Thus $\lim(f \mid X)(b_n) = \lim f(b_n) = f(b)$ and (H) gives that $f \mid X$ is continuous at b.

2. This follows from part 1. Another proof, using neighborhoods, is given in Proposition 6.1.5. $\square$

Next we consider the operation of composition.

Theorem 6.6.9 (continuity of composition) 1. *If g is continuous at the point $b \in M(f(g))$ and f is continuous at $g(b)$ then $f(g)$ is continuous at b.* 2. *If $f, g \in \mathcal{C}$ then $f(g) \in \mathcal{C}$.*

Proof. 1. Let f, g and b be as stated, and $(b_n) \subset M(f(g))$ be a sequence with $\lim b_n = b$. Then $\lim g(b_n) = g(b)$ by (H). Hence, again by (H), $\lim f(g)(b_n) = \lim f(g(b_n)) = f(g(b)) = f(g)(b)$. (H) shows that $f(g)$ is continuous at b.

2. This follows from the first part. $\square$

Exercise 6.6.10 *Prove part 1 of the previous theorem not by* (H) *but by means of neighborhoods.*

• *Inverses.* Now the situation is much more interesting compared to the previous operations. Inverting a function in general does *not* preserve continuity. For example, the function $f \colon \mathbb{N}_0 \to \mathbb{R}$ with values $f(0) \equiv 0$ and, for $n > 0$, $f(n) \equiv \frac{1}{n}$ is continuous, but the inverse $0 \mapsto 0$ and $\frac{1}{n} \mapsto n$ is not continuous at 0. Eventually we obtained the following theorem.

Theorem 6.6.11 (continuity of inverse) *Let $f \in \mathcal{C}(M)$ be injective. Then in each of the following five situations the inverse $f^{-1} \in \mathcal{F}(f[M])$ is continuous.* 1. *M is compact.* 2. *M is an interval.* 3. *M is open.* 4. *M is closed and f is monotone.* 5. *$M \subset (a, b)$, M is dense in (a, b) and f is monotone and uniformly continuous.*

Proof. 1. Let M be compact, $b \in f[M]$ and let $(b_n) \subset f[M]$ have $\lim b_n = b$. Let $a \equiv f^{-1}(b)$ and $a_n \equiv f^{-1}(b_n)$ $(\in M)$. We show that $\lim a_n = a$, which by (H) proves continuity of f^{-1} at b. We show that every subsequence of (a_n) has a subsequence with the limit a. Part 3 of Theorem 2.2.5 then implies that $\lim a_n = a$. Let (a_n') be a subsequence of (a_n). We use compactness of M and take a subsequence (a_{m_n}) of (a_n') with $\lim a_{m_n} = c \in M$. By (H) it holds that $\lim f(a_{m_n}) = f(c) = b$ because $(f(a_{m_n}))$ is a subsequence of (b_n). Since f is injective, $c = a$.

2. Let M be an interval. Corollary 6.3.6 says that f increases or decreases. Suppose that f decreases, the case of increasing f is similar. Theorem 6.3.1 says that $f[M]$ is an interval. Let $b \in f[M]$ and an ε be given. We show that f^{-1} is right-continuous at b. This is true when b is the right endpoint of the interval $f[M]$ because then $U^+(b, \delta) \cap f[M] = \{b\}$. We assume that b is not the right endpoint of $f[M]$. Since f^{-1} decreases, $a \equiv f^{-1}(b)$ $(\in M)$ is not the left

endpoint of the interval M. We may have ε is so small that $[a-\varepsilon, a] \subset M$. We set

$$\delta \equiv f(a-\varepsilon) - f(a) = f(a-\varepsilon) - b > 0 \ .$$

Theorem 6.3.1 implies that f is a (decreasing) bijection $[a-\varepsilon, a] \to [b, b+\delta]$, and hence also $(a-\varepsilon, a] \to [b, b+\delta)$. So $[b, b+\delta) \subset f[M]$ and $U^+(b,\, \delta) \cap f[M] = U^+(b,\, \delta) = [b, b+\delta)$. Hence

$$f^{-1}[U^+(b,\, \delta)] = U^-(a,\, \varepsilon) \subset U(a,\, \varepsilon) = U(f^{-1}(b),\, \varepsilon)$$

and f^{-1} is right-continuous at b. Left-continuity of f^{-1} at b is proven similarly. Exercise 5.2.11 gives that f^{-1} is continuous at b.

3. Let M be open. Let $b \in f(M)$, $a \equiv f^{-1}(b)$ ($\in M$) and an ε be given. For small enough ε one has that $U(a, \varepsilon) \subset M$. Proposition 6.4.9 says that the set $f[U(a, \varepsilon)]$ ($\ni b$) is open. So for some δ we have that $U(b, \delta) \subset f[U(a, \varepsilon)]$. Thus

$$f^{-1}[U(b,\, \delta)] \subset U(a,\, \varepsilon) = U(f^{-1}(b),\, \varepsilon)$$

and f^{-1} is continuous at b (exactly by Definition 5.2.1).

4. Let M be closed and f be increasing, for decreasing f we argue similarly. We assume for contradiction that for some $b \in f[M]$ with $a \equiv f^{-1}(b)$ ($\in M$) there is a sequence $(b_n) \subset f[M]$ such that $\lim b_n = b$ but $\lim f^{-1}(b_n)$ does not exist or differs from a. By part 2 of Theorem 2.2.5 and by Proposition 2.3.12 the sequence (b_n) has has a decreasing or an increasing subsequence (c_n) such that $\lim f^{-1}(c_n) = B$ ($\in \mathbb{R}^*$) and $B \neq a$. We assume that (c_n) decreases, the case of increasing (c_n) is similar. Then $b < \cdots < c_2 < c_1$, hence $a < \cdots < f^{-1}(c_2) < f^{-1}(c_1)$ (both f and f^{-1} increases). By part 2 of Theorem 3.3.1 we have that $B \in [a, f^{-1}(c_1))$. Thus, crucially, $B \in \mathbb{R}$ (here the argument fails for non-monotone f). Even $B \in M$ because M is closed. Due to the continuity of f in B we have that $f(B) = \lim f(f^{-1}(c_n)) = \lim c_n = b = f(a)$. But this contradicts the injectivity of f because $B \neq a$.

5. Let M, a, b and f be as stated. We use Theorem 6.5.6 and continuously extend f to a function $\overline{f}\colon [a, b] \to \mathbb{R}$. By Proposition 3.3.6 and by denseness of M in (a, b) the function $\overline{f}$ is strictly monotone and therefore injective. By part 1 or part 2 of this theorem the function $(\overline{f})^{-1}$ is continuous. Proposition 6.1.5 gives that also $(\overline{f})^{-1} \,|\, f[M] = f^{-1}$ is continuous. □

In *MA 1*$^+$ we prove the previous theorem in a more general context and generalize part 5.

Exercise 6.6.12 *Present examples showing that in part 4 of the theorem none of the assumptions (closedness of M, monotonicity of f) can be omitted.*

- *Continuity of elementary functions.* We prove that $\mathrm{EF} \subset \mathcal{C}$.

Exercise 6.6.13 *Use the previous theorem and prove the following corollary.*

Corollary 6.6.14 (continuity of some BEF) *These functions are continuous:* $\log x$, $\arccos x$, $\arcsin x$, $\arctan x$ *and* $\operatorname{arccot} x$.

Proposition 6.6.15 (continuity of x^b) *For every* $b \in (0, +\infty)$ *the function* $x^b \colon [0, +\infty) \to [0, +\infty)$ *is continuous.*

Proof. Let $b > 0$ and $x > 0$. We get that x^b is continuous at x by using the expression $x^b = \exp(b \log x)$, continuity of e^x (Corollary 6.6.6), continuity of logarithm (Corollary 6.6.14), continuity of the constant function k_b (Exercise 6.1.3), continuity of product (Theorem 6.6.1) and continuity of composition (Theorem 6.6.9). Continuity at $x = 0$ follows with the help of Proposition 5.2.5 from the limit

$$\lim_{x \to 0} x^b = \lim_{x \to 0} \exp(b \log x) = \lim_{y \to -\infty} \exp y = 0 = 0^b \,.$$

Here the second equality follows from Theorem 5.4.1 and from part 2 of Proposition 4.3.10. The third equality follows from part 3 of Proposition 4.3.7. □

We conclude this chapter by proving continuity of elementary functions.

Theorem 6.6.16 (EF $\subset \mathcal{C}$) *Every elementary function is continuous.*

Proof. We proceed by induction on the length of a generating word of the given elementary function f (Definition 4.4.14). If f is a constant function, exponential, logarithm, x^b with non-integral exponent $b > 0$, sine or arcsine, it is continuous by, respectively, Exercise 6.1.3, Corollaries 6.6.6 and 6.6.14, Proposition 6.6.15 and Corollaries 6.6.6 and 6.6.14. If f is a sum, a product, a ratio or a composition of two simpler elementary functions, it is continuous by induction and Theorems 6.6.1 and 6.6.9. □

Chapter 7

Derivatives

The seventh lecture is important because derivatives enter the stage. I gave it on April 4, 2024, as

`https://kam.mff.cuni.cz/~klazar/MAI24_pred7.pdf` .

It is considerably extended and reworked here.

In Section 7.1 we define ordinary and one-sided derivatives of functions with arbitrary definition domains. We consider both point-wise derivatives $b \mapsto f'(b)$ and global derivatives $f \mapsto f'$. Theorem 7.1.8 is the well-known criterion of local extremes for general functions. If f is differentiable at a point b then f is continuous at b (Proposition 7.1.11) and $f(b)$ is a limit point of the image of f (Proposition 7.1.15 and Exercise 7.1.17). Theorem 7.1.25 describes a discontinuous derivative. A better known example is in Exercise 7.5.8.

Section 7.2 starts with Definition 7.2.1 of standard tangents. Definition 7.2.7 introduces limit tangents — they formalize the intuition of a tangent at a point as the limit of sequences of secant lines going through that point. In Theorem 7.2.9 we demonstrate the equivalence of standard and limit tangents. In Theorem 7.2.11 we show how to define the tangent at a point B without actually using B — as the limit of sequences of secant lines going through pairs A, C of points of the graph such that A and C converge to B from the opposite sides. We hope to revisit tangents in *MA 1*$^+$.

Section 7.3 is devoted to the arithmetic of derivatives. Theorem 7.3.1 describes the point-wise and global derivatives of sums — we investigate the relation between f', g' and $(f+g)'$ for any pair of functions $f, g \in \mathcal{R}$. In general $(f+g)' \ne f' + g'$. Theorem 7.3.4 presents both the point-wise and global Leibniz formula for derivatives of products, and Theorem 7.3.8 does the same for ratios. Corollaries 7.3.3, 7.3.7, 7.3.10, 7.4.3 and 7.4.7 describe situations where the respective equalities $(f+g)' = f' + g'$, $(fg)' = f'g + fg'$, $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$, $(f(g))' = f'(g) \cdot g'$ and $(f^{-1})' = \frac{1}{f'(f^{-1})}$ do hold.

Section 7.4 treats derivatives of composite functions (Theorem 7.4.1) and inverses (Theorem 7.4.4 and Corollary 7.4.6). We always give both point-wise and global forms of the formula, and allow for any definition domain. Proofs make use of Heine's definition of derivatives. In Section 7.5 in Theorem 7.5.1 we differentiate power series. By this we obtain derivatives of e^x, $\sin x$, and $\cos x$. We also differentiate logarithm, but derivatives of other Basic Elementary Functions (Definition 4.3.1) are left as exercises. In Section 7.6 we pose Problem 7.6.1: show that Elementary Functions have elementary derivatives. In Theorem 7.6.3 we prove that the subset of Simple Elementary Functions, SEF, is closed to derivatives.

7.1 Point-wise and global derivatives

We introduce point-wise derivatives for functions with arbitrary definition domains and then the unary operation $f \mapsto f'$ for $f \in \mathcal{R}$. In this and subsequent chapters, we will see how to find, with the help of derivatives, extremes of functions, intervals of monotony, and intervals of convexity and concavity. Derivatives provide approximations of functions by polynomials and their expansions in power series.

• *Point-wise derivatives.* The next definition is fundamental.

Definition 7.1.1 (point-wise derivative) *Let $f \in \mathcal{F}(M)$ and $b \in M \cap L(M)$. The derivative of f at b is the limit*

$$\lim_{x \to b} \frac{f(x) - f(b)}{x - b} \overset{(*)}{=} \lim_{h \to 0} \frac{f(b+h) - f(b)}{h} \quad (\in \mathbb{R}^*)\,,$$

if it exists. We denote it by $f'(b)$ or by $\frac{df}{dx}(b)$.

Due to the uniqueness of limits of functions, derivatives are unique.

Exercise 7.1.2 *How would you prove the equality $(*)$?*

Corollary 7.1.3 (locality of derivatives) *If $f, g \in \mathcal{R}$, $b \in \mathbb{R}$ and there is a θ such that $f = g$ on $U(b, \theta)$ then $f'(b) = g'(b)$, if one side is defined.*

Proof. This is immediate from Proposition 4.2.9 and Definition 7.1.1. □

This corollary differs from Proposition 4.2.9: $f'(b)$ is the limit of a function at the point b, but the function involves $f(b)$ and therefore equality on $P(b, \delta)$ does not suffice.

If $f'(b) \in \mathbb{R}$, we say that f is differentiable at b. Then f has near b local approximation by a linear function, so called tangent (at the point $(b, f(b))$ of the graph of f):

$$f(x) = \underbrace{f(b) + f'(b) \cdot (x - b)}_{\text{the tangent}} + \underbrace{o(x - b)}_{\text{error}} \quad (x \to b)\,.$$

A useful tool is Heine's definition of derivatives (HDD).

Proposition 7.1.4 (HDD) *Let $f \in \mathcal{F}(M)$ and $b \in L(M) \cap M$. Then $f'(b) = B \iff$ for every sequence $(a_n) \subset M \setminus \{b\}$ with $\lim a_n = b$ it is true that $\lim \frac{f(a_n)-f(b)}{a_n-b} = B$.*

Proof. This follows from Definition 7.1.1 and Theorem 4.2.13. □

The existence of $f'(b)$ implies that $b \in M(f) \cap L(M(f))$, and we will not always explicitly state this.

• *One-sided derivatives.* They are defined by one-sided limits. Let f be in $\mathcal{F}(M)$ and $b \in L^-(M) \cap M$. Then we call the limit $f'_-(b) \equiv \lim_{x \to b^-} \frac{f(x)-f(b)}{x-b}$ $(\in \mathbb{R}^*)$ the left-sided derivative of f at b. By changing the signs $^-$ to $^+$ we get the right-sided derivative $f'_+(b)$ of f at b.

Exercise 7.1.5 *The following hold.*
1. $f'(a) = L \Rightarrow f'_-(a) = L$ *or* $f'_+(a) = L$.
2. $f'_-(a) = f'_+(a) = L \Rightarrow f'(a) = L$.
3. $f'_-(a) = K \neq L = f'_+(a) \Rightarrow f'(a)$ *does not exist.*

Exercise 7.1.6 *As in Proposition 5.1.13 concerning limits of functions, one-sided derivatives of functions can be reduced via restrictions to ordinary derivatives. State this result precisely and prove it.*

• *Derivative and extremes.* We employ the two-sided limit points defined above. Recall that a is a two-sided limit point of a real set M iff for every δ both $P^-(a, \delta) \cap M$ and $P^+(a, \delta) \cap M$ are non-empty. The set of these points is denoted by $L^{\mathrm{TS}}(M)$ $(\subset \mathbb{R})$.

Exercise 7.1.7 $L^{\mathrm{TS}}(M) \subset L(M)$. *The opposite inclusion in general does not hold.*

We adapt the well-known result linking derivatives and local extremes for functions with arbitrary definition domains.

Theorem 7.1.8 (derivatives and extremes) *Suppose that $f \in \mathcal{F}(M)$, that $b \in M \cap L^{\mathrm{TS}}(M)$ and that $f'(b)$ $(\in \mathbb{R}^*)$ exists and is not 0. Then f does not have local extreme at b — for every δ there exist points $c, d \in U(b, \delta) \cap M$ such that $f(c) < f(b) < f(d)$.*

Proof. Let f, M and b be as stated, and a δ be given. Let $f'(b) < 0$, the case with $f'(b) > 0$ is similar. We take an ε small enough so that $U(f'(b), \varepsilon) < \{0\}$. By Definition 7.1.1 there is a $\theta \leq \delta$ such that

$$x \in P(b, \theta) \cap M \Rightarrow D \equiv \tfrac{f(x)-f(b)}{x-b} \in U(f'(b), \varepsilon), \text{ hence } D < 0\,.$$

For these $x < b$ it holds that $f(x) > f(b)$ because $x - b < 0$ and $D < 0$. Similarly for these $x > b$ we have that $f(x) < f(b)$. We take any numbers

$$c \in P^+(b, \theta) \cap M \text{ and } d \in P^-(b, \theta) \cap M\,.$$

These numbers exist because $b \in L^{\mathrm{TS}}(M)$. Hence $c, d \in U(b, \delta) \cap M$ and $f(c) < f(b) < f(d)$. □

Exercise 7.1.9 *The function $f(x) = x \colon [0, 1] \to \mathbb{R}$ has a strict global minimum at 0, and at 1 it has a strict global maximum. At the same time it has nonzero derivatives $f'(0) = f'(1) = 1$. Does it contradict the theorem?*

We restate the previous theorem by reversed implication in the better-known form of necessary condition for local extremes.

Theorem 7.1.10 (NCLE) *Let $f \in \mathcal{F}(M)$, $b \in M$ and let f have a local extreme at b. Then $b \notin L^{\mathrm{TS}}(M)$ or $f'(b)$ does not exist or $f'(b) = 0$.*

So $f \in \mathcal{R}$ may have local, thus also global, extremes only in the following set of "suspicious" points.

$$\underline{\mathrm{SUSP}(f)} \equiv \{b \in M(f) : \ b \notin L^{\mathrm{TS}}(M) \vee \neg \exists f'(b) \vee f'(b) = 0\} \ \ (\subset M(f))\,.$$

• *Derivative and continuity.* Differentiability of a function strengthens its point-wise continuity.

Proposition 7.1.11 (derivative and continuity) *Every function $f \in \mathcal{R}$ is continuous at every point $b \in M(f)$ where it has a finite derivative $f'(b) \in \mathbb{R}$.*

Proof. Suppose that $f \in \mathcal{R}$, $b \in M(f)$ and that $f'(b)$ exists and is in $\mathbb{R}$. Thus $b \in L(M(f))$ and by Theorem 5.3.3 we have that

$$\lim_{x \to b} f(x) = \lim_{x \to b} \big(f(b) + (x - b) \cdot \tfrac{f(x) - f(b)}{x - b}\big) = f(b) + 0 \cdot f'(b) = f(b)\,.$$

The first equality follows from Proposition 4.2.9 and the second from Theorem 5.3.3. By Proposition 5.2.5, the function f is continuous at b. □

Exercise 7.1.12 *Are the functions on the two sides of the first equality the same?*

Exercise 7.1.13 *Show that* $\mathrm{sgn}'(0) = +\infty$.

Thus existence of infinite derivative at a point does not imply continuity at the point.

Exercise 7.1.14 *Show that* $(|x|)'_-(0) = -1$ *and* $(|x|)'_+(0) = +1$.

By part 3 of Exercise 7.1.5 the derivative $(|x|)'(0)$ does not exist. Continuity at a point therefore, of course, does not imply existence of derivative at the point.

Besides Theorem 7.1.8, nonzero point-wise derivative gives the next result which is helpful for derivatives of inverses.

Proposition 7.1.15 (limit points of images) *Suppose that $f \in \mathcal{F}(M)$ and that $f'(b)$ exists, is nonzero and finite. Then $f(b) \in L(f[M])$.*

Proof. Let f and $b \in M \cap L(M)$ be as stated and let an ε be given. Since $f'(b) \neq 0$, by Definition 7.1.1 there is a δ such that for every $x \in P(b, \delta) \cap M$ one has that $f(x) \neq f(b)$. By Proposition 7.1.11 we can take this δ so small that for these x also $f(x) \in U(f(b), \varepsilon)$. Since $b \in L(M)$ we can take an $a \in P(b, \delta) \cap M$. Then $f(a) \in P(f(b), \varepsilon) \cap f[M]$. Hence $f(b) \in L(f[M])$. $\square$

Exercise 7.1.16 *For $f'(b) = \pm\infty$ the proposition does not hold.*

Exercise 7.1.17 *The proposition holds even with the derivative $f'(b) = 0$, if the function f is non-constant on every neighborhood $U(b, \delta)$.*

Exercise 7.1.18 *Adapt Proposition 7.1.11 to one-sided derivatives and one-sided continuity.*

• *Examples of derivatives.* Let us differentiate $\sqrt{x}$ $(\in \mathcal{F}([0, +\infty)))$. Let $a \geq 0$. Then

$$(\sqrt{x})'(a) = \lim_{x \to a} \tfrac{\sqrt{x} - \sqrt{a}}{x - a} = \lim_{x \to a} \tfrac{x - a}{(x-a)(\sqrt{x} + \sqrt{a})} = \lim_{x \to a} \tfrac{1}{\sqrt{x} + \sqrt{a}}\,.$$

For $a > 0$ we get $(\sqrt{x})'(a) = \frac{1}{2\sqrt{a}}$. For $a = 0$ we have $(\sqrt{x})'(0) = +\infty$. Thus infinite derivative may coexist with continuity at a point.

Exercise 7.1.19 *For every $a \geq 0$ compute $(\sqrt{x})'_-(a)$ and $(\sqrt{x})'_+(a)$.*

• *Global derivative.* We add to six operations in Definition 4.4.1 a seventh one.

Definition 7.1.20 (global derivative) $\mathcal{R} \ni f \mapsto f' \in \mathcal{R}$ *is a new unary operation on $\mathcal{R}$. The function $f' \colon D(f) \to \mathbb{R}$ with values $f'(b) \equiv \frac{df}{dx}(b)$ is called the <u>derivative</u> of f. Here <u>$D(f)$</u> $\equiv \{b \in M :\ \exists \frac{df}{dx}(b) \in \mathbb{R}\}$ $(\subset L(M) \cap M)$.*

Thus the notation $f'(b)$ is not ambiguous. The derivative may have smaller definition domain than the original function. For instance $M(\sqrt{x}) = [0, +\infty)$ but $D(\sqrt{x}) = M((\sqrt{x})') = (0, +\infty)$. We investigate interactions of derivative with the operations in Definition 4.4.1 and begin with the operation of restriction.

Proposition 7.1.21 (derivative and restriction) *Let $f \in \mathcal{R}$, X be any set and $M \equiv M(f) \cap X$. <u>Then</u> the following hold.*
1. *If $f'(b)$ exists and $b \in M \cap L(M)$ then $(f \mid X)'(b) = f'(b)$.*
2. *The function $f' \mid M \cap L(M)$ is a restriction of the function $(f \mid X)'$.*

Proof. Let f, X, M and b be as stated. By Proposition 4.2.11, $(f \mid X)'(b)$ equals

$$\lim_{x \to b} \tfrac{(f \mid X)(x) - (f \mid X)(b)}{x - b} = \lim_{x \to b} \left(\tfrac{f(x) - f(b)}{x - b} \mid X\right)(x) = \lim_{x \to b} \tfrac{f(x) - f(b)}{x - b}$$

which is $f'(b)$.

2. Let $N \equiv M \cap L(M)$, $g \equiv f' \mid N$ and $c \in M(g)$. Then $c \in D(f) \cap N$ and by part 1 we have that $(f \mid X)'(c) = f'(c) = g(c)$. Hence $f' \mid N$ is a restriction of $(f \mid X)'$. $\square$

Exercise 7.1.22 (derivative of constants) *For every* $c \in \mathbb{R}$, $k_c' = k_0$.

Exercise 7.1.23 (derivative of identity plus constant) *For every* $c \in \mathbb{R}$, $(\mathrm{id} + k_c)' = k_1$.

Exercise 7.1.24 *For every* $f \in \mathcal{R}$ *we have that* $f \mid D(f) \in \mathcal{C}$.

• *A function with discontinuous derivative.* We find a function $f \in \mathcal{R}$ such that $f' \notin \mathcal{C}$.

Theorem 7.1.25 (discontinuous derivative) *There exists a function* $f \in \mathcal{R}$ *such that* $M(f) = D(f)$ *and* f' *is discontinuous.*

Proof. Let (a_n) and (b_n) be sequences such that $a_1 > b_1 > a_2 > b_2 > \cdots > 0$, $\lim a_n = \lim b_n = 0$ and $a_n - b_n = o(b_n)$ $(n \to \infty)$. Let $N \equiv \{0\} \cup \bigcup_{n=1}^{\infty}(b_n, a_n)$ and $f \in \mathcal{F}(N)$ be given by $f(0) \equiv 0$ and $f(x) \equiv x - b_n$ for $x \in (b_n, a_n)$. Then for any $x \in (b_n, a_n)$ we have that $f'(x) = (x - b_n)' = 1$ (Exercise 7.1.23 and Corollary 7.1.3). But $0 \in L(N)$ and $f'(0) = \lim_{x \to 0} \frac{f(x)}{x} = 0$ because for $x \in (b_n, a_n)$ it holds that $\left|\frac{f(x)}{x}\right| \le \frac{a_n - b_n}{b_n} \to 0$ $(n \to \infty)$. Hence $D(f) = N$ and since $f'(0) = 0$ but $f' = 1$ on $N \setminus \{0\}$, the derivative f' is discontinuous. □

By the previous exercise, $f \in \mathcal{C}$.

Exercise 7.1.26 *Why is* $\lim \frac{a_n - b_n}{b_n} = 0$*?*

7.2 Standard and limit tangents

We give two definitions of the tangent (line) to the graph G_f of a function $f \in \mathcal{R}$ at a point $(b, f(b)) \in G_f$.

• *Standard tangents.* We already mentioned them in connection with differentiability of a function at a point.

Definition 7.2.1 (standard tangents) *Suppose that* $f \in \mathcal{F}(M)$ *is differentiable at* $b \in M \cap L(M)$. *The* <u>*tangent*</u> *to the graph of* f *at the point* $(b, f(b))$ *is the line* ℓ $(\subset \mathbb{R}^2)$ *given by the equation*

$$y = f'(b) \cdot (x - b) + f(b) = f'(b) \cdot x + f(b) - f'(b) \cdot b,\ x \in \mathbb{R}\,.$$

The line ℓ has the slope $f'(b)$ $(\in \mathbb{R})$ and goes through the point $(b, f(b))$.

Exercise 7.2.2 *Let* f *and* b *be as in the definition. Then the function* $x \mapsto f'(b) \cdot (x - b) + f(b)$ *is the only linear function approximating* f *near* b *with the precision* $o(x - b)$ $(x \to b)$.

Exercise 7.2.3 *Write the equation of the tangent to the graph of* $\sqrt{x}$ *at the point* $(a, \sqrt{a})$.

We often read about tangents that they are certain limits of secant lines, but it is never clearly said what these limits exactly are. In the spirit of the initial motto, let us say it clearly.

• *Non-vertical lines.* By plane geometry every non-vertical line ℓ has a unique expression $\ell = \{(x, sx + t) : \ x \in \mathbb{R}\}$ where $s, t \in \mathbb{R}$. We call s the slope of ℓ. Let $\mathcal{N}$ $(\subset \mathcal{P}(\mathbb{R}^2))$ be the set of non-vertical lines in the plane.

Exercise 7.2.4 *The function $p\colon \mathcal{N} \to \mathbb{R}^2$, given by $p(\ell) = (p_1(\ell), p_2(\ell)) \equiv (s, t)$, is a bijection.*

Definition 7.2.5 (limits in $\mathcal{N}$) *For $(\ell_n) \subset \mathcal{N}$ and $\ell \in \mathcal{N}$ we write $\lim \ell_n = \ell$ iff $\lim p_1(\ell_n) = p_1(\ell)$ and $\lim p_2(\ell_n) = p_2(\ell)$. Here $p_i(\cdot)$ are as in the previous exercise.*

Exercise 7.2.6 *Let $A = (a, b)$ and $A' = (a', b')$ be in $\mathbb{R}^2$ & $a \neq a'$. Then there is a unique line $\ell \in \mathcal{N}$ such that $A \in \ell$ and $A' \in \ell$. This line ℓ has slope $\frac{b'-b}{a'-a}$.*

We denote this unique non-vertical line ℓ going through the points $A = (a, b)$ and $A' = (a', b')$ with $a \neq a'$ by $\kappa(A, A')$ or by $\kappa(a, b, a', b')$. If A and A' lie in the graph of f, we speak of ℓ as a secant of this graph.

• *Limit tangents.* We propose a rigorous definition of tangents to graphs of functions as limits of sequences of secants.

Definition 7.2.7 (limit tangents) *Let $f \in \mathcal{F}(M)$, $b \in M \cap L(M)$ and $\ell \in \mathcal{N}$. If for every sequence $(a_n) \subset M \setminus \{b\}$ with $\lim a_n = b$ it holds by Definition 7.2.5 that*

$$\lim_{n\to\infty} \kappa(b,\, f(b),\, a_n,\, f(a_n)) = \ell\,,$$

we say that the line ℓ is the limit tangent to G_f at $(b, f(b))$.

This tangent does not need the derivative $f'(b)$.

Exercise 7.2.8 *If ℓ is a limit tangent to G_f at $(b, f(b))$ then $(b, f(b)) \in \ell$.*

We show that standard and limit tangents coincide.

Theorem 7.2.9 (on limit tangents) *Let $f \in \mathcal{F}(M)$, $b \in M \cap L(M)$ and $\ell \in \mathcal{N}$. Then ℓ is a tangent to G_f at $(b, f(b))$ by Definition 7.2.1 $\iff$ ℓ is a limit tangent to G_f at $(b, f(b))$ by Definition 7.2.7.*

Proof. Let f, M, b and ℓ be as stated. Implication $\Rightarrow$. We assume that there exists $f'(b) \in \mathbb{R}$ and that ℓ is given by the equation

$$y = f'(b) \cdot x + f(b) - f'(b)f(b)\,.$$

Let $(a_n) \subset M \setminus \{b\}$ have $\lim a_n = b$. For $c_n \equiv \frac{f(a_n)-f(b)}{a_n-b}$ the line $\kappa_n \equiv \kappa(b, f(b), a_n, f(a_n))$ is given by the equation

$$y = c_n(x-b) + f(b) = c_n x + f(b) - c_n b\,.$$

By HDD we have that $\lim c_n = f'(b)$. Thus $\lim(f(b) - c_n b) = f(b) - f'(b)f(b)$ and $\lim \kappa_n = \ell$ in the sense of Definition 7.2.5.

Implication $\Leftarrow$. Let ℓ be given by the equation $y = sx + t$ and let (a_n), c_n and κ_n be as above. We assume that for every such sequence (a_n) it holds that $\lim \kappa_n = \ell$ in the sense of Definition 7.2.5. Thus always $\lim c_n = s$ and $\lim(f(b) - c_n b) = f(b) - sb = t$. Then by HDD we have $s = f'(b)$. Hence $t = f(b) - f'(b)b$ and ℓ is a tangent to G_f at $(b, f(b))$. □

• *Tangents in missing points.* We show how to define the tangent to the graph of a function at a point without actually using the point. The next lemma is clear.

Lemma 7.2.10 (a convex combination) *For nonnegative real numbers r, s, t, v, where $s, v > 0$, we define $\alpha \equiv \frac{s}{s+v}$ and $\beta \equiv \frac{v}{s+v}$. Then*

$$\frac{r+t}{s+v} = \alpha \cdot \frac{r}{s} + \beta \cdot \frac{t}{v}$$

is a convex combination with coefficients α and β because $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

Theorem 7.2.11 (tangents in missing points) *Let $b \in L^{\mathrm{TS}}(M)$ for a set $M \subset \mathbb{R}$, $f \in \mathcal{F}(M \setminus \{b\})$ and $\ell \in \mathcal{N}$. Then the equivalence holds that f can be extended to b by a value $f(b)$ so that ℓ is a tangent to G_f at $(b, f(b))$ $\iff$ for every two sequences (x_n), $(y_n) \subset M$ with $\lim x_n = \lim y_n = b$ such that always $x_n < b < y_n$ we have $\lim \kappa(x_n, f(x_n), y_n, f(y_n)) = \ell$ by Definition 7.2.5.*

Proof. Let b, M, f and ℓ be as stated. Implication $\Rightarrow$. We assume that f has been extended to b by a value $f(b)$, that $f'(b) \in \mathbb{R}$ exists and that the line ℓ is given by the equation

$$y = f'(b) \cdot x + f(b) - f'(b)b\,.$$

Suppose that (x_n) and (y_n) are as stated. We set $r_n \equiv f(b) - f(x_n)$, $s_n \equiv b - x_n$, $t_n \equiv f(y_n) - f(b)$ and $v_n \equiv y_n - b$. By Lemma 7.2.10 the slope u_n of the secant

$$\ell_n = \kappa(x_n,\, f(x_n),\, y_n,\, f(y_n))$$

of G_f is a convex combination of the slopes $\frac{r_n}{s_n}$ and $\frac{t_n}{v_n}$ of the two secants $\kappa(x_n, f(x_n), b, f(b))$ and $\kappa(b, f(b), y_n, f(y_n))$:

$$u_n = \frac{f(y_n)-f(x_n)}{y_n-x_n} = \frac{r_n+t_n}{s_n+v_n} = \alpha_n \cdot \frac{r_n}{s_n} + \beta_n \cdot \frac{t_n}{v_n}\,,$$

where $\alpha_n, \beta_n \geq 0$ and $\alpha_n + \beta_n = 1$. Since $\lim \frac{r_n}{s_n} = \lim \frac{t_n}{v_n} = f'(b)$, by Theorem 3.3.10 also $\lim u_n = f'(b)$. The secant ℓ_n is given by the equation

$$y = u_n x + f(x_n) - u_n x_n\,.$$

Since $\lim u_n = f'(b)$, $\lim x_n = b$ and $\lim f(x_n) = f(b)$ (f is continuous at b because $f'(b) \in \mathbb{R}$), we have that $\lim \ell_n = \ell$ in the sense of Definition 7.2.5.

Implication $\neg \Rightarrow \neg$. Suppose that f cannot be extended to b by any value $f(b)$ so that ℓ be a tangent to G_f at $(b, f(b))$. This means that if we take the only number $f(b) \in \mathbb{R}$ such that $(b, f(b)) \in \ell$ and if $s \in \mathbb{R}$ is the slope of the line ℓ, then it is not true that $\lim_{x \to b} \frac{f(x)-f(b)}{x-b} = s$. We show that there are two sequences in M that converge from different sides to b and are such that it is not true that the limit of the corresponding secants is ℓ.

The first case is that the extended function f is not continuous at b. Then there are sequences $(x_n), (y_n) \subset M$ satisfying that always $x_n < b < y_n$, that $\lim x_n = \lim y_n = b$, $\lim f(x_n) = K$, $\lim f(y_n) = L$, but it is not true that $K = L = f(b)$ (Exercise 7.2.12). If $K \neq L$ then the slopes of secants $\ell_n \equiv \kappa(x_n, f(x_n), y_n, f(y_n))$ go to $\pm\infty$ and the stated limit of lines does not hold. If $K = L \neq f(b)$ then the intersections of secants ℓ_n with the vertical line $(x = b)$ converge to (possibly infinite) point different from $(b, f(b))$. By Exercise 7.2.13 the limit $\lim \ell_n$, if it exists, cannot be a line going through $(b, f(b))$ and the stated limit of lines again does not hold.

The remaining case is that the extended function f is continuous at b, but it is not true that $\lim_{x \to b} \frac{f(x)-f(b)}{x-b} = s$. Then there is an $A \in \mathbb{R}^* \setminus \{s\}$ and a sequence $(x_n) \subset M \setminus \{b\}$ lying on one side of b such that $\lim x_n = b$ and $\lim \frac{f(x_n)-f(b)}{x_n-b} = A$. We may assume that always $x_n < b$, the case that always $x_n > b$ is similar. We take any sequence $(y_n) \subset M$ such that always $y_n > b$ and $\lim y_n = b$. Then $\lim f(y_n) = f(b)$ and we can choose from (y_n) a subsequence (y_{m_n}) such that

$$\lim_{n \to \infty} \frac{f(x_n)-f(y_{m_n})}{x_n-y_{m_n}} = A$$

(Exercise 7.2.14). Since $A \neq s$, the stated limit of lines again does not hold. □

The next three exercises are the lemmas used in the previous proof, and the fourth exercise shows that the assumption that b lies between x_n and y_n cannot be omitted.

Exercise 7.2.12 *If $b \in M$ is a two-sided limit point of $M \subset \mathbb{R}$ and $f \in \mathcal{F}(M)$ is not continuous at b, then there exist sequences $(x_n), (y_n) \subset M \setminus \{b\}$ converging from different sides to b such that there exist limits $\lim f(x_n) = K$ and $\lim f(y_n) = L$ but $K \neq f(b)$ or $L \neq f(b)$.*

Exercise 7.2.13 *Let $\ell_n, \ell \in \mathcal{N}$, $(b, c) \in \ell$, $\lim \ell_n = \ell$ and $(x = b) \cap \ell_n = \{(b, c_n)\}$. Then $\lim c_n = c$.*

Exercise 7.2.14 *Suppose that (x_n), (y_n), (z_n) and (u_n) are sequences such that $\lim x_n = \lim z_n = b$, always $x_n \neq b$, $\lim y_n = \lim u_n = c$ and $\lim \frac{y_n - c}{x_n - b} = A$. Then there exists a sequence $(m_n) \subset \mathbb{N}$ such that $\lim \frac{y_n - u_{m_n}}{x_n - z_{m_n}} = A$.*

Exercise 7.2.15 *The condition in Theorem 7.2.11 that always $x_n < b < y_n$ cannot be removed. Give an example of a function $f \in \mathcal{F}(M)$ with the tangent ℓ to G_f at $(b, f(b))$ and of two sequences $(x_n), (y_n) \subset M \setminus \{b\}$ such that always $x_n \neq y_n$ and that $\lim x_n = \lim y_n = b$, but that the limit of lines $\lim \kappa(x_n, f(x_n), y_n, f(y_n)) = \ell$ does not hold.*

7.3 Arithmetic of derivatives

The derivative $f \mapsto f'$ is a unary operation on $\mathcal{R}$. We describe its interactions with the binary operation $+$, $\cdot$ and $/$. In the next section we consider interactions with composition and inverses. Point-wise formulas for these interactions allow infinities as values of derivatives. Global formulas use only finite values.

• *Sums.* We differentiate point-wisely and globally sums of two functions.

Theorem 7.3.1 **($(f+g)'$)** *Let $f, g \in \mathcal{R}$ and $M \equiv M(f) \cap M(g)$. Then the following hold.*
1. *If $f'(b) = K$, $g'(b) = L$, $b \in L(M)$ and $K + L$ is not an indefinite expression then $(f+g)'(b) = K + L$.*
2. *The function $(f' + g') \,|\, L(M)$ is a restriction of the function $(f+g)'$.*

Proof. 1. Suppose that f, g, M, b, K and L are as stated, and that $K + L$ is not an indefinite expression. Let $h \equiv f + g$. Then $b \in M(h) \cap L(M(h))$ and by Theorem 5.3.3 one has

$$h'(b) = \lim_{x \to b} \tfrac{h(x)-h(b)}{x-b} = \lim_{x \to b} \tfrac{f(x)-f(b)}{x-b} + \lim_{x \to b} \tfrac{g(x)-g(b)}{x-b} = K + L\,.$$

2. Let $h \equiv (f' + g') \,|\, L(M)$ and $c \in M(h) = D(f) \cap D(g) \cap L(M)$. By part 1, $(f+g)'(c) = f'(c) + g'(c) = h(c)$. Hence h is a restriction of $(f+g)'$. □

We illustrate part 2 by two examples. *First*, let $f \equiv k_0 \,|\, (-\infty, 0]$ and $g \equiv k_0 \,|\, [0, +\infty)$. Then $f' + g'$ is $k_0 \,|\, \{0\}$ which is $(\{0\}, \mathbb{R}, \{(0,0)\})$, but $(f+g)' = (k_0 \,|\, \{0\})'$ is the empty function $(\emptyset, \mathbb{R}, \emptyset)$. Thus the restriction to $L(M)$ cannot be omitted. *Second*, let $f(x) = |x|$ and $g(x) = -|x|$ (both in $\mathcal{F}(\mathbb{R})$). Then $M = \mathbb{R}$, $L(M) = \mathbb{R}^*$, $(f' + g') \,|\, L(M) = k_0 \,|\, (\mathbb{R} \setminus \{0\})$ and $(f+g)' = k_0$. Thus $(f' + g') \,|\, L(M)$ may be a proper restriction of $(f+g)'$.

As an application of part 1 of Theorem 7.3.1 we compute one derivative. Let $f(x) \equiv \mathrm{sgn}(x)$, $g(x) \equiv \sqrt{x}$ and $b \equiv 0$. Then $M = [0, +\infty)$ and

$$(\mathrm{sgn}(x) + \sqrt{x})'(0) = \mathrm{sgn}'(0) + (\sqrt{x})'(0) = +\infty + (+\infty) = +\infty\,.$$

Exercise 7.3.2 $(\mathrm{sgn}(x) - \sqrt{x})'(0) =$*?*

Theorem 7.3.1 answers for any pair of functions $f, g \in \mathcal{R}$ the question what is the relation between f', g' and $(f+g)'$. It may seem unusual to someone who is used to the slogan $(f+g)' = f' + g'$. We therefore describe a common situation when this slogan speaks truth. Also, we need a result working in the situation of the proof of Theorem 7.6.3.

Corollary 7.3.3 ($(f+g)' = f' + g'$) *Let $f, g \in \mathcal{R}$ and $M \equiv M(f) \cap M(g)$. If $D(f) = M(f)$, $D(g) = M(g)$ and $M \subset L(M)$ then $(f+g)' = f' + g'$.*

Proof. Let f, g and M be as stated, and let $h \equiv (f' + g') \,|\, L(M)$. From the assumptions on f and g it follows that $M(h) = M$ and that $h = f' + g'$. Since $M((f+g)') \subset M$, part 2 of Theorem 7.3.1 gives that $f' + g' = h = (f+g)'$. $\square$

• *Products.* We derive two Leibniz formulas (LF) for the derivatives of the products, the point-wise and global one. *Gottfried W. Leibniz (1646–1716)* "was a German polymath active as a mathematician, philosopher, scientist and diplomat who is disputed with Sir Isaac Newton to have invented calculus in addition to many other branches of mathematics, such as binary arithmetic, and statistics." ([17])

Theorem 7.3.4 ($(fg)'$) *Let $f, g \in \mathcal{R}$ and $M \equiv M(f) \cap M(g)$. Then the following hold.*
1st LF *If $f'(b) = K$, $g'(b) = L$, f or g is continuous at b, $b \in L(M)$ and the expression $K \cdot g(b) + f(b) \cdot L$ is defined, then $(fg)'(b) = K \cdot g(b) + f(b) \cdot L$.*
2nd LF *The function $(f'g + fg') \,|\, L(M)$ is a restriction of the function $(fg)'$.*

Proof. 1. Let the expression $K \cdot g(b) + f(b) \cdot L$ be defined, $h \equiv fg$ and g be continuous at b. The case when f is continuous at b is Exercise 7.3.5. Then $b \in M(h) \cap L(M(h))$ and

$$h'(b) = \lim_{x \to b} \tfrac{f(x)g(x) - f(b)g(b)}{x-b} = \lim_{x \to b} \tfrac{(f(x)-f(b))g(x) + f(b)(g(x)-g(b))}{x-b}\,.$$

This by the assumptions, Proposition 5.2.5 and Theorem 5.3.3 equals to

$$\lim_{x \to b} \tfrac{f(x)-f(b)}{x-b} \cdot \lim_{x \to b} g(x) + f(b) \lim_{x \to b} \tfrac{g(x)-g(b)}{x-b} = Kg(b) + f(b)L\,.$$

2. Let $h \equiv (f'g + fg') \,|\, L(M)$ a $c \in M(h)$. Then $c \in D(f) \cap D(g) \cap L(M)$, g is continuous at c, because $g'(c) \in \mathbb{R}$, and by the first part we have that $(fg)'(c) = f'(c)g(c) + f(c)g'(c) = h(c)$. Hence h is a restriction of $(fg)'$. $\square$

Like for sum, it is not hard to produce examples showing that the restriction to $L(M)$ cannot be omitted and that $(f'g + fg') \,|\, L(M)$ can be a proper restriction of $(fg)'$.

Exercise 7.3.5 *Solve quickly the case when f is continuous at b.*

The next exercise shows that the assumption of continuity of one of the functions at b cannot be omitted.

Exercise 7.3.6 *Let $f, g \in \mathcal{F}(\mathbb{R})$, where for $x \neq 0$ we set $f(x) = -g(x) \equiv \operatorname{sgn} x$ and for $x = 0$ it is $f(0) \equiv -\frac{1}{2}$ and $g(0) \equiv \frac{1}{2}$. Show that then for $b \equiv 0$ the right side of the first Leibniz formula equals $(+\infty) \cdot \frac{1}{2} + (-\frac{1}{2}) \cdot (-\infty) = +\infty$, but $(fg)'(b)$ does not exist.*

We give a corollary with a simple form of the second Leibniz formula; we need it later.

Corollary 7.3.7 **($(fg)' = f'g + fg'$)** *Let $f, g \in \mathcal{R}$ and $M \equiv M(f) \cap M(g)$. If $D(f) = M(f)$, $D(g) = M(g)$ and $M \subset L(M)$ <u>then</u> $(fg)' = f'g + fg'$.*

Proof. Let f, g and M be as stated, and let $h \equiv (f'g + fg') \,|\, L(M)$. From the assumptions on f and g it follows that $M(h) = M$ and $h = f'g + fg'$. Since $M((fg)') \subset M$, part 2 of Theorem 7.3.4 gives that $f'g + fg' = h = (fg)'$. $\square$

• *Ratios.* We derive point-wise and global formulas for derivatives of ratios.

Theorem 7.3.8 **($(\frac{f}{g})')$)** *Let $f, g \in \mathcal{R}$ and $M \equiv M(f) \cap M(g) \setminus Z(g)$. <u>Then</u> the following hold.*
1. If $f'(b) = K$, $g'(b) = L$, g is continuous at b, $b \in L(M)$ and the expression $\frac{K \cdot g(b) - f(b) \cdot L}{g(b)^2}$ is defined then $(\frac{f}{g})'(b) = \frac{K \cdot g(b) - f(b) \cdot L}{g(b)^2}$.
2. The function $\frac{f'g - fg'}{g^2} \,|\, L(M)$ is a restriction of the function $(\frac{f}{g})'$.

Proof. 1. Let the expression $\frac{K \cdot g(b) - f(b) \cdot L}{g(b)^2}$ be defined, $h \equiv \frac{f}{g}$ and let g be continuous at b. Then $b \in M(h) \cap L(M(h))$ and

$$h'(b) = \lim_{x \to b} \frac{f(x)/g(x) - f(b)/g(b)}{x - b} = \lim_{x \to b} \frac{f(x)g(b) - f(b)g(b) + f(b)g(b) - f(b)g(x)}{g(x)g(b)(x-b)} \,.$$

Due to the assumptions, Proposition 5.2.5 and Theorem 5.3.3 this equals

$$\lim_{x \to b} \frac{f(x) - f(b)}{x - b} \lim_{x \to b} \frac{g(b)}{g(x)g(b)} - \lim_{x \to b} \frac{f(b)}{g(x)g(b)} \lim_{x \to b} \frac{g(x) - g(b)}{x - b}$$
$$= \frac{f'(b)g(b) - f(b)g'(b)}{g(b)^2} \,.$$

2. Let $h \equiv \frac{f'g - fg'}{g^2} \,|\, L(M)$ and $c \in M(h)$. Then $c \in D(f) \cap D(g) \cap L(M) \setminus Z(g)$, g is continuous at c, because $g'(c) \in \mathbb{R}$, and by the first part we have that $(\frac{f}{g})'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2} = h(c)$. Hence h is a restriction of $(\frac{f}{g})'$. $\square$

Again, the restriction to $L(M)$ cannot in general be omitted and $\frac{f'g - fg'}{g^2} \,|\, L(M)$ can be a proper restriction of $(\frac{f}{g})'$.

Exercise 7.3.9 *As in Exercise 7.3.6 show that the assumption of continuity of g at b cannot be omitted.*

For later use we again give a corollary with a simple form of the formula for derivatives of ratios.

Corollary 7.3.10 **($(\frac{f}{g})' = \frac{f'g - f'g}{g^2}$)** *Let $f, g \in \mathcal{R}$ and $M \equiv M(f) \cap M(g)$. If $D(f) = M(f)$, $D(g) = M(g)$ and $M \subset L(M)$ <u>then</u> $(\frac{f}{g})' = \frac{f'g - f'g}{g^2}$.*

Proof. Let f, g and M be as stated, and let $h \equiv \frac{f'g-fg'}{g^2} \,|\, L(M)$. From the assumptions on f and g it follows that $M(h) = M \setminus Z(g)$ and $h = \frac{f'g-fg'}{g^2}$. Since $M((\frac{f}{g})') \subset M \setminus Z(g)$, part 2 of Theorem 7.3.8 gives that $\frac{f'g-fg'}{g^2} = h = (\frac{f}{g})'$. $\Box$

7.4 Composite functions and inverses

Recall that for $f, g \in \mathcal{R}$ the composite function

$$f(g)\colon \{x \in M(g):\ g(x) \in M(f)\} \to \mathbb{R}$$

has values $f(g)(x) = f(g(x))$. Thus $M(f(g)) \subset M(g)$ and in general this may be a proper inclusion. Any injective $f \in \mathcal{R}$ has the inverse

$$f^{-1}\colon f[M(f)] \to \mathbb{R}\,,$$

with values $f^{-1}(y) = x \iff f(x) = y$. Hence $M(f^{-1}) = f[M(f)]$. For noninjective f the inverse is not defined.

• *Composite functions.* We obtain formulas for point-wise and global derivatives of composite functions.

Theorem 7.4.1 $((f(g))')$ *Let $f, g \in \mathcal{R}$ and $M \equiv M(f(g))$. Then the following hold.*
1. If $f'(g(b)) = K$, $g'(b) = L$, g is continuous at b, $b \in L(M)$ and $K \cdot L$ is not an indefinite expression, then $f(g)'(b) = K \cdot L$.
2. The function $(f'(g) \cdot g') \,|\, L(M)$ is a restriction of the function $(f(g))'$.

Proof. 1. Let f, g, M and b be as stated. We employ HDD. We assume that the product $f'(g(b)) \cdot g'(b)$ is not indefinite and that $(a_n) \subset M \setminus \{b\}$ is any sequence with $\lim a_n = b$. Then by the assumption $\lim g(a_n) = g(b)$. We partition (a_n) in two (possibly finite or empty) subsequences (b_n) and (c_n) so that always (i.e., for every n) it holds that $g(b_n) = g(b)$ and $g(c_n) \ne g(b)$. We show in both cases $(x_n) \equiv (b_n)$ and $(x_n) \equiv (c_n)$ that if (x_n) is infinite then we have the same limit

$$\lim_{n\to\infty} \tfrac{f(g)(x_n)-f(g)(b)}{x_n-b} = f'(g(b)) \cdot g'(b)\,.$$

Then also $\lim \frac{f(g)(a_n)-f(g)(b)}{a_n-b} = f'(g(b)) \cdot g'(b)$ and by HDD one has that $f(g)'(b) = f'(g(b)) \cdot g'(b)$.

Let (b_n) be infinite. Then $\lim b_n = b$. By HDD we have $g'(b) = \lim \frac{g(b_n)-g(b)}{b_n-b} = \lim \frac{g(b)-g(b)}{b_n-b} = 0$. Thus

$$\lim \tfrac{f(g)(b_n)-f(g)(b)}{b_n-b} = \lim \tfrac{f(g(b))-f(g(b))}{b_n-b} = 0 = f'(g(b)) \cdot 0 = f'(g(b)) \cdot g'(b)\,.$$

Let (c_n) be infinite. Then $\lim c_n = b$, $\lim g(c_n) = g(b)$ and by Theorem 5.3.3 and HDD again $\lim \frac{f(g)(c_n)-f(g)(b)}{c_n-b} = \lim \frac{f(g(c_n))-f(g(b))}{c_n-b}$ equals to

$$\lim\left(\frac{f(g(c_n))-f(g(b))}{g(c_n)-g(b)} \cdot \frac{g(c_n)-g(b)}{c_n-b}\right) = \lim \frac{f(g(c_n))-f(g(b))}{g(c_n)-g(b)} \cdot \lim \frac{g(c_n)-g(b)}{c_n-b}\,.$$

By HDD this is $f'(g(b)) \cdot g'(b)$.

2. Let $h \equiv (f'(g)\cdot g') \,|\, L(M)$ a $c \in M(h)$. Then $c \in M(f'(g)) \cap D(g) \cap L(M)$, g is continuous at c, because $g'(c) \in \mathbb{R}$, and by the first part $(f(g))'(c) = f'(g(c)) \cdot g'(c) = h(c)$. Hence h is a restriction of $(f(g))'$ □

Again, the restriction to $L(M)$ cannot be omitted and $(f'(g)\cdot g') \,|\, L(M)$ can be a proper restriction of $f(g)'$.

Exercise 7.4.2 *In general part 1 does not hold when the continuity of g at b is dropped.*

A corollary we use later gives a simple form of the formula for $(f(g))'$.

Corollary 7.4.3 **($(f(g))' = f'(g) \cdot g'$)** *Let $f, g \in \mathcal{R}$ and $M \equiv M(f(g))$. If $D(f) = M(f)$, $D(g) = M(g)$ and $M \subset L(M)$ <u>then</u> $(f(g))' = f'(g) \cdot g'$.*

Proof. Let f, g and M be as stated, and let $h \equiv (f'(g) \cdot g') \,|\, L(M)$. From the assumptions on f and g it follows that $M(h) = M$ and $h = f'(g) \cdot g'$. Since $M((f(g))') \subset M$, part 2 of Theorem 7.3.4 gives that $f'(g) \cdot g' = h = (f(g))'$. □

• *Inverses.* We take both point-wise and global derivatives of inverses, but now we split the two formulas between the theorem and its corollary. A function $f \in \mathcal{F}(M)$ <u>increases</u>, respectively <u>decreases, at a point</u> $b \in M$ if there exists a δ such that for every x and x' with $b - \delta < x < b < x' < b + \delta$ it holds that $f(x) < f(b) < f(x')$, respectively $f(x) > f(b) > f(x')$.

Theorem 7.4.4 **($(f^{-1})'$)** *Suppose that $f \in \mathcal{F}(M)$ is injective, $f'(b)$ ($\in \mathbb{R}^*$) exists and the inverse f^{-1} is continuous at $c \equiv f(b)$. <u>Then</u> the following hold.*
1. *If $f'(b) \in \mathbb{R} \setminus \{0\}$ then $\left(f^{-1}\right)'(c) = \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(c))}$.*
2. *If $f'(b) = 0$ and f increases, respectively decreases, at b then $(f^{-1})'(c) = +\infty$, respectively $-\infty$.*
3. *If $f'(b) = \pm\infty$ and $c \in L(f[M])$ then $(f^{-1})'(c) = 0$.*

Proof. Let f, M, b and c be as stated. We use again HDD. We take a sequence $(b_n) \subset f[M] \setminus \{c\}$ with $\lim b_n = c$ and set $a_n \equiv f^{-1}(b_n)$. Thus $(a_n) \subset M \setminus \{b\}$ and by the assumption $\lim a_n = b$.

1. Let $f'(b) \in \mathbb{R} \setminus \{0\}$. Then $c \in L(f[M])$ by Proposition 7.1.15. By Theorem 5.3.3 and HDD one has

$$\lim \frac{f^{-1}(b_n)-f^{-1}(c)}{b_n-c} = \lim \frac{1}{\frac{f(a_n)-f(b)}{a_n-b}} = \frac{1}{\lim \frac{f(a_n)-f(b)}{a_n-b}} = \frac{1}{f'(b)}\,.$$

By HDD we have $(f^{-1})'(c) = \frac{1}{f'(b)}$.

2. Let $f'(b) = 0$. Then $c \in L(f[M])$ by Exercise 7.1.17. Suppose that f decreases (respectively increases) at b. Then there is an n_0 such that $n \geq n_0 \Rightarrow \frac{f(a_n)-f(b)}{a_n-b} < 0$ (respectively $\cdots > 0$). The previous computation and part 5 of Proposition 3.1.4 show that $(f^{-1})'(b) = \frac{1}{0^-} = -\infty$ (respectively $\cdots = +\infty$).

3. Let $f'(b) = \pm\infty$ and $c \in L(f[M])$. Then we have that $(f^{-1})'(c) = \frac{1}{\pm\infty} = 0$ by part 1. □

Exercise 7.4.5 *What happens when f^{-1} is not continuous at c?*

Corollary 7.4.6 (global $(f^{-1})'$) *Suppose that $f \in \mathcal{R}$ is injective and set $M \equiv \{x \in f[M(f)] : f^{-1}$ is continuous at $x\}$. Then the function $\frac{1}{f'(f^{-1})} \,|\, M$ is a restriction of the function $(f^{-1})'$.*

Proof. Let $h \equiv \frac{1}{f'(f^{-1})} \,|\, M$ and $c \in M(h)$. Then $c \in M(f'(f^{-1})) \cap M \setminus Z(f'(f^{-1}))$, f^{-1} is continuous at c because $c \in M$, and by part 1 of Theorem 7.4.4 it holds that $(f^{-1})'(c) = \frac{1}{f'(f^{-1}(c))} = h(c)$. Hence h is a restriction of $(f^{-1})'$. □

A similar remark applies here as in previous cases of global derivatives. We do not need derivatives of inverses for the proof of Theorem 7.6.3 but for completeness we still give the corollary with simple formula for $(f^{-1})'$.

Corollary 7.4.7 ($(f^{-1})' = \frac{1}{f'(f^{-1})}$) *Let $f \in \mathcal{R}$ be injective and $M \equiv f[M(f)]$. If $D(f) = M(f)$, $f^{-1} \in \mathcal{C}$ and $M \subset L(M)$ then $(f^{-1})' = \frac{1}{f'(f^{-1})}$.*

Proof. Let f and M be as stated and let $h \equiv \frac{1}{f'(f^{-1})}$. It follows from the assumptions on f and from Corollary 7.4.6 that $M(h) = M \setminus Z(f'(f^{-1}))$ and that h is a restriction of $(f^{-1})'$. Suppose that $c \in M$ is such that $f'(f^{-1}(c)) = 0$. If $c \in M((f^{-1})')$ then part 1 of Theorem 7.4.1 gives that

$$1 = (\mathrm{id} \,|\, M)'(c) = (f(f^{-1}))'(c) = f'(f^{-1}(c)) \cdot (f^{-1})'(c) = 0 \cdot (f^{-1})'(c)\,,$$

which is impossible. Thus $c \notin M((f^{-1})')$ and we deduce that $M((f^{-1})') \subset M \setminus Z(f'(f^{-1}))$. Hence $\frac{1}{f'(f^{-1})} = h = (f^{-1})'$. □

7.5 Derivatives of Basic Elementary Functions

We take derivatives of functions in the set BEF introduced in Definition 4.3.1.

• *Exponential, sine and cosine.* We apply differentiation of power series.

Theorem 7.5.1 (derivatives of PS) *Let the sequence $(a_0, a_1, \dots) \subset \mathbb{R}$ satisfy that $\lim |a_n|^{1/n} = 0$. Then for every number $x \in \mathbb{R}$ the series $S(x) \equiv \sum_{n=0}^{\infty} a_n x^n$ is abscon and $S'(x) = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n \equiv T(x) \in \mathcal{F}(\mathbb{R})$.*

Proof. Let a_n, S and T be as stated, and let $x, c \in \mathbb{R}$ with $c \neq 0$. By Theorem 6.6.3 we know that $S, T \in \mathcal{F}(\mathbb{R})$ (because $\lim(n+1)^{1/n} = 1$). We estimate $U = \left|\frac{1}{c}(S(x+c) - S(x)) - T(x)\right|$: $U \leq \sum_{n=1}^{\infty} |a_{n+1}| \cdot |\sum_{j=0}^{n}(x+c)^j x^{n-j} - (n+1)x^n|$. This with $y \equiv |c| + |x|$ is

$$\begin{aligned}&\leq |c| \textstyle\sum_{n=1}^{\infty} |a_{n+1}| \cdot \sum_{j=1}^{n} \sum_{i=1}^{j} \binom{j}{i} y^{i-1} y^{n-i} \\ &\leq |c| \textstyle\sum_{n=1}^{\infty} |a_{n+1}| \cdot y^{n-1} \cdot \sum_{j=1}^{n} 2^j \leq |c| \cdot \sum_{n=1}^{\infty} |a_{n+1}| \cdot (2y)^{n+1}\,.\end{aligned}$$

For $c \to 0$ this goes to 0. Hence $S'(x) = \lim_{c \to 0} \frac{S(x+c)-S(x)}{c} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = T(x)$. □

Exercise 7.5.2 *Explain the estimates of U in the proof.*

Corollary 7.5.3 ($\exp x$, $\cos x$ and $\sin x$) *Thus* $(\exp x)' = \exp x$*,* $(\cos x)' = -\sin x$ *and* $(\sin x)' = \cos x$*.*

Proof. By the previous theorem one has

$$\begin{aligned}&(\exp x)' = \big(\textstyle\sum_{n\geq 0} \frac{x^n}{n!}\big)' = \sum_{n\geq 0} \frac{(n+1)x^n}{(n+1)!} = \exp x\,, \\ &(\cos x)' = \big(\textstyle\sum_{n\geq 0}(-1)^n \frac{x^{2n}}{(2n)!}\big)' = \sum_{n\geq 0}(-1)^{n+1} \frac{(2n+2)x^{2n+1}}{(2n+2)!} = -\sin x \text{ and} \\ &(\sin x)' = \big(\textstyle\sum_{n\geq 0}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\big)' = \sum_{n\geq 0}(-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = \cos x\,.\end{aligned}$$

□

We obtain derivatives of remaining functions in BEF. By Exercise 7.1.22 we have for every $c \in \mathbb{R}$ that $k_c' = k_0$.

- *Logarithm.* Since $\log x = (\mathrm{e}^x)^{-1}$ and $(\mathrm{e}^x)' = \mathrm{e}^x$, Corollary 7.4.7 gives that

$$(\log x)' = \frac{1}{(\mathrm{e}^x)' \circ (\log x)} = \frac{1}{(\mathrm{e}^x) \circ (\log x)} = \frac{1}{x} \mid (0, +\infty)\,.$$

So $(\log x)' = (k_1(x)/\mathrm{id}(x)) \mid (0, +\infty)$, $\frac{1}{x} = k_1(x)/\mathrm{id}(x)$ is a different function.

Exercise 7.5.4 *What is* $(\log |x|)'$*?*

- *Real power.* We leave their derivatives to an exercise.

Exercise 7.5.5 *Prove the following derivatives. 1. For $a > 0$ it holds that $(a^x)' = a^x \cdot \log a$. 2. For $b \neq 1$ it holds that $(x^b)' = bx^{b-1}$. 3. For $b = 1$ it holds that $(x^b)' = k_1 \mid [0, +\infty)$. 4. $(0^x)' = k_0 \mid (0, +\infty)$. 5. For $m \in \mathbb{Z}$ with $m \neq 0$ it holds that $(x^m)' = mx^{m-1}$. 6. For $m = 0$ it holds that $(x^m)' = k_0$.*

- *Tangent and cotangent.* Same here.

Exercise 7.5.6 *Show that* $(\tan x)' = \frac{1}{\cos^2 x}$ *and* $(\cot x)' = -\frac{1}{\sin^2 x}$*.*

• *Inverse trigonometric functions.* Same here.

Exercise 7.5.7 *Show that* $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$*, that* $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$*, that* $(\arctan x)' = \frac{1}{1+x^2}$ *and that* $(\mathrm{arccot}\, x)' = -\frac{1}{1+x^2}$*. Recall that on the right sides* x *is the identity function* id *and* 1 *is in fact* k_1*.*

• *We summarize derivatives of functions in* BEF *by their definition domains.*

1. The domain $\mathbb{R}$. Here $(\exp x)' = \exp x$, $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\arctan x)' = \frac{1}{1+x^2}$ and $(\mathrm{arccot}\, x)' = -\frac{1}{1+x^2}$. For $m \in \mathbb{N}$ we have that $(x^m)' = mx^{m-1}$, for $m = 0$ that $(x^m)' = k_0$ and for $c \in \mathbb{R}$ that $(k_c(x))' = k_0(x)$.
2. The domain $\mathbb{R} \setminus \{0\}$. For $m \in \mathbb{Z}$ with $m < 0$ it holds that $(x^m)' = mx^{m-1}$. Also, $(\log |x|)' = \frac{1}{x}$.
3. The domain $[0, +\infty)$. For $b > 1$ one has that $(x^b)' = bx^{b-1}$ and for $b = 1$ that $(x^b)' = k_1 \mid [0, +\infty)$.
4. The domain $(0, +\infty)$. For $b < 1$ it holds that $(x^b)' = bx^{b-1}$ and that $(\log x)' = \frac{1}{x} \mid (0, +\infty)$.
5. The domain $\mathbb{R} \setminus \{k\pi + \frac{\pi}{2} :\ k \in \mathbb{Z}\}$. Here $(\tan x)' = \frac{1}{\cos^2 x}$.
6. The domain $\mathbb{R} \setminus \{k\pi :\ k \in \mathbb{Z}\}$. Here $(\cot x)' = -\frac{1}{\sin^2 x}$.
7. The domain $(-1, 1)$. Here $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ and $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ (with $\sqrt{x} \equiv x^{1/2}$).

We give another example of a discontinuous derivative, but now with the definition domain $\mathbb{R}$. The first example is in Theorem 7.1.25.

Exercise 7.5.8 *Let* $f \in \mathcal{F}(\mathbb{R})$ *be given by* $f \equiv x^2 \sin(\frac{1}{x}) \cup \{(0,0)\}$ *(with* $x \equiv \mathrm{id}$*). In other words,* $f(0) = 0$ *and* $f(x) = x^2 \sin(\frac{1}{x})$ *for* $x \neq 0$*. Show that* $D(f) = \mathbb{R}$ *but that* $f' \notin \mathcal{C}$*.*

Here $f = \mathrm{id} \cdot \mathrm{id} \cdot \sin(k_1/\mathrm{id}) \cup \{(0,0)\}$. Probably $f \notin \mathrm{EF}$ because $f' \notin \mathcal{C}$.

7.6 Derivatives of Simple Elementary Functions

What about derivatives of Elementary Functions, the set of functions EF ($\subset \mathcal{R}$)? See their Definitions 4.4.5 and 4.4.14.

Problem 7.6.1 (derivatives in EF) *Prove (or disprove) that for every function* $f \in \mathrm{EF}$ *also* $f' \in \mathrm{EF}$*, that is, the derivative of every elementary function is elementary.*

This is not as clear as one might think. For example, let $f(x) \equiv \arcsin x$ and $g(x) \equiv -\arcsin x$, which are both elementary functions in $\mathcal{F}([-1,1])$. Then

$$f'(x) = \tfrac{1}{\sqrt{1-x^2}} \text{ and } g'(x) = -\tfrac{1}{\sqrt{1-x^2}}$$

are elementary functions in $\mathcal{F}((-1,1))$. Is $(f+g)'$ elementary? Yes, it is, because $(f+g)' = (k_0 \,|\, [-1,1])' = k_0 \,|\, [-1,1]$ and

$$k_0 \,|\, [-1,1] = \sqrt{1-x} - \sqrt{1-x} + \sqrt{1+x} - \sqrt{1+x}\,,$$

but this justification has nothing to do with f' and g'. Moreover, these derivatives have definition domains smaller than $M((f+g)')$, and it is not clear how they could be used to justify $(f+g)' \in \mathrm{EF}$.

We show that when the troublesome functions x^b and $\arcsin x$ are dropped from RBEF then the obtained subset SEF of EF can be by induction relatively easily proven to be closed to derivatives.

Definition 7.6.2 (VBEF & SEF) *Very Basic Elementary Functions are the elementary functions*

$$\underline{\mathrm{VBEF}} \equiv \{k_c(x):\ c \in \mathbb{R}\} \cup \{\mathrm{e}^x,\, \log x,\, \sin x\}\,.$$

Simple Elementary Functions, SEF*, is the subset of* EF *obtained by replacing in Definition 4.4.14 the starting set of functions* RBEF *with* VBEF.

Thus Simple Elementary Functions arise from constants, exponential, logarithm and sine by repeated addition, multiplication, division and composition.

We close this hopefully interesting chapter with the following partial result toward affirmative solution of Problem 7.6.1.

Theorem 7.6.3 (derivatives in SEF) *For every function $f \in \mathrm{SEF}$ it holds that $M(f)$ is an open set, $D(f) = M(f)$ and that $f' \in \mathrm{SEF}$.*

Proof. Let $f \in \mathrm{SEF}$. The proof goes by induction on the length of a generating word of f. First recall that the identity $x = \mathrm{id}(x) \in \mathrm{SEF}$ because it is $\log(\mathrm{e}^x)$ and that instead of the constant function $k_c = k_c(x)$ we often write just c. To begin with, we see that every function in VBEF has the three stated properties: $M(k_c) = M(\mathrm{e}^x) = M(\sin x) = \mathbb{R}$ and $M(\log x) = (0,+\infty)$ are open sets, $(k_c)' = k_0$, $(\mathrm{e}^x)' = \mathrm{e}^x$, $(\log x)' = \frac{1}{x} \,|\, (0,+\infty) = \frac{1}{x} + \log x + (-1)\cdot\log x$, $(\sin x)' = \cos x = \sin(x + \frac{\pi}{2})$ are in SEF and each of these derivatives has the same definition domain as the original function.

In the induction step, we have to show in the four cases when (i) $f = g + h$ or (ii) $f = g \cdot h$ or (iii) $f = g/h$ or (iv) $f = g(h)$ for some $g, h \in \mathrm{SEF}$ with the three stated properties that also f has them. We make use of the facts that every open set $M \subset \mathbb{R}$ satisfies that $M \subset L(M)$ (Exercise 6.4.6) and that open sets are closed to finite intersections (Exercise 6.4.5). Then in the case (i) indeed $M(f) = M(g+h) = M(g) \cap M(h)$ is open, and by Corollary 7.3.3 also

$f' = g' + h' \in \mathrm{SEF}$ with $M(f') = M(g') \cap M(h') = M(g) \cap M(h) = M(f)$. In the case (ii) the proof is very similar and uses Corollary 7.3.7.

In cases (iii) and (iv) we employ the fact that $\mathrm{EF} \subset \mathcal{C}$ (Theorem 6.6.16) and two results on open sets and continuous functions: the zero set of a continuous function is relatively closed (Proposition 6.4.8) and the preimage of an open set by a continuous function is relatively open (Proposition 6.4.10). Then in the case (iii) there is a closed set $U \subset \mathbb{R}$ such that

$$\begin{aligned} M(f) &= M(g) \cap M(h) \setminus Z(h) = M(g) \cap M(h) \cap (\mathbb{R} \setminus Z(h)) \\ &= M(g) \cap M(h) \cap \big(\mathbb{R} \setminus (M(h) \cap U)\big) \\ &= M(g) \cap M(h) \cap \big((\mathbb{R} \setminus M(h)) \cup (\mathbb{R} \setminus U)\big) \\ &= M(g) \cap M(h) \cap (\mathbb{R} \setminus U) \end{aligned}$$

and this is an open set. By Corollary 7.3.10 also $f' = \frac{g'h - gh'}{h^2} \in \mathrm{SEF}$ and $M(f') = \cdots = M(g) \cap M(h) \setminus Z(h) = M(f)$. Finally, in the case (iv) the set $M(f) = h^{-1}[M(g)]$ is open. By Corollary 7.4.3 also $f' = g'(h) \cdot h' \in \mathrm{SEF}$ and

$$\begin{aligned} M(f') &= h^{-1}[M(g')] \cap M(h') = h^{-1}[M(g)] \cap M(h) \\ &= h^{-1}[M(g)] = M(g(h)) \\ &= M(f)\,. \end{aligned}$$

The theorem is proven by induction. □

We plan to address Problem 7.6.1 again in *MA 1*$^{+}$.

Chapter 8

Applications of mean value theorems

I gave the eighth lecture

`https://kam.mff.cuni.cz/~klazar/MAI24_pred8.pdf`

on April 11, 2024. In Section 8.1 we meet three mean value theorems, Rolle's Theorem 8.1.1, Lagrange's Theorem 8.1.4 (strengthened in Theorem 8.1.6) and Cauchy's Theorem 8.1.7. Sections 8.2–8.4 deal with three applications. In Section 8.2 we show by means of Rolle's theorem that the sequence $(\log n) = (0, \log 2, \log 3, \dots)$ is not P-recurrent; Theorem 8.2.6 proves in fact a more general result. In Sections 8.3 and 8.4 we show with the help of Lagrange's theorem in two effective ways that real transcendental numbers exist. Theorem 8.5.1 and Proposition 8.5.2 deal with the relation of the sign of f' and monotonicity of f. Theorems 8.5.9 and 8.5.10 are l'Hospital rules for computing limits of indefinite functional expressions $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Proposition 8.6.3 relates the sign of $(f')'(b)$ and the type of the local extreme of f at b. In Theorem 8.6.9 we prove that any convex or concave function f defined on a set $M \subset \mathbb{R}$ with no minimum or maximum element has on $L^{\pm}(M)$ finite one-sided derivatives, hence f is continuous. Theorem 8.6.14 relates the sign of f'' and the convexity/concavity of f (defined on an interval). Theorems 8.6.20 and 8.6.21 provide necessary and sufficient conditions for existence of inflection points. In Section 8.7 we give in twelve steps a procedure for determining main geometric features of the graph of f; step 0 places f in the hierarchy SEL $\subset$ EL $\subset \mathcal{R}$. We exemplify the procedure on three functions: $\operatorname{sgn} x$, $\tan x$ and $\arcsin\left(\frac{2x}{x^2+1}\right)$.

8.1 Three mean value theorems

These theorems concern relations between values of functions and values of their derivatives. In this section we assume that $a < b$ are real numbers.

• *Rolle's theorem.* It is the most basic of the three mean value theorems, the other two are deduced from it.

Theorem 8.1.1 (Rolle) *Suppose that $f \in \mathcal{C}([a,b])$, $f(a) = f(b)$ and that for every $c \in (a,b)$ the derivative $f'(c)$ in $\mathbb{R}^*$ exists. Then there is a $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. If f is constant, the theorem is trivially valid. Suppose that for some c in (a,b) we have $f(c) > f(a) = f(b)$ (the case with $\ldots < \ldots$ is similar). By Theorem 6.4.1, f has a maximum $c \in [a,b]$. Clearly, $c \neq a, b$ and $c \in L^{\mathrm{TS}}([a,b])$. Since $f'(c)$ exists, Theorem 7.1.8 gives $f'(c) = 0$. □

Michel Rolle (1652–1719) was a French mathematician.

Exercise 8.1.2 *Which assumption of the Rolle theorem is not satisfied for the function $|x| \mid [-1,1]$ and the interval $[-1,1]$?*

This exercise inspired us to the following derivatives-free generalization of Rolle's theorem.

Theorem 8.1.3 (generalized RT) *Let $f \in \mathcal{C}([a,b])$ with $f(a) = f(b)$. Then there is a $c \in (a,b)$ such that for every ε there exist $d_1, d_2 \in (a,b)$ such that*

$$d_1 < c < d_2,\ d_2 - d_1 \le \varepsilon \ \text{ and } \ f(d_1) = f(d_2)\,.$$

In other words, it is possible to cut the graph by horizontal secants in two points lying arbitrarily close to and on opposite sides of the point $(c, f(c))$.

Proof. If f is constant on $[a,b]$, the result holds trivially with any $c \in (a,b)$. Else like in the proof of Theorem 8.1.1 we take, say, a maximum $c \in (a,b)$ of $f(x)$ (for a minimum we argue similarly). There are three cases: (i) $f(x)$ is constantly $f(c)$ on $[c-\delta, c]$ ($\subset [a,b]$) or (ii) $f(x)$ is constantly $f(c)$ on $[c, c+\delta]$ ($\subset [a,b]$) or (iii) $f(x)$ has arbitrarily close to c both to the left and right of c values smaller than $f(c)$. In case (i), respectively (ii), we replace c with $c - \frac{\delta}{2}$, respectively $c + \frac{\delta}{2}$, and it is clear that then c has the stated property. In case (iii) we keep c and for every given ε find by means of Theorem 6.3.1 the required points d_1 and d_2. □

• *Lagrange's theorem.* This mean value theorem is used most often and has an interesting geometric interpretation and strengthening.

Theorem 8.1.4 (Lagrange) *Suppose that $f \in \mathcal{C}([a,b])$ and that for every c in (a,b) the derivative $f'(c)$ in $\mathbb{R}^*$ exists. Then there is a $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

Proof. Let $z \equiv \frac{f(b)-f(a)}{b-a}$. Then the function $g(x) \equiv f(x) - z(x-a)$ $(\in \mathcal{C}([a,b]))$ satisfies all assumptions of Theorem 8.1.1, especially $g(a) = g(b) = f(a)$. Thus for some $c \in (a,b)$ one has $g'(c) = f'(c) - z = 0$ and $f'(c) = z$. □

Geometrically, there is a point $(c, f(c)) \in G_f$, $c \in (a,b)$, at which the tangent to G_f is parallel to the secant $\kappa(a, f(a), b, f(b))$. *Joseph-Louis Lagrange (1736–1813)* was an Italian-French mathematician, physicist, and astronomer.

Tangents to graphs are of two kinds. Let $f \in \mathcal{F}(M)$, $b \in M$ and f be differentiable at b, so that $f'(b) \in \mathbb{R}$ exists. Let $\ell(x) = f'(b)(x-b) + f(b)$ $(\in \mathcal{F}(\mathbb{R}))$ be the tangent to G_f at $B \equiv (b, f(b))$. If there exists a δ such that $\ell(x) \geq f(x)$ on $U(b,\delta) \cap M$, or $\ell(x) \leq f(x)$ on $U(b,\delta) \cap M$, we say that $\ell(x)$ is a non-cutting tangent. Otherwise, if there is no such δ, we call $\ell(x)$ a cutting tangent. For cutting tangents the graph contains, arbitrarily close to the contact point, points both below and above the tangent. For non-cutting tangents points in the graph sufficiently close to the contact point lie on the same side of the tangent.

Exercise 8.1.5 *Prove the following more precise version of Lagrange's theorem.*

Theorem 8.1.6 (more precise Lagrange) *Suppose that $f \in \mathcal{C}([a,b])$ and that for every $c \in (a,b)$ the derivative $f'(c)$ in $\mathbb{R}^*$ exists. Then for some $c \in (a,b)$ the tangent to G_f at $(c, f(c))$ is parallel to the secant $\kappa(a, f(a), b, f(b))$ and is non-cutting.*

- *Cauchy's theorem* involves two functions.

Theorem 8.1.7 (Cauchy) *Suppose that $f, g \in \mathcal{C}([a,b])$, $g(b) \neq g(a)$ and that for every $c \in (a,b)$ the derivatives $f'(c)$ in $\mathbb{R}^*$ and $g'(c)$ in $\mathbb{R}$ (so $g'(c) \neq \pm\infty$) exist. Then there is a $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{g(b) - g(a)} \cdot g'(c)\,.$$

Proof. Let $z \equiv \frac{f(b)-f(a)}{g(b)-g(a)}$. The function $h(x) \equiv f(x) - z(g(x) - g(a))$ $(\in \mathcal{C}([a,b]))$ satisfies all assumptions of Theorem 8.1.1, in particular $h(a) = h(b) = f(a)$. Hence there is a $c \in (a,b)$ with $h'(c) = f'(c) - zg'(c) = 0$ and $f'(c) = zg'(c)$. □

Exercise 8.1.8 *Where in the proof would $g'(c) = \pm\infty$ cause problems?*

8.2 The sequence $(\log n)$ is not P-recurrent

We show with the help of Rolle's Theorem 8.1.1 that the sequence

$$(\log n) = (0,\ \log 2,\ \log 3,\ \dots)$$

is not P-recurrent. We begin by defining this class of sequences.

• P-*recurrent sequences* generalize C-recurrent (constantly recurrent) sequences. These sequences $(a_n) \subset \mathbb{R}$ satisfy for some $k \in \mathbb{N}$ real coefficients $c_1,\ \dots,\ c_k$, not all of them zero, for every $n = k, k+1, \dots$ the relations

$$\textstyle\sum_{i=1}^{k} c_i a_{n-i+1} = c_1 a_n + c_2 a_{n-1} + \cdots + c_k a_{n-k+1} = 0\,.$$

A well known constantly recurrent sequence is the Fibonacci numbers

$$(F_n) = (1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \dots)\,.$$

Exercise 8.2.1 *What C recurrence do they satisfy?*

In P-recurrent sequences (a_n) constant coefficients are generalized to polynomials in n. So every C-recurrent sequence is P-recurrent.

Definition 8.2.2 (P-recurrence) *A sequence $(a_n) \subset \mathbb{R}$ is P-recurrent if there exist $k \in \mathbb{N}$ polynomials $p_i \in \mathrm{POL}$, $i \in [k]$, not all of them zero, such that for every integer $n \geq k$ we have the equality*

$$\textstyle\sum_{i=1}^{k} p_i(n) \cdot a_{n-i+1} = 0\,.$$

For instance, the sequence

$$(a_n) \equiv (n!) = (1,\ 2,\ 6,\ 24,\ 120,\ 720,\ 5040,\ \dots)$$

of factorials is P-recurrent: for every $n \geq 2$ we have $n! = n \cdot (n-1)!$, that is, $1 \cdot a_n + (-n) \cdot a_{n-1} = 0$.

Exercise 8.2.3 *If the inequality $n \geq k$ in Definition 8.2.2 is weakened to $n \geq n_0$, where $n_0 \geq k$, then we still get the same, equivalent definition of* P*-recurrent sequences.*

Exercise 8.2.4 *If $\sum_{i=1}^{k} p_i(n) \cdot a_{n-i+1} = 0$ holds just for every $n \geq n_0$, where $n_0 \geq k$, then there exist polynomial coefficients $q_1(x),\ \dots,\ q_k(x)$ for which the recurrence holds for every $n \geq k$.*

See [37] for use of P-recurrent sequences in enumerative combinatorics.

• *The sequence* $(\log n)$ *is not* P*-recurrent.* We begin the proof with the next proposition.

Proposition 8.2.5 (nonzero derivative) *Suppose that $r(x) \in \mathrm{RAC}$ is a rational function, that $c_1, \dots, c_k$ are $k \in \mathbb{N}$ real numbers, not all zero, and that*

$$f(x) \equiv r(x) + \textstyle\sum_{i=1}^{k} c_i \log(x - i + 1)\,.$$

<u>Then</u> the derivative f' is a non-zero rational function.

Proof. We may assume that $c_k \neq 0$. Clearly, $f'(x) = r'(x) + \sum_{i=1}^{k} \frac{c_i}{x-i+1}$. If $r(x)$ is a zero rational function, then so is $r'(x)$,

$$\lim_{x \to (k-1)^+} f'(x) = \lim_{x \to (k-1)^+} \tfrac{c_k}{x-k+1} = c_k \cdot (+\infty) = \pm\infty$$

and $f'(x) \neq 0$. If $r(x) \neq 0$, we write $r(x) = s(x) \cdot (x-k+1)^m$ where $s(x) \in \mathrm{RAC}$, $s(k-1) \neq 0$ and $m \in \mathbb{Z}$. Then

$$r'(x) = s'(x) \cdot (x-k+1)^m + m s(x) \cdot (x-k+1)^{m-1}\,.$$

If $m \geq 0$ then again $\lim_{x \to (k-1)^+} f'(x) = c_k \cdot (+\infty) = \pm\infty$. If $m < 0$ then we still have the infinite limit

$$\begin{aligned}\lim_{x \to (k-1)^+} f'(x) &= \lim_{x \to (k-1)^+} (x-k+1)^{m-1}\big(s'(x) \cdot (x-k+1) + m s(x) + \\ &\quad + c_k (x-k+1)^{-m}\big) = (+\infty) \cdot (0 + m s(k-1) + 0) \\ &= (+\infty) \cdot m s(k-1) = \pm\infty\end{aligned}$$

and see that $f'(x)$ is nonzero. □

In other words, rational functions that are derivatives of other rational functions are disjoint from rational functions that are derivatives of linear combinations of shifted logarithms.

The following theorem is actually more general than the result on $(\log n)$ which we are proving.

Theorem 8.2.6 (finitely many zeros) *Consider elementary functions $f(x)$ of the form*

$$f(x) \equiv r(x) + \textstyle\sum_{j=1}^{k} p_j(x) \log(x - j + 1)$$

where $r(x) \in \mathrm{RAC}$, $k \in \mathbb{N}$, $p_j(x) \in \mathrm{POL}$ and not all polynomials $p_j(x)$ are zero. <u>Then</u> $Z(f)$ is finite, such functions always have only finitely many zeros.

Proof. For each of these functions we define its degree $\deg f$ ($\in \mathbb{N}_0$) as the minimum value of the sum

$$\textstyle\sum_{j \in [k] \wedge p_j \neq 0} \deg p_j\,,$$

where we minimize over all possible representations of $f(x)$ in the displayed form. We argue by contradiction and take a function $f(x)$ of the considered form with infinitely many zeros and with the minimum degree. We choose an infinite strictly monotone sequence $(a_n) \subset Z(f)$ (Exercise 8.2.7) and may assume that for every $n \in \mathbb{N}$ the interval (a_n, a_{n+1}), respectively (a_{n+1}, a_n), is contained in $M(f)$ (Exercise 8.2.8). We use Rolle's Theorem 8.1.1 and get a sequence $(b_n) \subset Z(f')$ such that

$$a_1 < b_1 < a_2 < b_2 < a_3 < \dots, \text{ respectively } a_1 > b_1 > a_2 > b_2 > a_3 > \dots$$

(Exercise 8.2.9). Thus (b_n) is injective and also $f'(x)$ has infinitely many zeros. But this is a contradiction. If $\deg f = 0$, we get a contradiction by Proposition 8.2.5 because then f' is a nonzero rational function which has only finitely many zeros. If $\deg f > 0$, we have a contradiction with the minimality of $\deg f$ because f' is again a function of the considered type but $\deg f' < \deg f$ (Exercise 8.2.10). □

Exercise 8.2.7 *How do we select from the infinite set $Z(f)$ an increasing, or a decreasing, sequence (a_n)?*

Exercise 8.2.8 *Why can we assume that the gaps between consecutive terms in (a_n) are contained in $M(f)$?*

Exercise 8.2.9 *How do we exactly apply Rolle's theorem to f and (a_n) so that we get the interleaving zeros (b_n) of f'?*

Exercise 8.2.10 *Why is f' of the considered form and why for $\deg f > 0$ we have $\deg f' < \deg f$?*

The fact that the sequence $(\log n)$ is not P-recurrent is an immediate corollary of Theorem 8.2.6.

Corollary 8.2.11 (on $(\log n)$) *The sequence $(\log n)$ is not P-recurrent.*

Proof. If it is P-recurrent then there exist $k \in \mathbb{N}$ polynomials $p_j(x) \in \mathrm{POL}$, $j \in [k]$, not all zero, such that for every $n \geq k$,

$$\textstyle\sum_{j=1}^{k} p_j(n) \log(n - j + 1) = 0\,.$$

But then, contrary to Theorem 8.2.6, the function

$$f(x) \equiv \textstyle\sum_{j=1}^{k} p_j(x) \log(x - j + 1)$$

has infinitely many zeros because $Z(f) \supset \{k, k+1, \dots\}$. □

Exercise 8.2.12 *Generalize it to sequences $(\log(n + c))$ for any real $c > -1$.*

Exercise 8.2.13 *Prove by means of Rolle's theorem the following proposition.*

Proposition 8.2.14 *Suppose that $I \subset \mathbb{R}$ is a nontrivial interval, $f \in \mathcal{C}(I)$, $f'(x) \in \mathbb{R}^*$ exists for every $x \in I$ and $Z(f)$ is infinite. Then $Z(f')$ is infinite.*

This section is based on the preprint [22] of the author.

8.3 Cantor's transcendental numbers

We give with the help of Lagrange's Theorem 8.1.4 an effective version of Cantor's proof of the existence of transcendental numbers. We begin by defining them.

• *Algebraic and transcendental numbers.* A number $\alpha \in \mathbb{C}$ is algebraic if it is a root of a nonzero polynomial with rational coefficients. This means that there exist fractions $a_0, a_1, \dots, a_n$, $n \in \mathbb{N}_0$, such that $a_n \neq 0$ and

$$\textstyle\sum_{i=0}^{n} a_i\alpha^i = a_0 + a_1\alpha + \cdots + a_n\alpha^n = 0\,.$$

Exercise 8.3.1 *Rational numbers and the number $\sqrt{2}$ are algebraic.*

Exercise 8.3.2 *The polynomial in the definition of an algebraic number can always be modified so that its degree is preserved and* (i) *the polynomial is rational and monic (leading coefficient is* 1*) or* (ii) *the polynomial is integral (all coefficients are integers).*

Algebraic numbers for which both forms (i) and (ii) are simultaneously achievable, that is, roots of monic integral polynomials, are called algebraic integers.

Exercise 8.3.3 *Which fractions are algebraic integers?*

Exercise 8.3.4 *Is the golden ratio $\phi \equiv \frac{1+\sqrt{5}}{2}$ an algebraic integer?*

Exercise 8.3.5 *What is the relation of the golden ratio and the Fibonacci numbers?*

Definition 8.3.6 (transcendental numbers) *We say that a complex number is transcendental if it is not algebraic.*

• *Cantor's proof of the existence of real transcendental numbers.* In 1870s G. Cantor gave a simple proof of existence of transcendental numbers. We give it here as an exercise.

Exercise 8.3.7 (Cantor's proof) *Show that the set of algebraic numbers is countable. Deduce from this the existence of real transcendental numbers.*

• *An effective version of Cantor's proof.* We give an effective, algorithmic version of Cantor's proof and begin with an auxiliary result. It effectivizes, by means of Lagrange's Theorem 8.1.4, the observation that around any nonzero value of a continuous function there is a neighborhood on which it does not vanish. A decimal fraction is any fraction of the form $\frac{a}{10^k}$ where $a \in \mathbb{Z}$ and $k \in \mathbb{N}_0$.

Proposition 8.3.8 **($p(x) \neq 0$ on I)** *Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$, with $n \in \mathbb{N}_0$, $a_i \in \mathbb{Z}$ and $a_n \neq 0$, be a nonzero integral polynomial and let $\alpha = \frac{a}{10^k}$, with $a \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, be a decimal fraction such that $p(\alpha) \neq 0$. Let*

$$b \equiv (n+1)^2 \cdot \max(|a_0|, \ldots, |a_n|) \cdot (|a|+1)^n \quad (\in \mathbb{N}) \quad \textit{and} \quad l \equiv kn + b \quad (\in \mathbb{N}).$$

Then

$$\forall\, x \in \big[\alpha,\, \alpha + 10^{-l}\big]\, \big(p(x) \neq 0\big).$$

Proof. Let $p(x)$ and α be as stated. Since α has denominator 10^k and $p(x)$ is integral, the assumption that $p(\alpha) \neq 0$ gives that $|p(\alpha)| \geq 10^{-kn}$. For every $x \in [\alpha, \alpha+1]$ we have by Lagrange's Theorem 8.1.4 some $y \in (\alpha, x)$ such that

$$p(x) = p(\alpha) + p'(y) \cdot (x - \alpha).$$

Now $|p'(y)| \leq b = (n+1)^2 \cdot \max(|a_0|, \ldots, |a_n|) \cdot (|a|+1)^n$ (Exercise 8.3.9). If $x - \alpha \leq 10^{-l} = 10^{-kn-b} < \frac{10^{-kn}}{b}$ then

$$|p(x)| \geq |p(\alpha)| - |p'(y)| \cdot (x - \alpha) > 10^{-kn} - b \cdot \tfrac{10^{-kn}}{b} = 0,$$

as stated. □

Exercise 8.3.9 *Explain the estimate of the derivative.*

Theorem 8.3.10 (Cantor's transcendents) *We describe an algorithm*

$$\mathcal{A}\colon \mathbb{N} \to \{0, 1, \ldots, 9\}$$

such that the number

$$\kappa \equiv \sum_{n=1}^{\infty} \mathcal{A}(n) \cdot 10^{-n} \quad (\in \mathbb{R})$$

is not a root of any nonzero integral polynomial and therefore is transcendental.

Proof. We start from an algorithm $\mathcal{B}\colon \mathbb{N} \to \bigcup_{n=1}^{\infty} \mathbb{Z}^n \equiv Z$ such that

$$\mathcal{B}[\mathbb{N}] = \{(a_0, \ldots, a_n) \in Z :\ n \in \mathbb{N}_0 \wedge a_n \neq 0\}$$

—$\mathcal{B}$ lists all nonzero integral polynomials as tuples of their coefficients. We write

$$\mathcal{B}(m) = p_m(x) = \textstyle\sum_{j=0}^{n_m} a_{j,m} x^j,$$

with $n_m \in \mathbb{N}_0$, $a_{j,m} \in \mathbb{Z}$ and $a_{n_m,m} \neq 0$. $\mathcal{A}$ constructs a sequence of decimal fractions

$$\alpha_1 = \tfrac{z_1}{10^{k_1}} = \tfrac{0}{10^0} = 0,\ \alpha_2 = \alpha_1 + \tfrac{z_2}{10^{k_2}},\ \alpha_3 = \alpha_2 + \tfrac{z_3}{10^{k_3}},\ \ldots,$$

with $k_j \in \mathbb{N}_0$, such that $k_1 = 0 < k_2 < \ldots$, for every $j \in \mathbb{N}$ it holds that $z_j \in \{0, 1, \ldots, 10^{k_j - k_{j-1}} - 1\}$ (with $k_0 \equiv 0$) and for every $m \in \mathbb{N}$,

$$\forall\, x \in \big[\alpha_m,\, \alpha_m + 10^{-k_m}\big]\, \big(p_1(x) p_2(x) \ldots p_{m-1}(x) \neq 0\big).$$

Suppose that $m \in \mathbb{N}$ and that $\mathcal{A}$ already generated $\alpha_1, \ldots, \alpha_m$ (for $m = 1$ the displayed condition is void and holds trivially). To obtain α_{m+1}, $\mathcal{A}$ calls $\mathcal{B}$, considers the polynomial $p_m(x)$ and takes a $k \in \mathbb{N}$ such that $k > k_m$ and $10^{k-k_m} > \deg p_m$. Then there is a $j \in \{0, 1, \ldots, 10^{k-k_m} - 1\}$ such that $p_m(\alpha_m + \frac{j}{10^k}) \neq 0$, and $\mathcal{A}$ takes one. $\mathcal{A}$ applies Proposition 8.3.8 with $p(x) = p_m(x)$ and $\alpha = \alpha_m + \frac{j}{10^k}$, and gets the $l \in \mathbb{N}$. Then $\mathcal{A}$ computes

$$k_{m+1} \equiv \max(k, l),\ z_{m+1} \equiv j \cdot 10^{k_{m+1}-k} \text{ and } \alpha_{m+1} \equiv \alpha_m + \tfrac{z_{m+1}}{10^{k_{m+1}}}\ .$$

Thus if $\overline{z_j}$ denotes the $k_j - k_{j-1}$-tuple of decimal digits of z_j (for example, if $k_j - k_{j-1} = 5$ and $z_j = 27$ then $\overline{z_j} = 0, 0, 0, 2, 7,$) then $\mathcal{A}$ actually computes the sequence of digits

$$\mathcal{A}(1),\ \mathcal{A}(2),\ \dots = \overline{z_1}\,\overline{z_2}\ \dots\ .$$

It follows that

$$\kappa \equiv \textstyle\sum_{n=1}^{\infty} \mathcal{A}(n) \cdot 10^{-n} = \bigcap_{m=1}^{\infty} \left[\alpha_m,\, \alpha_m + 10^{-m}\right]$$

and so $p_m(\kappa) \neq 0$ for every $m \in \mathbb{N}$. □

In *MA 1*$^+$ we hope to generalize the construction of κ so that κ will not be a zero of any nonzero function in $\mathrm{EF}_{\mathbb{Q}}$ ($\subset$ EF). The latter functions are the elementary functions generated by rational constants $k_\alpha(x)$, $\alpha \in \mathbb{Q}$.

8.4 Liouville's transcendental numbers

The French mathematician and physicist *Joseph Liouville (1809–1882)* was the first who proved, in 1844, that transcendental numbers exist. We explain his proof in this section. It is constructive, not complicated and produces simpler examples of transcendental numbers than Theorem 8.3.10. J. Liouville proved that irrational algebraic numbers cannot be too closely approximated by fractions. The corollary is that every irrational real number with excellent rational approximations, for example every irrational sum of very quickly converging series with rational summands, is transcendental.

• *Liouville's inequality.* Lagrange's Theorem 8.1.4 is again an important tool in the proof of the next theorem.

Theorem 8.4.1 (Liouville's inequality) *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational algebraic number. Then there is an $n \in \mathbb{N}$ and a real constant $c > 0$ such that for every fraction $\frac{p}{q} \in \mathbb{Q}$ with $q \in \mathbb{N}$ we have*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^n}\ .$$

Proof. Suppose that α is as stated and that $f(x)$ is a nonzero integral polynomial with minimum degree $\deg f$ $(\in \mathbb{N})$ such that $f(\alpha)=0$. Let $n \equiv \deg f$ and $I \equiv [\alpha-1, \alpha+1]$. If $\frac{p}{q} \in \mathbb{Q} \setminus I$ then $|\alpha - \frac{p}{q}| \ge 1 \ge \frac{1}{q^n}$. In $\frac{p}{q} \in I$ then $\frac{p}{q} \ne \alpha$ and by Lagrange's Theorem 8.1.4 there is a real number x lying between α and $\frac{p}{q}$ such that (recall that $f(\alpha)=0$)

$$f(\alpha) - f(\tfrac{p}{q}) = f'(x)(\alpha - \tfrac{p}{q}), \text{ hence } \left|\alpha - \tfrac{p}{q}\right| = \tfrac{|f(\frac{p}{q})|}{|f'(x)|}\,.$$

The crucial fact is that $f(\frac{p}{q}) \ne 0$. If $f(\frac{p}{q}) = 0$ then $g(x) \equiv \frac{f(x)}{x-\frac{p}{q}}$ would be a rational polynomial with $g(\alpha)=0$ (Exercise 8.4.2) but $\deg g = \deg f - 1$, in contradiction with the choice of $f(x)$. Thus $f(\frac{p}{q}) \ne 0$ and, as we know, this implies that $|f(\frac{p}{q})| \ge \frac{1}{q^n}$. We take a real number $d>0$ such that $|f'(y)| \le \frac{1}{d}$ for every $y \in I$ (Exercise 8.4.3). Then

$$\left|\alpha - \tfrac{p}{q}\right| \ge \tfrac{d}{q^n}$$

and we get Liouville's inequality with the constant $c \equiv \min(1,d)$. □

Exercise 8.4.2 *Why is α a root of $g(x)$?*

Exercise 8.4.3 *Where does the constant d come from?*

• *Liouville's transcendental numbers.* Here is an example of a real transcendental number constructed by Liouville's method.

Corollary 8.4.4 (λ is transcendental) *The real number*

$$\lambda \equiv \sum_{n=1}^{\infty} \frac{1}{10^{n!}} = 0.110001000000000000000001000\ldots$$

is transcendental.

Proof. The number λ is irrational because it does not have eventually periodic decimal expansion (Exercise 8.4.5). For $m \in \mathbb{N}$ we set

$$\textstyle\sum_{n=1}^{m} \frac{1}{10^{n!}} \equiv \frac{z_m}{10^{m!}} \equiv \frac{z_m}{q_m} \quad (\in \mathbb{Q})\,,$$

$z_m, q_m \in \mathbb{N}$ and $q_m \ge 2$. Then $|\lambda - \frac{z_m}{q_m}| \le \frac{1}{10^{(m+1)!}} \frac{1}{1-10^{-(m+1)!}} < \frac{2}{q_m^{m+1}}$. It is easy to see that for any $n \in \mathbb{N}$ and any $c>0$, for sufficiently large m the fraction $\frac{z_m}{q_m}$ violates Liouville's inequality in Theorem 8.4.1 (Exercise 8.4.6). Thus λ is transcendental. □

Exercise 8.4.5 *Show that every rational number has an eventually periodic decimal expansion.*

Exercise 8.4.6 *Explain in detail why fractions $\frac{z_m}{q_m}$ violate for large m any Liouville's inequality.*

Exercise 8.4.7 *What is the complexity of the natural algorithm $\mathcal{L}$ that computes the decimal expansion of λ?*

Exercise 8.4.8 *Prove that for every $k \in \mathbb{N}$, $k \geq 2$, the number $\sum_{n\geq 1} k^{-n!}$ is transcendental.*

8.5 Monotonicity and l'Hospital rules

We show with the help of Lagrange's Theorem 8.1.4 how intervals of monotonicity of a function can be determined from its derivative. We also prove l'Hospital rules; one computes by them limits of indefinite expressions $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

• *Derivatives and intervals of monotonicity of functions.* For any set $M \subset \mathbb{R}$,

$$M^0 \equiv \{a \in M:\ \exists\,\delta\,\big(U(a,\delta) \subset M\big)\}$$

denotes its *interior*. The interior of an interval $I \subset \mathbb{R}$ is the open interval $I^0 \subset I$ obtained from I by deleting its endpoints.

Theorem 8.5.1 (monotonicity 1) *Suppose that $f \in \mathcal{C}(I)$, where $I \subset \mathbb{R}$ is a nontrivial interval, and that for every $c \in I^0$ there exists $f'(c)$ in $\mathbb{R}^*$. Then the following implications hold.*

$$\begin{aligned}
\forall\, c \in I^0 \cap D(f)\,\big(f'(c) \geq 0\big) \quad &\Rightarrow \quad \big(x,\, y \in I \wedge x < y \Rightarrow f(x) \leq f(y)\big)\,,\\
\forall\, c \in I^0 \cap D(f)\,\big(f'(c) \leq 0\big) \quad &\Rightarrow \quad \big(x,\, y \in I \wedge x < y \Rightarrow f(x) \geq f(y)\big),\\
\forall\, c \in I^0 \cap D(f)\,\big(f'(c) > 0\big) \quad &\Rightarrow \quad \big(x,\, y \in I \wedge x < y \Rightarrow f(x) < f(y)\big) \text{ and}\\
\forall\, c \in I^0 \cap D(f)\,\big(f'(c) < 0\big) \quad &\Rightarrow \quad \big(x,\, y \in I \wedge x < y \Rightarrow f(x) > f(y)\big)\,.
\end{aligned}$$

In words, if f' is nonnegative on $I^0 \cap D(f)$ then f weakly increases on I, and similarly in the other three cases.

Proof. We only prove the last part, other proofs are similar. Let $f'(c) < 0$ for every $c \in I^0$ and $x < y$ be in I. By Theorem 8.1.4 there is a $z \in (x, y)$ $(\subset I^0)$ such that $\frac{f(y)-f(x)}{y-x} = f'(z) < 0$. From $y - x > 0$ we get $f(x) > f(y)$. Hence f decreases on I. □

In Proposition 8.5.5 it is shown that on I^0 the infinite values $f'(c) = \pm\infty$ cannot have different signs than the finite values $f'(c) \in \mathbb{R}$.

Proposition 8.5.2 (monotonicity 2) *Suppose that $f \in \mathcal{F}(M)$ and that the respective one-sided derivative of f at b in $\mathbb{R}^*$ exists. Then the following implications hold.*

$$\begin{aligned} f'_-(b) < 0 \ &\Rightarrow\ \exists\,\delta\left(f[P^-(b,\,\delta)] > \{f(b)\}\right), \\ f'_-(b) > 0 \ &\Rightarrow\ \exists\,\delta\left(f[P^-(b,\,\delta)] < \{f(b)\}\right), \\ f'_+(b) < 0 \ &\Rightarrow\ \exists\,\delta\left(f[P^+(b,\,\delta)] < \{f(b)\}\right) \ \text{and} \\ f'_+(b) > 0 \ &\Rightarrow\ \exists\,\delta\left(f[P^+(b,\,\delta)] > \{f(b)\}\right). \end{aligned}$$

Moreover, each of the four sets $f[\cdots]$ is nonempty. In words, if $f'_-(b)$ is negative (possibly $-\infty$) then there is a δ such that for every $x \in M \cap P^-(b,\delta)$ the value $f(x)$ (and there is one) is smaller than $f(b)$, and similarly in the other three cases.

Proof. We only prove the last part, other proofs are similar. So suppose that $f \in \mathcal{F}(M)$, $b \in M \cap L^+(M)$ and that $f'_+(b)$ ($\in \mathbb{R}^*$) exists and is positive. Then there is a δ such that

$$x \in P^+(b,\,\delta) \cap M \Rightarrow \tfrac{f(x)-f(b)}{x-b} > 0, \text{ hence } f(x) > f(b)\,.$$

The set $f[P^+(b,\delta)]$ is nonempty because $b \in M$ is a right limit point of M. □

Exercise 8.5.3 *Can we say more strongly that each of the four sets $f[\cdots]$ is infinite?*

Exercise 8.5.4 *Prove the following proposition.*

Proposition 8.5.5 *If $a < b$ are in $\mathbb{R}$, $f \in \mathcal{C}([a,b])$, for every $c \in (a,b)$ there exists $f'(c)$ in $\mathbb{R}^*$ and every finite derivative $f'(c) \geq 0$, then every infinite derivative $f'(c) = +\infty$. The same holds for $\geq$ replaced with $\leq$ and $+\infty$ with $-\infty$.*

Exercise 8.5.6 (anti-monotonicity 1) *Describe a function $f \in \mathcal{F}(\mathbb{Q} \cap [0,1])$ such that $D(f) = M(f)$ and $f' = 1$ on $D(f)$, but f does not increase on $M(f)$.*

• *Limit extensions of derivatives.* Again, $a < b$ are real numbers. In the following proposition we have the same assumptions as in Theorem 8.1.4.

Proposition 8.5.7 (extending the derivative) *Suppose that $f \in \mathcal{C}([a,b])$ and that for every $c \in (a,b)$ the derivative $f'(c)$ in $\mathbb{R}^*$ exists. Then a and b are in $L(D(f))$,*

$$\lim_{x \to a} f'(x) = f'(a) \ \text{and}\ \lim_{x \to b} f'(x) = f'(b)\,,$$

if these limits exist.

Proof. By Theorem 8.1.4 for every $c \in (a,b)$ we have $(a,c) \cap D(f) \neq \emptyset$ and $(c,b) \cap D(f) \neq \emptyset$, thus $a, b \in L(D(f))$. Let $\lim_{x \to b} f'(x) = K$ and an ε be given. Then there is a δ such that $f'[(b-\delta, b)] \subset U(K, \varepsilon)$. By Theorem 8.1.4 for every $x \in (b-\delta, b)$ there is a $y \in (x, b)$ such that $\frac{f(x)-f(b)}{x-b} = f'(y) \in U(K, \varepsilon)$. Hence $f'(b) = K$. Similarly for $f'(a)$. □

Exercise 8.5.8 *Show that under the assumptions of the previous proposition it is possible that the derivative $f'(a)$ exists but the limit $\lim_{x \to a} f'(x)$ does not exist.*

- *Two l'Hospital rules.* These concern limits $\lim_{x \to A} \frac{f(x)}{g(x)}$ of types $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$. The idea is to compute the limit by means of the algebraic transformation

$$\frac{f(x)}{g(x)} = \frac{f(x)/x}{g(x)/x} .$$

We begin with a simple version for any definition domain.

Theorem 8.5.9 (LHP 1) *Let $b \in L(M(f/g)) \cap M(f) \cap M(g)$ and $f(b) = g(b) = 0$. Then the equality*

$$\lim_{x \to b} \frac{f(x)}{g(x)} = \frac{f'(b)}{g'(b)} \quad (\in \mathbb{R}^*)$$

holds, if the right-hand side is defined.

Proof. We assume that the derivatives $f'(b)$ and $g'(b)$ $(\in \mathbb{R}^*)$ exist and that $\frac{f'(b)}{g'(b)}$ is not an indefinite expression. Then by Theorem 5.3.3 we have

$$\lim_{x \to b} \frac{f(x)}{g(x)} = \lim_{x \to b} \frac{\frac{f(x)-f(b)}{x-b}}{\frac{g(x)-g(b)}{x-b}} = \frac{\lim_{x \to b} \frac{f(x)-f(b)}{x-b}}{\lim_{x \to b} \frac{g(x)-g(b)}{x-b}} = \frac{f'(b)}{g'(b)} .$$ □

For example, $\lim_{x \to 0} \frac{\sin x}{x} = \frac{\cos 0}{k_1(0)} = \frac{1}{1} = 1$ and $\lim_{x \to 0} \frac{\exp x - 1}{x} = \frac{\exp 0}{k_1(0)} = \frac{1}{1} = 1$.

Theorem 8.5.10 (LHP 2) *Suppose that $b < c$ are in $\mathbb{R}$, $f, g \in \mathcal{F}((b,c))$, that for every $x \in (b,c)$ the derivatives $f'(x) \in \mathbb{R}$ and $g'(x) \in \mathbb{R} \setminus \{0\}$ exist and that*
1. $\lim_{x \to b} f(x) = \lim_{x \to b} g(x) = 0$ *or*
2. $\lim_{x \to b} g(x) = \pm\infty$. *Then the equality*

$$\lim_{x \to b} \frac{f(x)}{g(x)} = \lim_{x \to b} \frac{f'(x)}{g'(x)} \quad (\in \mathbb{R}^*) ,$$

holds, if the right-hand side is defined.

Proof. 1. Suppose that $\lim_{x \to b} \frac{f'(x)}{g'(x)} = K$ and that an ε is given. With $f(b) = g(b) \equiv 0$ we have $f, g \in \mathcal{C}([b,c))$. Theorem 8.1.1 implies that $g \neq 0$ on (b,c). There is a δ such that $(\frac{f'}{g'})[(b, b+\delta)] \subset U(K, \varepsilon)$. By Cauchy's Theorem 8.1.7

for every $x \in (b, b+\delta)$ there is a $y \in (b, x)$ such that $\frac{f(x)}{g(x)} = \frac{f(x)-f(b)}{g(x)-g(b)} = \frac{f'(y)}{g'(y)}$ is in $U(K, \varepsilon)$. Hence $\lim_{x \to b} \frac{f(x)}{g(x)} = K$.

2. We prove this part later by integrals. □

For instance, by part 2 for every $\varepsilon > 0$ the limit $\lim_{x \to 0} x^\varepsilon \log x$ equals

$$\lim_{x \to 0} \tfrac{\log x}{x^{-\varepsilon}} = \lim_{x \to 0} \tfrac{1/x}{-\varepsilon x^{-\varepsilon-1}} = \lim_{x \to 0} \tfrac{1}{-\varepsilon x^{-\varepsilon}} = \tfrac{1}{-\infty} = 0\,.$$

Marquis *Guillaume de l'Hospital (1661–1704)* published in 1696 the historically first textbook of differential calculus *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes.*

Exercise 8.5.11 *Show that the theorem holds also for the definition domains* (c, b)*,* $P(b, \delta)$ *and* $U(\pm\infty, \delta)$*.*

• *A counter-example to* LHP 2. In the spirit of Exercise 8.5.6 we show that the assumption in LHP 2 that $M(f) = M(g)$ is an interval is substantial. For any interval $I \subset \mathbb{R}$ we set $I_{\mathbb{Q}} \equiv I \cap \mathbb{Q}$.

Theorem 8.5.12 (a counter-example to LHP 2) *There exist functions* f *and* g *in* $\mathcal{F}((0,1)_{\mathbb{Q}})$ *with* $D(f) = D(g) = (0,1)_{\mathbb{Q}}$*,* $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0$ *and such that*

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 1, \ \ \textit{but} \ \lim_{x \to 0} \frac{f(x)}{g(x)} = 0\,.$$

Proof. Let $(c_n) \subset (0, 1)$ be any sequence of irrational numbers such that $c_0 \equiv 1 > c_1 > c_2 > \cdots > 0$, $\lim c_n = 0$ and $\lim \frac{c_{n-1}}{c_n} = 1$. For $n \in \mathbb{N}$ and $\alpha \in (c_n, c_{n-1})_{\mathbb{Q}}$ we define $f(\alpha) \equiv c_n^2 + \alpha - c_n$ and $g(\alpha) \equiv \alpha$. Thus G_f consists of short pierced segments beginning on the parabola $y = x^2$ and each with slope 1, and $g(x) = \mathrm{id}(x) \,|\, (0,1)_{\mathbb{Q}}$. It is clear that $f' = g' = k_1 \,|\, (0,1)_{\mathbb{Q}}$ (Exercise 8.5.13) and therefore $\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{1}{1} = 1$. For $\alpha \in (c_n, c_{n-1})_{\mathbb{Q}}$ we have

$$0 < \tfrac{f(\alpha)}{g(\alpha)} \le \tfrac{c_n^2 + c_{n-1} - c_n}{c_n} = c_n + \tfrac{c_{n-1}}{c_n} - 1 \to 0 \ \ (n \to \infty)\,.$$

Therefore $\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$. □

Exercise 8.5.13 *Why are the derivatives of* f *and* g *constantly* 1*?*

8.6 Applications of second-order derivatives

We begin with the definition of higher-order derivatives. Then we turn to applications of second-order derivatives.

• *Derivatives of order* $k \in \mathbb{N}_0$*.* We simply iterate the derivative operation in Definition 7.1.20. Recall that $\mathcal{R} = \{f : M \to \mathbb{R} : \ M \subset \mathbb{R}\}$.

Definition 8.6.1 ($f^{(k)}(x)$) *Let $k \in \mathbb{N}$. We define a unary operation $f^{(k)}$ on $\mathcal{R}$,*

$$\mathcal{R} \ni f \mapsto f^{(k)} \equiv (\ldots((f')')' \ldots)' \in \mathcal{R}\,,$$

by applying the derivative k times. We call it the derivative of order k. We set $f^{(0)} \equiv f$ and write f' for $f^{(1)}$, f'' for $f^{(2)}$ and f''' for $f^{(3)}$.

For example, $(x \sin x)'' = (\sin x + x \cos x)' = 2\cos x - x \sin x$. We know that the notation $f'(b)$, $b \in \mathbb{R}$, is a little ambiguous because it may refer to $\pm\infty$. Similarly for $k \in \mathbb{N}_0$ and $b \in \mathbb{R}$, the value $f^{(k+1)}(b)$ is, if defined, in $\mathbb{R}$, but $(f^{(k)})'(b)$ is, if defined, in $\mathbb{R}^*$. If $b \in M(f^{(k+1)})$ then $f^{(k+1)}(b) = (f^{(k)})'(b)$.

Exercise 8.6.2 *Determine the two sequences in $\mathcal{R}$ that are given as $\big((\sin x)^{(n)}\big)$ and $\big((\frac{1}{x})^{(n)}\big)$, $n \in \mathbb{N}_0$.*

• *Second-order derivatives and extremes.* We give the well-known criterion for the type of the local extreme of f at b in terms of the sign of $(f')'(b)$.

Proposition 8.6.3 (**$(f')'$ and local extremes**) *Let $f \in \mathcal{R}$. Suppose that*

$$U(b,\,\delta) \subset D(f),\ f'(b) = 0 \ \textit{and that}\ (f')'(b)\ \textit{in}\ \mathbb{R}^*\ \textit{exists}\,.$$

Then for $(f')'(b) < 0$ (respectively $(f')'(b) > 0$) the function f has at b a strict local maximum (respectively minimum).

Proof. Let f, b and δ be as stated and let $(f')'(b) > 0$ (the case with $(f')'(b) < 0$ is similar). By parts 2 and 4 of Proposition 8.5.2 there is a $\theta < \delta$ such that for every $x \in P^-(b,\theta)$ (respectively $x \in P^+(b,\theta)$) we have $f'(x) < 0 = f'(b)$ (respectively $f'(x) > 0 = f'(b)$). By Theorem 8.5.1 the function f decreases on $[b-\theta, b]$ and increases on $[b, b+\theta]$. Hence f has at b a strict local minimum. □

Exercise 8.6.4 *Show that under the assumptions of the previous proposition and with $f''(b) = 0$, it is possible that f does not have at b a local extreme.*

• *Convexity and concavity of (graphs of) functions.* Visually speaking, convex graphs are bulging downward and concave graphs upward.

Definition 8.6.5 (**convex and concave**) *A function $f \in \mathcal{F}(M)$ is convex (respectively concave) iff for all $a < b < a'$ in M we have $f(b) \le sb + c$ (respectively $f(b) \ge sb + c$), where $y = sx + c$ is the secant $\kappa(a, f(a), a', f(a'))$ of G_f. If these inequalities hold as strict, we speak of strict convexity, respectively strict concavity, of f.*

So f is strictly convex iff the middle point $(b, f(b))$ of G_f always lies below the secant that passes through the extreme points $(a, f(a))$ and $(a', f(a'))$ of G_f. For convex f, the point $(b, f(b))$ may lie also on the secant. Similarly for concavity.

Exercise 8.6.6 *Function $f(x) = x^2$ is strictly convex. Function $f(x) = |x|$ is convex, but not strictly convex. Function $f(x) = \log x$ is strictly concave.*

Exercise 8.6.7 *(Strict) convexity and (strict) concavity are preserved under restrictions of functions.*

Exercise 8.6.8 *A function f is (strictly) convex $\iff$ $-f$ is (strictly) concave.*

Convexity and concavity force one-sided derivatives in existence. We say that a set $M \subset \mathbb{R}$ is <u>end-free</u> if it has neither minimum nor maximum.

Theorem 8.6.9 (existence of $f'_\pm$) *Let $f \in \mathcal{F}(M)$ be convex, respectively concave, on a nonempty and end-free set $M \subset \mathbb{R}$. <u>Then</u>, with equal signs, for every $b \in M \cap L^\pm(M)$ there exists finite one-sided derivative $f'_\pm(b) \in \mathbb{R}$, and both functions f'_- and f'_+ are weakly increasing, respectively weakly decreasing.*

Proof. We prove that convex f has at every point $b \in M \cap L^-(M)$ finite left-sided derivative $f'_-(b) \in \mathbb{R}$, and that the function f'_- weakly increases; the other three cases are treated similarly. By part 1 of Theorem 5.3.1, the limit

$$\lim_{x \to b^-} \tfrac{f(x)-f(b)}{x-b} \equiv f'_-(b)$$

exists and is finite: for any $a \in M$ with $a < b$, the function $g(x)$, defined by

$$[a, b) \cap M \ni x \mapsto \tfrac{f(x)-f(b)}{x-b} \,,$$

weakly increases and $g(x) \le \frac{f(b')-f(b)}{b'-b}$ for every $x \in [a, b) \cap M$ and any fixed $b' \in M$ with $b' > b$ (such b' exists because M has no maximum). These two properties of $g(x)$ easily follow from the convexity of f if we recall that $\frac{f(x)-f(b)}{x-b}$ is the slope of the secant $\kappa(x, f(x), b, f(b))$ of G_f, and similarly for $\frac{f(b')-f(b)}{b'-b}$. As for the monotonicity of f'_-, for every $b < b'$ in $M \cap L^-(M)$ we get the inequality $f'_-(b) \le f'_-(b')$ again from the convexity of f: for every $x, y \in M$ with $x < b < y < b'$ we have two inequalities between slopes

$$\tfrac{f(x)-f(b)}{x-b} \le \tfrac{f(y)-f(b)}{y-b} \le \tfrac{f(y)-f(b')}{y-b'} \,,$$

hence $\frac{f(x)-f(b)}{x-b} \le \frac{f(y)-f(b')}{y-b'}$ and this inequality is preserved in the limit transitions $x \to b^-$ and $y \to (b')^-$. $\Box$

But for concave (convex) f the ordinary derivative $f'(b)$ may not always exist because we may have $f'_-(b) \ne f'_+(b)$, for example for $|x|$ at 0.

Corollary 8.6.10 (implied continuity) *Let $M \subset \mathbb{R}$ be a nonempty end-free set and $f \in \mathcal{F}(M)$ be convex or concave. <u>Then</u> $f \in \mathcal{C}(M)$.*

Proof. We show that f is right-continuous at every $b \in M$; left-continuity is proven in a very similar way. Then by Exercise 5.2.11, f is continuous at b. If $b \in M$ but is not the right limit point of M, the function f is right-continuous at b trivially. If $b \in M \cap L^+(M)$ then by the previous theorem there exists $f'_+(b) \in \mathbb{R}$. By Exercise 7.1.18, then f is right-continuous at b. $\square$

Exercise 8.6.11 *Prove the next proposition.*

Proposition 8.6.12 (at endpoints) *Let $M \subset \mathbb{R}$, $b = \max(M)$, $b \in L^-(M)$ and let $f \in \mathcal{F}(M)$ be convex or concave. Then there exist the one-sided, possibly infinite, derivative $f'_-(b)$. The same holds if we replace max with min and the sign $-$ with $+$.*

Exercise 8.6.13 *Is it true that if $I \subset \mathbb{R}$ is a nontrivial interval and $f \in \mathcal{F}(I)$ is convex or concave, then f is continuous?*

• *Convexity, concavity and $(f')'$.* Convex and concave parts of the graph of a function can be determined by means of its second-order derivative.

Theorem 8.6.14 (f'' vs. convex, concave) *Suppose that $f \in \mathcal{C}(I)$, where I is a nontrivial interval, $D(f) \supset I^0$ and that for every $c \in I^0$ there exists $(f')'(c)$ in $\mathbb{R}^*$. Then the following implications hold.*

$$\begin{aligned}
\forall\, c \in I^0 \cap M(f'')\,\big(f''(c) \ge 0\big) \quad &\Rightarrow \quad f \text{ is convex},\\
\forall\, c \in I^0 \cap M(f'')\,\big(f''(c) > 0\big) \quad &\Rightarrow \quad f \text{ is strictly convex},\\
\forall\, c \in I^0 \cap M(f'')\,\big(f''(c) \le 0\big) \quad &\Rightarrow \quad f \text{ is concave and}\\
\forall\, c \in I^0 \cap M(f'')\,\big(f''(c) < 0\big) \quad &\Rightarrow \quad f \text{ is strictly concave}.
\end{aligned}$$

We prove this theorem with the help of the next lemma whose proof we leave to the reader.

Exercise 8.6.15 *Prove the next lemma.*

Lemma 8.6.16 (on slopes) *Let (a, a'), (b, b') and (c, c') be in $\mathbb{R}^2$, $a < b < c$ and $\frac{b'-a'}{b-a} \le \frac{c'-b'}{c-b}$. Then the point (b, b') lies below or on the line $\kappa(a, a', c, c')$. Three analogous claims hold when the $\le$ is replaced with any of the three inequalities $\{<, \ge, >\}$.*

Proof of Theorem 8.6.14. Let f and I be as stated and let $f'' \ge 0$ on $I^0 \cap M(f'')$, the other three cases can be treated similarly. Let $a < b < c$ be in I. By Theorem 8.1.4 there exist numbers $y \in (a, b)$ and $z \in (b, c)$ such that

$$s = \tfrac{f(b)-f(a)}{b-a} = f'(y) \text{ and } t = \tfrac{f(c)-f(b)}{c-b} = f'(z)\,.$$

By Theorem 8.5.1 f' weakly increases on I^0 as $(f')' \ge 0$. From $y < z$ it follows that $s = f'(y) \le f'(z) = t$. By Lemma 8.6.16 the point $(b, f(b))$ lies below or on the line $\kappa(a, f(a), c, f(c))$. Hence f is convex by Definition 8.6.5. $\square$

• *Inflection points.* In these points the graph of a function crosses the tangent.

Definition 8.6.17 (inflection points) *Let $f \in \mathcal{F}(M)$, $b \in M \cap L^{\mathrm{TS}}(M)$, $\ell(x)$ be the tangent line to G_f at $(b, f(b))$ and $z \in \{-1, 1\}$. If there is a δ such that*

$$x \in P^-(b,\,\delta) \cap M \Rightarrow zf(x) \le z\ell(x) \ \text{ and } \ x \in P^+(b,\,\delta) \cap M \Rightarrow zf(x) \ge z\ell(x)\,,$$

we call $(b, f(b))$ an <u>*inflection point*</u> *of G_f. If the inequalities hold as strict, we call it a* <u>*strict inflection point*</u> *of G_f.*

Compare inflection points with cutting and non-cutting tangents of Section 8.1.

Exercise 8.6.18 *The origin $(0, 0)$ is a strict inflection point of the graph of $f(x) = x^3$.*

Exercise 8.6.19 *Which points of the graph of the constant function $k_1(x)$ are its inflection points?*

Theorem 8.6.20 (no inflection) *Suppose that $f \in \mathcal{R}$, $D(f) \supset U(b, \delta)$ and that $(f')'(b) \in \mathbb{R}^* \setminus \{0\}$.* <u>*Then*</u> *$(b, f(b))$ is not an inflection point of G_f.*

Proof. Let $(f')'(b) > 0$, the case with $(f')'(b) < 0$ is similar. Let ℓ be the tangent to G_f at $(b, f(b)$. By Proposition 8.5.2 there is a $\theta \le \delta$ such that for every $x \in P^-(b, \theta)$ and every $x' \in P^+(b, \theta)$ we have

$$f'(x) < f'(b) < f'(x')\,. \tag{1}$$

Let $x \in P^-(b, \theta)$, $x' \in P^+(b, \theta)$ and let s and t be the respective slopes of the secants

$$\kappa(x,\, f(x),\, b,\, f(b)) \ \text{ and } \ \kappa(b,\, f(b),\, x',\, f(x'))$$

of G_f. Inequalities (1) and <u>Theorem 8.1.4</u> give that $s < f'(b) < t$. Thus both points $(x, f(x))$ and $(x', f(x'))$ lie above ℓ. The condition in Definition 8.6.17 is not satisfied. □

Next we obtain a sufficient condition for the existence of an inflection point.

Theorem 8.6.21 ($\exists$ inflection) *Let $f \in \mathcal{R}$, $M(f'') \supset U(b, \delta)$ and $z \in \{-1, 1\}$. If $zf''[P^-(b, \delta)] \ge \{0\}$ and $zf''[P^+(b, \delta)] \le \{0\}$* <u>*then*</u> *$(b, f(b))$ is an inflection point of G_f. If these inequalities hold strictly* <u>*then*</u> *$(b, f(b))$ is a strict inflection point of G_f.*

Proof. Let f, b, δ and z be as stated. We assume that for every $x \in P^-(b, \delta)$ and $x' \in P^+(b, \delta)$ we have $f''(x) \le 0$ and $f''(x') \ge 0$, the other three cases are treated similarly. We have $M(f') \supset U(b, \delta)$ and denote the tangent to G_f at $(b, f(b))$ by ℓ. By Theorem 8.5.1 the derivative f' weakly decreases on $[b - \delta, b]$ and weakly increases on $[b, b + \delta]$, for every $x \in [b - \delta, b)$ and $x' \in (b, b + \delta]$ we have $f'(x) \ge f'(b) \le f'(x')$. <u>Theorem 8.1.4</u> implies that

$$\tfrac{f(b)-f(x)}{b-x} \ge f'(b) \le \tfrac{f(x')-f(b)}{x'-b}\,.$$

Thus the slopes of $\kappa(x, b, f(x), f(b))$ and $\kappa(b, x', f(b), f(x'))$ are at least the slope $f'(b)$ of ℓ. So $(x, f(x))$ lies below or on ℓ, and $(x', f(x'))$ above or on ℓ. Hence $(b, f(b))$ is an inflection point. $\square$

8.7 Drawing the graph of a function

We describe twelve steps for determining the main geometric features of the graph of an (elementary) function. But first we define asymptotes.

• *Asymptotes.* The graph of a function gets arbitrarily close to these lines.

Definition 8.7.1 (vertical asymptotes) *If for $f \in \mathcal{F}(M)$ and $b \in L^-(M)$ we have $\lim_{x\to b^-} f(x) = \pm\infty$, we call the line $x = b$ a left vertical asymptote (of f). Right vertical asymptotes are obtained by replacing the two signs $^-$ by two signs $^+$.*

Exercise 8.7.2 *The axis y is both a left and right vertical asymptote of $f(x) = \frac{1}{x}$. It is a right vertical asymptote of $f(x) = \log x$.*

Definition 8.7.3 (asymptotes at infinity) *Let $s, b \in \mathbb{R}$, $f \in \mathcal{F}(M)$ and $\pm\infty \in L(M)$. If*

$$\lim_{x\to\pm\infty}(f(x) - sx - b) = 0\,,$$

we call the line $y = sx + b$ $(\in \mathcal{N})$ an asymptote (of f) at $\pm\infty$ (equal signs).

Exercise 8.7.4 *The line $y = sx + b$ is an asymptote of a function f at $\pm\infty$ $\iff$ $\lim_{x\to\pm\infty} \frac{f(x)}{x} = s$ and $\lim_{x\to\pm\infty}(f(x) - sx) = b$ (equal signs).*

Exercise 8.7.5 *Find the asymptote of $f(x) = \frac{1}{x}$ at $+\infty$ and at $-\infty$.*

Definition 8.7.1 and Exercise 8.7.4 imply that asymptotes are unique.

• *Geometry of graphs of elementary and other functions.* Recall (from Definition 4.4.14) that an elementary function can be obtained from the constant functions $k_c(x) = c$ with $c \in \mathbb{R}$ and the functions $\exp x$, $\log x$, $\sin x$, $\arcsin x$ and x^b with $b \in (0, +\infty) \setminus \mathbb{N}$ by repeated (binary) addition, multiplication, division, and composition. Let $f \in \mathrm{EF}$, but the steps below can be applied to any $f \in \mathcal{R}$. We determine the following main geometric features of G_f.

0. *Elementary?* We begin with determining if $f \in \mathrm{EF}$. If it is the case, we determine if $f \in \mathrm{SEF}$. Memberships of f in these sets of functions have bearing on $M(f)$, the continuity of f and on $D(f)$.

1. *Definition domain.* We find $M(f)$ $(\subset \mathbb{R})$. If $f \in \mathrm{EF}$, we start from $M(\mathrm{e}^x) = M(\sin x) = M(k_c) = \mathbb{R}$, $M(x^b) = [0, +\infty)$, $M(\log x) = (0, +\infty)$ and $M(\arcsin x) = [-1, 1]$, and then apply the relations $M(f + g) = M(fg) =$

$M(f) \cap M(g)$, $M(f/g) = M(f) \cap M(g) \setminus Z(g)$ and $M(f(g)) = \{x \in M(g) : g(x) \in M(f)\}$.

2. *Is f of a special type?* Even ($f(-x) = f(x)$), odd ($f(-x) = -f(x)$), c-periodic ($f(c+x) = f(x)$), ...?

Exercise 8.7.6 *Precisely define these types of functions.*

3. *Derivatives and continuity.* We determine the derivative f' of f and find the sets $\{a \in M(f) :\ \exists\, f'(a) \in \mathbb{R}^*\}$ and $\{a \in M(f) :\ f$ is continuous at $a\}$. Recall that $\mathrm{EF} \subset \mathcal{C}$ by Theorem 6.6.16 and that $D(f) = M(f)$ if $f \in \mathrm{SEF}$ by Theorem 7.6.3.

4. *One-sided limits.* We find one-sided limits of f at points of discontinuity and at the elements of $L(M(f)) \setminus M(f)$. For example, $\lim_{x\to-\infty} \exp x = 0$ and $\lim_{x\to+\infty} \exp x = +\infty$.

5. *Intersections with coordinate axes and the image.* We determine the set $\{x \in M(f) :\ f(x) = 0\} = Z(f)$ and the value $f(0)$, and find $f[M(f)]$ ($\subset \mathbb{R}$).

6. *One-sided derivatives.* For points $a \in M(f)$ where $f'(a)$ does not exist we compute one-sided derivatives $f'_-(a)$ and $f'_+(a)$. Proposition 8.5.7 can help. For instance, it implies that $(|x|)'_-(0) = -1$ and $(|x|)'_+(0) = 1$, but these values are easily computed directly.

7. *Monotonicity and extremes.* We find inclusion-wise maximal subsets of $M(f)$ where f is monotone. If these are intervals, we can usually use Theorem 8.5.1. We find local and global extremes of f. Now Theorems 6.4.1 and 7.1.8, and Proposition 8.6.3 are relevant.

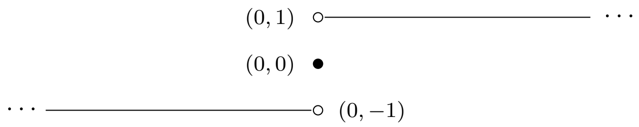
8. *Convexity and concavity.* We find inclusion-wise maximal subsets of $M(f)$ where f is convex or concave. If these are intervals, we can usually use Theorem 8.6.14.

9. *Inflections.* We find points of inflection in G_f. Now Theorems 8.6.20 and 8.6.21 are relevant.

10. *Asymptotes.* We find asymptotes of f. Now Definitions 8.7.1 and 8.7.3, and Exercise 8.7.4 are relevant.

11. *Sketching the graph.* By hand, the computer or the Internet we sketch G_f.

• *Example* 1. *Let $f(x) \equiv \operatorname{sgn} x$.* Recall that signum has value -1 for $x < 0$, 1 for $x > 0$ and $\operatorname{sgn} 0 = 0$. **0.** $\operatorname{sgn} x \notin \mathrm{EF}$. **1.** $M(\operatorname{sgn} x) = \mathbb{R}$. **2.** Signum is an odd function. **3.** Signum is continuous at every $x \neq 0$ and $\operatorname{sgn}'(x) = 0$ for every $x \neq 0$. At zero signum is discontinuous and $\operatorname{sgn}'(0) = +\infty$. **4.** We have the one-sided limits $\lim_{x\to0^-} \operatorname{sgn} x = -1$ and $\lim_{x\to0^+} \operatorname{sgn} x = 1$. Also, $\lim_{x\to-\infty} \operatorname{sgn} x = -1$ and $\lim_{x\to+\infty} \operatorname{sgn} x = 1$. **5.** Signum intersects both coordinate axes at the origin $(0, 0)$ and has image $\{-1, 0, 1\}$. **6.** Since $\operatorname{sgn}'(x)$ exists for every $x \in \mathbb{R}$, there is nothing to compute; $\operatorname{sgn}'_-(x) = \operatorname{sgn}'_+(x) = \operatorname{sgn}'(x)$. **7.** We see directly from the definition of signum that it weakly increases on $\mathbb{R}$, that x is its global minimum iff $x < 0$, and that x is its global maximum iff $x > 0$. It has no strict extremes. **8.** The maximal interval of convexity of $\operatorname{sgn} x$ is $(-\infty, 0]$, and $[0, +\infty)$ is the maximal interval of concavity. **9.** Signum has no strict inflection point but has inflection at every point $(x, \operatorname{sgn} x)$ with $x \neq 0$. At $(0, 0)$ signum does not have tangent. **10.** Signum has no vertical asymptotes. The axis x, that is the line $y = 0$, is the asymptote of $\operatorname{sgn} x$ at both $-\infty$ and $+\infty$. **11.**



• *Example* 2. *Let* $f(x) \equiv \tan x = \frac{\sin x}{\cos x} = \frac{\sin x}{\sin(x+\pi/2)}$. **0.** $\tan x \in \mathrm{SEF}$. **1.** $M(\tan x) = \bigcup_{n\in\mathbb{Z}}(\pi n - \frac{\pi}{2}, \pi n + \frac{\pi}{2}) = \mathbb{R} \setminus \{n\pi + \frac{\pi}{2} :\ n \in \mathbb{Z}\}$. **2.** Tangent is π-periodic because $\sin(\pi + x) = -\sin x$ and $\cos(\pi + x) = -\cos x$. It is an odd function because sine is odd and cosine is even. **3.** Tangent is continuous and $D(\tan x) = M(\tan x)$ by Theorems 6.6.16 and 7.6.3. We have $(\tan x)' = \frac{1}{\cos^2 x}$. **4.** For $n \in \mathbb{Z}$ let $b_n \equiv \pi n + \frac{\pi}{2}$. Then $\lim_{x\to b_n^-} \tan x = +\infty$ and $\lim_{x\to b_n^+} \tan x = -\infty$. The limits of $\tan x$ at $\pm\infty$ do not exist. **5.** G_f intersects the axis y at $(0, 0)$, and the axis x at the points $(b_n - \frac{\pi}{2}, 0) = (\pi n, 0)$, $n \in \mathbb{Z}$. Theorem 6.3.1 and the above infinite limits show that the image $\tan[M(\tan)] = \tan[\,(b_n - \pi,\, b_n)\,] = \mathbb{R}$. **6.** $D(\tan x) = M(\tan x)$, there is nothing to compute. **7.** Since $(\tan x)' = \frac{1}{\cos^2 x} > 0$ on the definition domain, tangent increases on every interval $(\pi n - \frac{\pi}{2}, \pi n + \frac{\pi}{2})$. Thus tangent has no extremes. **8.** We have $(\tan x)'' = \frac{2\sin x}{\cos^3 x}$, with $M((\tan x)'') = M(\tan x)$ by Theorem 7.6.3. Since $(\tan x)'' < 0$ on $(\pi n - \frac{\pi}{2}, \pi n)$ and $(\tan x)'' > 0$ on $(\pi n, \pi n + \frac{\pi}{2})$, tangent is strictly concave on $(\pi n - \frac{\pi}{2}, \pi n]$, and strictly convex on $[\pi n, \pi n + \frac{\pi}{2})$. **9.** Due to the above sign of $\tan'' x$ the inflection points are exactly $(b_n - \frac{\pi}{2}, 0) = (\pi n, 0)$, $n \in \mathbb{Z}$, and are strict. **10.** The limits in step 4 show that every line $x = b_n = \pi n + \frac{\pi}{2}$, $n \in \mathbb{Z}$, is both right and left vertical asymptote of $\tan x$. At $\pm\infty$ there is no asymptote. **11.** `https://www.desmos.com/calculator`.

• *Example* 3. *Let* $f(x) \equiv \arcsin\big(\frac{2x}{1+x^2}\big)$. We follow the lecture notes [8, pp. 193–194]. **0.** $f(x) \in \mathrm{EF} \setminus \mathrm{SEF}$. **1.** $M(f) = \mathbb{R}$ because $M(\arcsin x) = [-1, 1]$ and $2|x| \le 1 + x^2$ for every $x \in \mathbb{R}$ as $x^2 \pm 2x + 1 = (x \pm 1)^2 \ge 0$. **2.** The function $f(x)$ is odd because the functions $\sin x$, $\arcsin x$, and $\frac{2x}{1+x^2}$ are odd. It is not periodic. **3.** The function $f(x)$ is continuous by Theorem 6.6.16. The formulas for derivatives of arkus sine, of composite functions and of ratios give that on the set

$$D(f) = \{x \in \mathbb{R} :\ \tfrac{2x}{1+x^2} \neq \pm 1\} = \mathbb{R} \setminus \{-1, 1\} = M(f) \setminus \{-1,\, 1\}$$

we have

$$f'(x) = \frac{1}{\sqrt{1-(2x/(1+x^2))^2}} \cdot \frac{2\cdot(1+x^2)-2x\cdot 2x}{(1+x^2)^2} = 2 \cdot \frac{(1-x^2)/(1+x^2)^2}{|(1-x^2)/(1+x^2)|} = 2 \cdot \frac{1-x^2}{|1-x^2|} \cdot \frac{1}{1+x^2}\,,$$

so that $f'(x) = \frac{2\cdot\operatorname{sgn}(1-x^2)}{1+x^2}$. In step 6 we will see that neither $f'(-1)$ nor $f'(1)$ exist. **4.** Clearly, $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = \arcsin 0 = 0$ because

$\frac{2x}{1+x^2} \to 0$ for $x \to \pm\infty$. **5.** G_f intersects both axes exactly at the origin $(0,0)$. Shortly we will see that $f[M(f)] = f[\mathbb{R}] = [-\frac{\pi}{2}, \frac{\pi}{2}]$. **6.** It is clear that $\lim_{x\to 1^\pm} f'(x) = \mp 1$. So Proposition 8.5.7 gives that $f'_\pm(1) = \mp 1$. Since $f(x)$ is odd, $f'_\pm(-1) = \pm 1$. **7.** Since $f' < 0$ on $(-\infty, -1)$, $f' > 0$ on $(-1, 1)$ and $f' < 0$ on $(1, +\infty)$, Theorem 8.5.1 implies that f decreases on $(-\infty, -1]$, increases on $[-1, 1]$ and decreases on $[1, +\infty)$. Also $f(x) < 0$ for $x < 0$ and $f(x) > 0$ for $x > 0$ (and $f(0) = 0$). Considering these (maximal) intervals of monotonicity, the above zero limits and the fact that f is odd we see that f has at $x = -1$ the strict global minimum with $f(-1) = -\frac{\pi}{2}$, at $x = 1$ the strict global maximum with $f(1) = \frac{\pi}{2}$ and that there are no other local extremes. Hence, using Theorem 6.3.1, we get the above image $f[M(f)]$. **8.** We have

$$f''(x) = \tfrac{-4x\cdot\mathrm{sgn}(1-x^2)}{(1+x^2)^2}\,,$$

with $M(f'') = \mathbb{R} \setminus \{-1, 1\}$. Since $f'' < 0$ on $(-\infty, -1)$, $f'' > 0$ on $(-1, 0)$, $f'' < 0$ on $(0, 1)$ and $f'' > 0$ on $(1, +\infty)$, Theorem 8.6.14 implies that f is strictly concave on $(-\infty, -1]$, strictly convex on $[-1, 0]$, strictly concave on $[0, 1]$ and strictly convex on $[1, +\infty)$. These intervals of convexity and concavity are clearly maximal. **9.** By the sign of f'' and since the second derivatives $f''(\pm 1)$ do not exist, by Theorems 8.6.20 and 8.6.21 the point $(0, 0)$ is the only inflection point of G_f (at $(-1, F(-1))$ and $(1, F(1))$ tangents do not exist). **10.** By the limits in step 4 the line $y = 0 = 0x + 0$ is an asymptote of $f(x)$ both at $-\infty$ and $+\infty$. There are no vertical asymptotes. **11.** `https://www.desmos.com/calculator`.

Exercise 8.7.7 *Draw the graph of the Riemann function $r(x)$. Recall that $r(x) \in \mathcal{F}(\mathbb{R})$, $r(\alpha) = 0$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $r(\frac{p}{q}) = \frac{1}{q}$ for $\frac{p}{q} \in \mathbb{Q}$ in lowest terms (then $q \in \mathbb{N}$).*

Exercise 8.7.8 *Draw the graph of the function $f(x) \equiv x^x$ $(= \mathrm{e}^{x \log x})$.*

Chapter 9

Taylor expansions. Primitives

9.1 Taylor polynomials

Appendix A

Solutions to exercises

Question marks ??? mean that no solution is known to the author.

1 Paradoxes. Real numbers

Exercise 1.1.1 Now the sequence of partial sums is $(s_n) = (a_1, b_1, c_1, a_2, b_2, c_2, \dots)$ where, with $d_n \equiv \sum_{i=1}^{n} \frac{1}{2i(2i-1)}$, one has that $a_n \equiv d_{n-1} + \frac{1}{2n-1}$, $b_n \equiv d_{n-1} + \frac{1}{2n-1} + \frac{1}{2n}$ and $c_n \equiv d_n$. It is clear that $\lim s_n$ is a positive real number or $+\infty$.

Exercise 1.1.2 Let $(a_{i,j})_{i,j=1}^{\infty}$ have entries $a_{i,j} \ge 0$. We show that the total sum by rows $R = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j}$ equals to the supremum

$$A = \sup\left(\{\textstyle\sum_{(i,j)\in I} a_{i,j} \mid I \subset \mathbb{N}^2 \text{ is finite}\}\right) ,$$

taken in $(\mathbb{R}^*, <)$. The same equality holds for the total sum by columns. Clearly, $A \le R$. Let $c < R$ be arbitrary. Then there is an m such that $c < \sum_{i=1}^{m} \sum_{j=1}^{\infty} a_{i,j}$. Thus there exist $n_1, \dots, n_m$ in $\mathbb{N}$ such that $c < \sum_{i=1}^{m} \sum_{j=1}^{n_i} a_{i,j}$. Hence $R \le A$ and $R = A$.

Exercise 1.2.1 alpha, beta, capital gamma, gamma, capital delta, delta, epsilon, zeta, eta, capital theta, theta, vartheta, iota, kappa, capital lambda, lambda, mu, nu, capital xi, xi, omicron, capital pi, pi, rho, capital sigma, sigma, tau, capital upsilon, upsilon, capital phi, phi, varphi, chi, capital psi, psi, capital omega, omega. Capitals not listed are identical with the Latin alphabet, for example A for α, H for η etc.

Exercise 1.2.2 The order is the standard one of the Latin alphabet: a, b, c, d, ..., x, y and z.

Exercise 1.2.3 The left side is F iff (φ is T and ψ is F). Which is the same as that $\neg\psi$ is T and $\neg\varphi$ is F which is the same as that the right side is F.

Exercise 1.2.4 If there is no element a in the domain of $\varphi(x)$ such that $\varphi(a)$ holds, it means that for every a in this domain the proposition $\varphi(a)$ does not hold and the proposition $\neg\varphi(a)$ holds. The other proposition holds for a similar reason.

Exercise 1.2.6 Using the set-theoretic axiom of foundation we forbid such sets, because no physical collection is a member of itself, but allowing their existence does not lead to any contradiction. Set theories allowing such sets are considered and investigated.

Exercise 1.2.8 The answer depends on equalities between elements of this set. If they are all distinct then the set has five distinct elements. If $a = b = 2 = \{\emptyset, \{\emptyset\}\}$ then, since usually $a \neq \{a\}$ by the axiom of foundation, the set has only two distinct elements.

Exercise 1.2.10 $x_n = \emptyset$.

Exercise 1.2.12 $\mathbb{P} \equiv \{n \in \mathbb{N} \colon\ n > 1 \wedge \forall\, l, m \in \mathbb{N}\,\big(lm = n \Rightarrow (l = 1 \vee m = 1)\big)\}$.

Exercise 1.2.13 If $M \in M$ then $M \notin M$. If $M \notin M$ then $M \in M$.

Exercise 1.2.15 $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi) \iff (\varphi \iff \psi)$ is a tautology.

Exercise 1.2.16 It follows from the axiom of extensionality. A and B are disjoint by the definition iff they have no common element which means iff their intersection is $\emptyset$.

Exercise 1.2.17 $\bigcap \emptyset$ is the proper class of all sets.

Exercise 1.2.18 No, for example if A and B are disjoint then $|A \setminus B| = |A|$.

Exercise 1.2.19 $\mathcal{P}(\emptyset) = \{\emptyset\}$ and $|\mathcal{P}(\{1, 2, \dots, n\})| = 2^n$, $n = 0$ included.

Exercise 1.2.20 We prove only the first formula, the proof for the second one is similar. An element x is in the left side iff $x \in A$ but $x \notin C$ for every $C \in B$ which holds iff for every $C \in B$ it holds that $x \in A \setminus C$ which holds iff x is in the right side.

Exercise 1.2.22 In general the set (X, Y) is for $X = Y$ the one-element set $\{\{X\}\} = \{\{Y\}\}$ and for $X \neq Y$ it is the two-element set with the elements $\{X\}$ (one-element set) and $\{X, Y\}$ (two-element set). From this we get the stated equivalence by repeated use of the axiom of extensionality.

Exercise 1.2.24 It follows from the axiom of extensionality, from the previous exercise and from the fact that any ordered k-tuple A has $|A| = k$ elements.

Exercise 1.3.4 Implication $\Leftarrow$ is trivial. Suppose that (A, B, f) and (C, D, g) are congruent, so that $f = g$ and f is a functional relation between A and B, as well as between C and D. For any $a \in A$ there is a $b \in B$ such that $(a, b) \in f$. But also $(a, b) \in C \times D$, so that $a \in C$. Hence $A \subset C$. In the same way $C \subset A$. Hence $A = C$.

Exercise 1.3.5 Neither equality in general holds.

Exercise 1.3.6 iff $X = Y$.

Exercise 1.3.7 For injectivity, constantness and identicalness it is true, for surjectivity and bijectivness not.

Exercise 1.3.8 It is not. Both symbols may appear at the same time only when f is injective, but then their meanings agree.

Exercise 1.3.9 Since $f^{-1} \colon f[X] \to X$ is bijective, we have $(f^{-1})^{-1} \colon X \to f[X]$. Hence for $f[X] \neq Y$ the functions $(f^{-1})^{-1}$ and f differ as sets. They are always congruent. But $((f^{-1})^{-1})^{-1} \colon f[X] \to X$. Hence $((f^{-1})^{-1})^{-1}$ and f^{-1} are even equal as sets.

Exercise 1.3.10 They are.

Exercise 1.3.11 Let f and g be injective and $f(g)(x) = f(g)(y)$. Then $f(g(x)) = f(g(y))$ and $g(x) = g(y)$. Then $x = y$ and $f(g)$ is injective. Let $g \colon X \to Y$ and $f \colon Y \to B$ be onto and $b \in B$. Since f is onto, there is a $y \in Y$ such that $f(y) = b$. Since g

is onto, there is an $x \in X$ such that $g(x) = y$. Thus $f(g)(x) = f(g(x)) = f(y) = b$ and $f(g)$ is onto. If $g\colon X \to Y$ and $f\colon A \to B$ are nonempty and surjective, and for example $Y \cap A = \emptyset$ then $f(g)\colon \emptyset \to B$ is not onto.

Exercise 1.3.12 Both the range of $f(g(h))$ on the left side and the range of $f(g)(h)$ on the right side equals to the range R of f. It suffices to show that $f(g(h)) = f(g)(h)$ as sets. This is true, both sets are equal to the set of pairs $(x, y) \in M(h) \times R$ such that there exist $a \in M(g)$ & $b \in M(f)$ such that $h(x) = a$, $g(a) = b$ and $f(b) = y$.

Exercise 1.3.13 We set $Y \equiv h[X]$, $(X, Y, g) \equiv (X, Y, h)$ and $f \equiv \mathrm{id}_Y$.

Exercise 1.3.14 If f is a bijection then $g \equiv f^{-1}$ has the required properties. Let $g\colon Y \to X$ be as stated. Since $g(f)(x) = x$, the function f is injective. Since $M(f(g)) = Y$, the function f is onto.

Exercise 1.3.15 Exactly when the definition domain is an empty or one-element set.

Exercise 1.3.16 $\bigcup_{i\in\mathbb{N}} A_i$ and $\bigcap_{i\in\mathbb{N}_0} A_i$.

Exercise 1.3.19 Let R be an equivalence relation on $A \neq \emptyset$ (for $A = \emptyset$ everything trivially holds) and $[a]_R$ be an equivalence block. Clearly, $a \in [a]_R$ so that the elements in A/R are nonempty and $\bigcup A/R = A$. Let $a, b \in A$, $[a]_R \cap [b]_R \neq \emptyset$ and $c \in [a]_R$. Hence there is a $d \in [a]_R \cap [b]_R$. From cRa, aRd and dRb we get by transitivity of R that cRb and therefore $c \in [b]_R$. Hence $[a]_R \subset [b]_R$. The opposite inclusion is proven in the same way and $[a]_R = [b]_R$. Thus the elements of A/R are pairwise disjoint.

If $b, c \in [a]_R$ then bRa and cRa, so that (since R is an equivalence relation) also bRc. If bRc, $b \in [a]_R$ and $c \in [a']_R$ then bRa, cRa', so that aRa'. Hence $[a]_R = [a']_R$ and b, c lie in a common block.

Exercise 1.3.20 Let X be a partition of $Y \neq \emptyset$ and $R \equiv Y/X$. For $y \in Y$ we take a block $Z \in X$ with $y \in Z$. So yRy and R is reflexive. For $y, y' \in Y$ with yRy' there is a block $Z \in X$ such that $y, y' \in Z$. So also $y'Ry$ and R is symmetric. Let $y, y', y'' \in Y$ with yRy' a $y'Ry''$. Thus there exist blocks $Z, Z' \in X$ such that $y, y' \in Z$ and $y', y'' \in Z'$. But then $Z \cap Z' \neq \emptyset$ and therefore $Z = Z'$. Hence yRy'' and R is transitive.

It follows from the definition that $x, y \in Z \in X$ iff $x(Y/X)y$.

Exercise 1.3.21 Let R, $A \neq \emptyset$ and B be as stated. We prove the first equality. We know that $C \equiv A/R$ is a partition of A, thus $S \equiv A/C$ is an equivalence relation on A. We show that $S = R$. Let $a, b \in A$. Then aSb iff there is a block $D \in C$ such that $a, b \in D$. As we know from Exercise 1.3.19, a, b lie in a common block of C iff aRb. Hence $S = R$.

We prove the second equality. We know that $S \equiv A/B$ is an equivalence relation on A. Thus $C \equiv A/S$ is a partition of A. We show that $C = B$. Again, $a, b \in A$ lie in a common block of C iff aSc. This is the case iff a, b lie in a common block of B. Hence $C = B$.

Exercise 1.4.2 Always $a \leq a$ because $a = a$. The transitivity of $\leq$ follows from the transitivity of $<$ and the same is true for trichotomy.

Exercise 1.4.3 Neither $a < b \wedge a = b$ nor $b < a \wedge a = b$ holds because $<$ is irreflexive. Nor $a < b \wedge b < a$ holds because the transitivity of $<$ produces a contradiction with irreflexivity.

Exercise 1.4.4. When m and n are maxima of B, then both $n \leq m$ and $m \leq n$. Hence $m = n$. For minima the same argument works.

Exercise 1.4.5 Let $(A, <)$ be a LO and $B = \{b_1, b_2, \ldots, b_n\} \subset A$, $|B| = n$, be a nonempty finite set. For $i = 1, 2, \ldots, n$ we compare by the trichotomy of $<$ the element b_i with the previous elements $b_1, \ldots, b_{i-1}$. This yields a permutation π of $[n]$ such that $b_{\pi(1)} < b_{\pi(2)} < \cdots < b_{\pi(n)}$. Then $b_{\pi(1)} = \min(B)$ and $b_{\pi(n)} = \max(B)$.

Exercise 1.4.7 This follows from uniqueness of maxima and minima.

Exercise 1.4.8 Let $c = \sup(B)$. Then c is an upper bound of B. Let $a < c$. Since c is the minimum upper bound of B, the element a is not an upper bound of B and there exists the stated b. In the other way, let c have the stated properties. They say that c is the smallest upper bound of B, so that $c = \sup(B)$.

Similarly one proves the equivalence that $c \in A$ is an infimum of B iff $c \le b$ for every $b \in B$ & for every $a \in A$ with $c < a$ there is a $b \in B$ such that $b < a$.

Exercise 1.5.1 Reflexivity and symmetry of $\sim$ are clear. We prove transitivity. Let $a/b \sim c/d$ and $c/d \sim e/f$. Thus $ad = bc$ and $cf = de$. But then $adf = bcf = bde$ and $adf = bde$. The $d \ne 0$ can be canceled ($\mathbb{Z}$ is an integral domain) and $af = be$. Hence $a/b \sim e/f$.

Exercise 1.5.3 Let $\alpha \in \mathbb{Q}$. We get the protofraction $p_\alpha \in \alpha$ by bringing any protofraction $\frac{m}{n} \in \alpha$ to lowest terms. The function $\alpha \mapsto p_\alpha$ is the desired bijection. It is bijective and unique because two different elements in Z_{z} are $\not\sim$. We prove it. If $\frac{k}{l}, \frac{m}{n} \in Z_{\mathrm{z}}$ are in lowest terms and $\frac{k}{l} \sim \frac{m}{n}$ then $kn = ml$. Thus any prime power dividing k divides m and vice versa. By the Fundamental Theorem of Arithmetic it holds that $k = m$. Hence also $l = n$.

Exercise 1.5.5 If 0_X and $0'_X$ are additively neutral elements then commutativity of addition yields that $0_X = 0_X + 0'_X = 0'_X + 0_X = 0'_X$. Similarly for multiplicatively neutral elements. If $\alpha + \beta = 0_X$ and $\alpha + \gamma = 0_X$ then due to associativity and commutativity of addition we have that $\gamma = (\alpha + \beta) + \gamma = (\alpha + \gamma) + \beta = \beta$. Similarly for multiplicative inverses.

Exercise 1.5.6 We define $T \equiv \langle \{0, 1\}, 0_T, 1_T, +, \cdot \rangle$ by $0_T \equiv 0$, $1_T \equiv 1$ and setting the operations $+$ and $\cdot$ to be the addition and multiplication in $\mathbb{Z}$ modulo 2. This is a field, by a theorem in number theory that more generally addition and multiplication in $\mathbb{Z}$ modulo a prime produce a field. This two-element field is unique, up to an isomorphism, which can be seen as follows. Clearly, $0 + 0 = 0 + 0_T = 0$ and $1 + 0 = 0 + 1 = 1$. The additive inverse to 1 must be 1 and so $1 + 1 = 0$. For multiplication the values $1 \cdot 0 = 0 \cdot 1 = 0$ and $1 \cdot 1 = 1$ are clear. By distributivity, $0 \cdot 0 = 0 \cdot (1 + 1) = 0 \cdot 1 + 0 \cdot 1 = 0 + 0 = 0$.

Since we require that in any field F we have $0_F \ne 1_F$, one-element field does not exist. But if you like mathematical mysticism, see [16].

Exercise 1.5.8 If $0_T < 1_T$ then using repeatedly the axiom of shift we get arbitrarily many elements: $0_T < 1_T < 1_T + 1_T < 1_T + 1_T + 1_T < \ldots$. If $1_T < 0_T$ we similarly get arbitrarily many elements $\cdots < 1_T + 1_T + 1_T < 1_T + 1_T < 1_T < 0_T$. In fact, always the former case occurs.

Exercise 1.5.10 $0 \cdot 1 = 0 \ne 1 = 1 \cdot 1$.

Exercise 1.5.11 For any protofraction $\frac{a}{b}$ it holds that $\frac{a}{b} \sim \frac{-a}{-b}$.

Exercise 1.5.13 Let $\alpha = [a/b]_\sim \in \mathbb{Q}$ be arbitrary. Then $\alpha \le [(|a| + 1)/1]_\sim$. Thus $\alpha \le m \equiv [(|a| + 1)/1]_\sim$ and m can be also viewed as an element of $\mathbb{N}$. Since $m = 1_\mathbb{Q} + 1_\mathbb{Q} + \cdots + 1_\mathbb{Q}$ with m summands, $\mathbb{Q}_{\mathrm{OF}}$ is Archimedean.

Exercise 1.5.15 In any nonempty LO $(A, <)$, $H(\emptyset) = A$. Thus $\sup(\emptyset) = \min(A)$, if this minimum exists. If $B \subset A$ is not bounded from above then $H(B) = \emptyset$ and $\min(H(B))$ is not defined. So neither is $\sup(B)$.

Exercise 1.5.16 If $F_{\mathrm{OF}} = \langle F, \dots \rangle$ is a complete ordered field and $A \subset F$ is nonempty and bounded from below then $\inf(A) = -\sup(-A)$ where $-A = \{-x:\ x \in A\}$.

Exercise 1.5.17 Let $T_{\mathrm{OF}} = \langle T, \dots \rangle$ be a complete ordered field. We set N ($\subset T$) to be the set of all finite sums of 1_Ts. Then $1_T \in N$ and we show that N is not bounded from above. If it were, we could take $a \equiv \sup(N)$. But then there would be a $b \in N$ such that $a - 1_T < b$ (it can be proven that $a - 1_T < a$). Adding 1_T we get by the axiom of shift that $a < b + 1_T$. But $b + 1_T \in N$ and we have a contradiction. Thus N is not bounded from above and this produces the stated upper bounds.

Exercise 1.5.20 If $\frac{a}{b}$ with coprime $a, b \in \mathbb{Z}$ and $b \neq 0$ were a solution, we would get the equality $a^n + a_{n-1}a^{n-1}b + \cdots + a_1ab^{n-1} + a_0b^n = 0$. Thus p divides a^n. It follows that p divides a and that p^2 divides $a_{n-i}a^{n-i}b^i$ for every $i = 0, 1, \dots, n-1$ (where $a_n = 1$). Thus p^2 divides a_0b^n. It follows that p divides b, which contradicts the coprimality of a and b.

Exercise 1.5.22 For fractions $s, r > 0$ with $s^2 > 2$ the inequality $(s-r)^2 > 2$ holds if $s^2 - 2 > 2sr - r^2$. Thus, for example, for every positive $r < \frac{s^2-2}{2s}$. For fractions r, s with $s^2 < 2$, $s > 0$ and $r \in (0, 1)$ (then $r^2 < r$) the inequality $(s+r)^2 < 2$ holds if $2sr + r^2 < 2 - s^2$. Thus, for example, for every positive $r < \min(\{1, \frac{2-s^2}{2s+1}\})$.

Exercise 1.6.1 Reflexivity and symmetry of $\sim$ are trivial. Transitivity easily follows from the triangle inequality.

Exercise 1.6.2 It suffices to show that if sequences (a_n) and (b_n) in C are congruent by the definition, then they satisfy the formally stronger condition. Suppose that for every large n it holds that $|a_n - b_n| \le \frac{1}{2k}$. Since (a_n) is Cauchy, for every large m and n it holds that $|a_m - a_n| \le \frac{1}{2k}$. By the triangle inequality for every large m and n we have that $|a_m - b_n| \le |a_m - a_n| + |a_n - b_n| \le \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}$.

Exercise 1.6.10 It is easy to check that for every two fractions a and b it holds that $f(a-b) = f(a) - f(b)$, $f(a \cdot b) = f(a) \cdot f(b)$ and that $a < b$ iff $f(a) < f(b)$.

Exercise 1.6.11 For $z = 0$ the inequality $x < y$ turns in the equality $xz = 0 = yz$.

Exercise 1.6.13 If $a_n, b \in \mathbb{Q}$, $n \in \mathbb{N}$, are such that $a_1 \le a_2 \le \cdots \le b$ then (a_n) is Cauchy. The proof is very similar to that of the theorem.

Exercise 1.6.15 Let $\alpha \in \mathbb{R}$ be arbitrary. Then $\alpha = [a_n]_\sim$ for some $(a_n) \in C$. Since (a_n) is bounded (and $\mathbb{Q}_{\mathrm{OF}}$ is Archimedean), there is an $m \in \mathbb{N}$ such that for every n one has that $a_n \le m$. Thus in $\mathbb{R}_{\mathrm{OF}}$ it holds that $\alpha \le \overline{m}$. But $\overline{m} = 1_\mathbb{R} + 1_\mathbb{R} + \cdots + 1_\mathbb{R}$ with m summands $1_\mathbb{R}$. Hence $\mathbb{R}_{\mathrm{OF}}$ is Archimedean.

Exercise 1.7.2 Suppose that X is a nonempty finite set (for $\emptyset$ it holds trivially with empty map $f = \emptyset$). Using the axiom of choice we chose elements $f(1) \in X$, $f(2) \in X \backslash \{f(1)\}$, $f(3) \in X \backslash \{f(1), f(2)\}$, $\dots$ as long as the remaining set is nonempty. Since X is finite, we cannot obtain an injecion from $\mathbb{N}$ to X. In some step $m \in \mathbb{N}$ we exhaust all elements in X and get a bijection $f\colon [m] \to X$. The values $f(n) \in X$ for $n > m$ are added arbitrarily.

Exercise 1.7.3 We found this bijection in the previous proof.

Exercise 1.7.6 The function $f\colon \mathbb{N} \to \mathbb{Z}$ given by $f(1) = 0,\ f(2) = 1,\ f(3) = -1,\ f(4) = 2,\ f(5) = -2,\ f(6) = 3,\ \dots$ is bijective.

Exercise 1.7.8 $Y = \emptyset$.

Exercise 1.7.9 Suppose that $f\colon \mathcal{P}(X) \to X$ is an injective map. We take the inverse $f^{-1}\colon f[\mathcal{P}(X)] \to \mathcal{P}(X)$ and set $Y \equiv \{x \in f[\mathcal{P}(X)] \mid x \notin f^{-1}(x)\}$ $(\subset X)$. For the element $f(Y) = y$ $(\in X)$ we get from $y \overset{?}{\in} Y$ the familiar contradiction.

Exercise 1.7.12 This is clear.

Exercise 1.7.15 These pairs (apart of the positive and negative zero) are uniquely determined by the digits before the maximal run of nines (or zeros).

Exercise 1.7.18 We redirect the values of the function f lying outside Y in Y, which gives f_0. The bijection g is clear. The function h is surjective by Exercise 1.3.11.

2 Existence of limits

Exercise 2.1.1 There is a positive real number epsilon such that for every positive real number delta there exist in M two real numbers a and b such that a and b are closer than delta, but the functional values $f(a)$ and $f(b)$ have distance at least epsilon.

Exercise 2.1.2 We treat the case $n = 2$, for larger n one uses induction. Let $a, b \in \mathbb{R}$. If they have the same sign or one of them is 0 then $|a + b| = |a| + |b|$. Else $|a + b| \le \max(\{|a|, |b|\}) \le |a| + |b|$.

Exercise 2.1.4 $+\infty$, $-\infty$, $-\infty$ and undefined.

Exercise 2.1.5 We extended the LO $(\mathbb{R}, <)$ by adding infinities so that $-\infty \not< -\infty$ and $+\infty \not< +\infty$, Thus the extended relation $<$ is irreflexive. Transitivity and trichotomy are clear.

Exercise 2.1.7 These sets are $\emptyset$ and $\{-\infty\}$.

Exercise 2.1.11 Neighborhoods of points and infinities are intervals and hence convex sets.

Exercise 2.1.12 For $A, B \in \mathbb{R}$ we may take any $\varepsilon < \frac{B-A}{2}$. If $A = -\infty$ and $B = +\infty$, we may take any ε. If $A = -\infty$ and $B \in \mathbb{R}$, we may take any $\varepsilon < \frac{1}{|B|+1}$.

Exercise 2.1.13 This is immediate from the definition of neighborhoods.

Exercise 2.1.14 This is again immediate from the definition of neighborhoods.

Exercise 2.1.16 $x \le \varepsilon \Rightarrow x < 2\varepsilon$ and $x < \varepsilon \Rightarrow x \le \varepsilon$.

Exercise 2.1.19 Except V_4 all other properties are robust.

Exercise 2.1.20 We only prove 3, the proofs for 1 and 2 are similar. Suppose that X is as stated, that (a_n) and (b_n) differ for only finitely many n and that $(a_n) \in \bigcap X$. Thus for every $Y \in X$ it holds that $(a_n) \in Y$. Since Y is robust, it holds also for (b_n). Hence $(b_n) \in \bigcap X$ and $\bigcap X$ is robust.

Exercise 2.1.21 Clearly, $\frac{\sqrt[3]{n}-\sqrt{n}}{\sqrt[4]{n}} = \frac{n^{-1/6}-1}{n^{-1/4}} \to \frac{0-1}{0^+} = \frac{-1}{0^+} = -\infty$, due to positivity of $n^{-1/4}$.

Exercise 2.2.2 Reflexivity follows from setting $m_n = n$. Transitivity is clear when one views subsequences as obtained by omitting terms in original sequences.

Exercise 2.2.3 For example, $(0, 1, 0, 1, \dots)$ a $(1, 0, 1, 0, \dots)$.

Exercise 2.2.7 Let (b_n) and (a_n) be the stated sequences and let an ε be given. Thus there is an n_0 such that $n \geq n_0 \Rightarrow a_n \in U(L, \varepsilon)$. Then there is an n_1 such that $n \geq n_1 \Rightarrow m_n \geq n_0$. Then for every $n \geq n_1$ we have that $b_n = a_{m_n} \in U(L, \varepsilon)$ and $\lim b_n = L$.

Exercise 2.2.8 It is easy to see that the sequence $(m_n) \subset \mathbb{N}$ witnessing $(b_n) \preceq^* (a_n)$ has an increasing subsequence.

Exercise 2.2.9 The coefficient of the monomial $a^j b^{n-j}$ is the number of ways how to obtain it: we chose j factors $a + b$ in the product $(a + b)^n$ from which we pick the number a, and pick b from the remaining $n - j$ factors. There are $\binom{n}{j}$ ways to do it because there are $\binom{n}{j}$ subsets of $[n]$ with j elements.

Exercise 2.2.11 Negate existence of the limit $n^{1/n} \to 1$.

Exercise 2.3.1 If such c exists then for every n it holds that $-c \leq a_n \leq c$ and (a_n) is bounded both from below and from above. Suppose that (a_n) is bounded by the definition, so that $d \leq a_n \leq c$ for every n and some numbers d and c. Then $|a_n| \leq \max(\{|d|, |c|\})$ for every n.

Exercise 2.3.2 The last three concerning boundedness.

Exercise 2.3.5. For example, suppose that (a_n) weakly increases for every $n \geq m$ and that $b_n = a_n$ for every $n \geq n_0$. Then (b_n) weakly increases for every $n \geq \max(\{m, n_0\})$.

Exercise 2.3.6 For example, if (a_n) weakly decreases then for every n the implication $a_m > a_n \Rightarrow m < n$ holds. Hence (a_n) goes down. Similarly for weakly increasing sequences.

Exercise 2.3.7 Consider, for example, the sequence $(1, 0, 2, 1, 3, 2, 4, 3, 5, \dots)$. It goes up but no tail is monotone.

Exercise 2.3.8 A sequence $(a_n) \subset \mathbb{R}$ is quasi-monotone iff

$$\forall\, l\, \exists\, m\, \big(n > m \Rightarrow a_n \geq a_l\big) \vee \forall\, l\, \exists\, m\, \big(n > m \Rightarrow a_n \leq a_l\big)\,.$$

Exercise 2.3.11 Suppose for example that (a_n) goes up starting from $n = m$ and that $b_n = a_n$ for every $n \geq n_0$. Then (b_n) goes up from $n = \max(\{m, n_0\})$.

Exercise 2.3.13 The generalization says that in any (finite or infinite) LO $(X, <)$ every sequence (a_n) has a monotone subsequence. The same proof works.

Exercise 2.3.16 It is easy to see that the limit c of this subsequence satisfies the inequalities $a \leq c \leq b$.

Exercise 2.3.18 Suppose that (a_n) is Cauchy and (b_n) is such that $b_n = a_n$ for $n \geq n_0$. If for a given ε for every $m, n \geq n_1$ it holds that $|a_m - a_n| \leq \varepsilon$, then for every $m, n \geq \max(\{n_0, n_1\})$ it holds that $|b_m - b_n| \leq \varepsilon$. Hence (b_n) is Cauchy.

Exercise 2.3.19 Let (a_n) be Cauchy. Then there is an n_0 such that for every $m, n \geq n_0$ one has that $|a_m - a_n| \leq 1$. By the Δ-inequality it holds for every n that $|a_n| \leq 1 + \max(\{|a_1|, \dots, |a_{n_0}|\})$. Hence (a_n) is bounded.

Exercise 2.3.21 Take for example the sequence $(1, 1.4, 1.41, 1.414, \dots)$ of truncations of the decimal expansion of the number $\sqrt{2}$.

Exercise 2.3.22 We used in the proof the B.–W. theorem whose proof uses the theorem on limits of monotone sequences. For this theorem we need the existence of suprema, respectively infima, of corresponding sets of real numbers.

Exercise 2.3.23 black

Exercise 2.3.24 Then one would have had to go to a library or to a bookshop to check the corresponding dictionary. Another option was to ask somebody from Slovakia.

Exercise 2.3.26 Let us show that $f(n)$ is superadditive. Let $m, n \in \mathbb{N}$, $a_1 \dots a_{f(m)}$ be a word over $[m]$ satisfying (i) and (ii) and with the maximum length, and $b_1 \dots b_{f(n)}$ be the analogous word over $\{m+1, \dots, m+n\}$. Then the word

$$u = a_1 \dots a_{f(m)} b_1 \dots b_{f(n)}$$

over $[m+n]$ satisfies (i) and (ii), so that $f(m) + f(n) \le f(m+n)$. How do we know that u avoids *abba*? This is due to the fact that the word *abba* cannot be split, it cannot be written as a concatenation of two words over disjoint alphabets.

Exercise 2.3.27 We argue as in the previous exercise. Again, the key point is that *abab* cannot be split.

Exercise 2.3.28 But *aabb* can be split, $aabb = aa\, bb$. Thus the previous argument using Fekete's lemma cannot be used, at least not in the simplest way.

Exercise 2.3.29 We show that for every fixed k the function $r_k(n)$ is subadditive. If $A \subset \mathbb{Z}$ contains no AP of length k we say that A is acceptable. Let $m, n \in \mathbb{N}$ and $A \subset [m+n]$ be an acceptable set with the maximum size $|A| = r_k(m+n)$. We easily see that the sets

$$A' = [m] \cap A \ \text{ and } \ A'' = \{x \in [n] \mid x + m \in A\}$$

are acceptable — acceptability is inherited by subsets and is preserved by shifts. Hence $r_k(m+n) = |A| = |A'| + |A''| \le r_k(m) + r_k(n)$.

3 Arithmetic of limits. AK series

Exercise 3.1.1 Since $|-b| = |b|$, it suffices to prove the first inequality. We apply to $a = (a+b) + (-b)$ the standard Δ-inequality and rearrange the result.

Exercise 3.1.3 1. Let $|a_n| \le d$ for every n, $L = -\infty$ and a $c < 0$ be given. It is clear that for every large n one has that $b_n \le c - d$. Thus for every large n we have that $a_n + b_n \le d + c - d = c$. Hence $a_n + b_n \to -\infty$. The case that $L = +\infty$ is similar.

2. Let $|a_n| \le d$ for every n, $b_n \to 0$ and an ε be given. Clearly we have for every large n that $|b_n| \le \frac{\varepsilon}{d}$. So for every large n it holds that $|a_n b_n| \le d \cdot \frac{\varepsilon}{d} = \varepsilon$. Hence $a_n b_n \to 0$.

3. Let a_n, c, $L = +\infty$ and b_n be as stated and let a $d > 0$ be given. One has for every large n that $b_n \ge \frac{d}{c}$. So for every large n it holds that $a_n b_n \ge c \cdot \frac{d}{c} = d$. Hence $a_n b_n \to +\infty = L$. The other case is similar.

4. Let $|a_n| \le d$ for every n, $b_n \to \pm\infty$ and an ε be given. For every large n we have that $|b_n| \ge \frac{d}{\varepsilon}$. So for every large n we have that $\left|\frac{a_n}{b_n}\right| = |a_n| \cdot \frac{1}{|b_n|} \le d \frac{1}{d/\varepsilon} = \varepsilon$. Hence $\frac{a_n}{b_n} \to 0$.

5. Let a_n, c and b_n be as stated and a $d > 0$ be given. For every large n one has that $0 < b_n \le \frac{c}{d}$. So for every large n we have that $\frac{a_n}{b_n} \ge \frac{c}{c/d} = d$. Hence $\frac{a_n}{b_n} \to +\infty$.

6. Let a_n, c, $L = -\infty$ and b_n be as stated and let a $d < 0$ be given. For every large n one has that $b_n \le dc$. So for every large n we have that $\frac{b_n}{a_n} \le \frac{dc}{c} = d$. Hence $\frac{b_n}{a_n} \to -\infty = L$. The other case is treated similarly.

Exercise 3.1.5 1. For $A \in \mathbb{R}$ we take $a_{n,1} \equiv n$, $b_{n,1} \equiv -n + A$. For $A = +\infty$ we take $a_{n,1} \equiv n$, $b_{n,1} \equiv -\sqrt{n}$. For $A = -\infty$ we take $a_{n,1} \equiv \sqrt{n}$, $b_{n,1} \equiv -n$.

2. For $A \in \mathbb{R} \setminus \{0\}$ we take $a_{n,2} \equiv \frac{\mathrm{sgn}(A)}{n}$, $b_{n,2} \equiv n|A|$. For $A = 0$ we take $a_{n,2} \equiv 0$, $b_{n,2} \equiv n$. For $A = +\infty$ we take $a_{n,2} \equiv \frac{1}{n}$, $b_{n,2} \equiv n^2$. For $A = -\infty$ we take $a_{n,2} \equiv -\frac{1}{n}$, $b_{n,2} \equiv n^2$.

3. For $A \in \mathbb{R}$ we take $a_{n,3} \equiv \frac{A}{n}$, $b_{n,3} \equiv \frac{1}{n}$. For $A = \pm\infty$ we take $a_{n,3} \equiv \frac{1}{n}$, $b_{n,3} \equiv \pm\frac{1}{n^2}$ (equal signs).

4. For $A \in \mathbb{R} \setminus \{0\}$ we take $a_{n,4} \equiv An$, $b_{n,4} \equiv n$. For $A = 0$ we take $a_{n,4} \equiv n$, $b_{n,4} \equiv n^2$. For $A = \pm\infty$ we take $a_{n,4} \equiv \pm n^2$, $b_{n,4} \equiv n$ (equal signs).

Exercise 3.2.1 This inequality is equivalent to the inequality $(\sqrt{a} - \sqrt{b})^2 \ge 0$.

Exercise 3.2.3 This follows from the identity $|a_n - 0| = ||a_n| - 0|$ $(= |a_n|)$.

Exercise 3.3.2 The set $\{((a_n), (b_n)) :\ \exists\, n_0\, \forall\, m, n \ge n_0 \big(a_m < b_n\big)\}$ (of pairs of sequences) is a proper subset of the set $\{((a_n), (b_n)) :\ \exists\, n_0\, \forall\, n \ge n_0 \big(a_n < b_n\big)\}$. In *MA 1*$^+$ we show that the former set is much smaller than the latter.

Exercise 3.3.4. For example $(a_n) \equiv (\frac{1}{n})$ and $(b_n) \equiv (0, 0, \dots)$.

Exercise 3.3.5. Let (a_n), (b_n), K and L be as stated. We take a number c such that $K < c < L$. By Exercise 2.1.11 there is an ε such that $U(K, \varepsilon) < U(c, \varepsilon) < U(L, \varepsilon)$. Then we take any two numbers $a, b \in U(c, \varepsilon)$ such that $a < b$. For every large m and n we have that $a_m \in U(K, \varepsilon)$ and $b_n \in U(L, \varepsilon)$. Hence $a_m \le a$ and $b \le b_n$.

Reversal of this implication is: if for every n_0 and every real numbers $a < b$ there exist m and n with $m, n \ge n_0$ such that $a_m > a$ or $b_n < b$, then $K \ge L$.

Exercise 3.3.8 Any singleton $\{a\}$ is such an interval.

Exercise 3.3.11 Let $\lim a_n = -\infty$, $b_n \le a_n$ for every large n and let a $c < 0$ be given. Then for every large n one has that $b_n \le a_n \le c$. Thus $b_n \le c$ and $\lim b_n = -\infty$. The case of the limit $+\infty$ is similar.

Exercise 3.4.2 By part 1 of Theorem 2.2.5 every sequence has a subsequence that has a limit.

Exercise 3.4.7 For every $n \ge 2$ it is true that $\tau(n) \ge 2$ because 1 and n always divide n. For infinitely many n, namely for the prime numbers, the equality holds.

Exercise 3.4.8 One can reduce parts 3 and 4 to parts 1 and 2 by means of the identity $\liminf a_n = -\limsup(-a_n)$.

Exercise 3.4.9 We let m run in $\mathbb{N}$ and in the m-th step the m-th segment of the sequence (a_n) runs through the numbers $-m$, $-m + \frac{1}{m}$, $-m + \frac{2}{m}$, $\dots$, m.

Exercise 3.4.10 Such sequence does not exist because $L(a_n) \cap \mathbb{R}$ is always a closed set.

Exercise 3.4.11 $L(a_n) = \{0, +\infty\}$.

Exercise 3.5.2 Suppose that the constant c witnesses that $\sum_{x \in X} r_x$ is an AK series and that $Z \subset Y$ is finite. Then $\sum_{x \in Z} |r_x| \le c$ because $Z \subset X$.

Exercise 3.5.7 The sets $Z'_1, \dots, Z'_{n-1}$ and Z''_n are pairwise disjoint and their union is X_0. Thus if x runs bijectively through all elements of X_0, it runs bijectively through all elements of the sets $Z'_1, \dots, Z'_{n-1}, Z''_n$.

Exercise 3.5.8 In the equality $\sum_{x\in X_0} r_x = \sum_{i=1}^{n-1} \sum_{x\in Z'_i} r_x + \sum_{x\in Z''_n} r_x$.

Exercise 3.5.9 Reflexivity is witnessed by the identity bijection, symmetry by the inverse bijection and transitivity by the composition of two bijections.

Exercise 3.5.10 Let $R = \sum_{x\in X} r_x$ and $R' = \sum_{x\in Y} s_x$ be congruent AK series. If X and Y are finite, the equality of their sums is trivial. Suppose that they are infinite and that $f\colon X \to Y$ is a bijection proving that $R \sim R'$. Let $g\colon \mathbb{N} \to X$ be any bijection. Then $S(R') = \lim \sum_{i=1}^{n} s_{f(g)(i)} = \lim \sum_{i=1}^{n} r_{g(i)} = S(R)$ because $f(g)$ is a bijection from $\mathbb{N}$ to Y.

Exercise 3.5.11 Suppose that the constant c witnesses that R is an AK series and that $Y \subset X$ is a finite set. Then $\sum_{x\in Y} |ar_x| \le |a|\cdot c$. Hence aR is an AK series. For finite X the identity for sums is trivial and we may assume that X is infinite. Let $f\colon \mathbb{N} \to X$ be any bijection. Then $S(aR) = \lim \sum_{i=1}^{n} ar_{f(i)} = a \lim \sum_{i=1}^{n} r_{f(i)} = aS(R)$.

Exercise 3.5.13 Let a, $R = \sum_{x\in X} r_x$ and $R' = \sum_{x\in Y} s_x$ be as stated and $f\colon X \to Y$ be a bijection witnessing that $R \sim R'$. Then for every $x \in X$ it holds that $r_x = s_{f(x)}$. Hence also $ar_x = as_{f(x)}$ and $aR \sim aR'$.

Exercise 3.5.15 Let $Q = \sum_{x\in X} r_x$, $Q' = \sum_{x\in X'} r'_x$, $R = \sum_{x\in Y} s_x$ and $R' = \sum_{x\in Y'} s'_x$ be as stated and let $f\colon X \to X'$, $g\colon Y \to Y'$ be bijections witnessing that $Q \sim Q'$ and $R \sim R'$. Thus for every $x \in X$ and $y \in Y$ we have that $r_x = r'_{f(x)}$ and $s_y = s'_{g(y)}$. Let $Z \equiv X \times \{0\} \cup Y \times \{1\}$ and $W \equiv X' \times \{0\} \cup Y' \times \{1\}$. We consider $Q + R = \sum_{z\in Z} t_z$ and $Q' + R' = \sum_{z\in W} t'_z$. We define the bijection $h\colon Z \to W$ by $h(z) \equiv (f(x), 0)$ if $z = (x, 0)$, and by $h(z) \equiv (g(y), 1)$ if $z = (y, 1)$. Then it follows that for every $z \in Z$ we have that $t_z = t'_{h(z)}$. Hence $Q + R \sim Q' + R'$.

Exercise 3.5.17 We proceed as in the previous exercise, with the modifications that $Z \equiv X \times Y$, $W \equiv X' \times Y'$ and that the bijection $h\colon X \times Y \to X' \times Y'$ is given by $h((x, y)) \equiv (f(x), g(y))$. Then for every $(x, y) \in X \times Y$ we have that $r_x s_y = r'_{f(x)} s'_{g(y)}$ because $r_x = r'_{f(x)}$ and $s_y = s'_{g(y)}$. Hence $Q \cdot R \sim Q' \cdot R'$.

Exercise 3.5.19 For $R = \sum_{x\in X} r_x$, $R' = \sum_{y\in Y} s_y$ and $R'' = \sum_{z\in Z} t_z$ the bijection sends $(x, 0)$ to $((x, 0), 0)$, $((y, 0), 1)$ to $((y, 1), 0)$ and $((z, 1), 1)$ to $(z, 1)$.

Exercise 3.5.20 For R and R' as in the previous exercise the bijection sends (x, y) to (y, x).

Exercise 3.5.21 For R, R' and R'' as in Exercise 3.5.19 the bijection sends $(x, (y, z))$ to $((x, y), z)$.

Exercise 3.5.23 In general this identity does not hold.

4 Infinite series. Elementary functions

Exercise 4.1.1 As the next solution shows, if we change in a series $\sum a_n$ finitely many summands and get $\sum a'_n$ then there is an m and a c such that for every $n \ge m$ it holds that $s'_n = s_n + c$. Then $\lim s_n$ exists and is finite iff $\lim s'_n$ exists and is finite.

Exercise 4.1.2 Let $\sum a_n$ and $\sum b_n$ be convergent series and let there be an m such that $b_n = a_n$ for $n \ne m$ and $b_m = a_m + c$ with $c \ne 0$. Let (s_n) and (t_n) be respective partial sums. Then $t_n = s_n$ for $n < m$ and $t_n = s_n + c$ for $n \geq m$, so that $\sum b_n = \lim t_n = c + \lim s_n = c + \sum a_n$.

Exercise 4.1.3 Partial summands weakly increase (resp. decrease) for $n \geq n_0$.

Exercise 4.1.4 Since $s_n = n$, we have that $\lim s_n = +\infty$.

Exercise 4.1.7 The first equality follows from the definition of partial sums, the second one from AL of sequences, the third one from the assumption and from the limit of a subsequence, and the fourth one is trivial.

Exercise 4.1.8 Due to the monotonicity it holds that $\lim a_n = L$. All subsequences have this limit and hence $L = +\infty$.

Exercise 4.1.9 Let (s_n) and (t_n) be partial sums of both series. There is a c such that for every $n \geq n_0$ we have $s_n \geq t_n + c$. Since $\lim(t_n + c) = +\infty$ the one-cop theorem shows that also $\lim s_n = +\infty$.

Exercise 4.1.12 Let $n \geq 2$. We assume that $1 + \frac{1}{2} + \cdots + \frac{1}{n} = m \in \mathbb{N}$ and deduce a contradiction. Following the hint we write every denominator $j = 1, 2, \ldots, n$ in the form $j = a(j) \cdot 2^{b(j)}$ where $a(j) \in \mathbb{N}$ is odd and $b(j) \in \mathbb{N}_0$. For $j_0 = 2^k$, where $k \in \mathbb{N}$ is the largest number with $2^k \leq n$, this expression takes the form $j_0 = 1 \cdot 2^k$. For every $j \in [n] \setminus \{j_0\}$ it holds that $b(j) < k$. Hence $1 + \frac{1}{2} + \cdots + \frac{1}{n} = \frac{a+b}{a \cdot 2^k}$, where $a \equiv a(1)a(2)\ldots a(n) \in \mathbb{N}$ is an odd number and $b \in \mathbb{N}$ is even, be cause it is the sum of $n-1$ even numbers. The numerator $a+b$ is therefore odd and the power $2^k \geq 2$ in the denominator cannot be canceled. Thus the quality $\frac{a+b}{a \cdot 2^k} = m$ is impossible. The same argument shows that for no $n \geq 2$ we have that $h_n = \frac{k}{l}$ with odd l.

Exercise 4.1.13 ???

Exercise 4.1.15 One changes the series $\sum a_n$ to $\sum(-a_n)$.

Exercise 4.1.20 This is a special case of Corollary 3.5.5.

Exercise 4.1.23 This follows from the equality $q^m + q^{m+1} + \cdots = q^m \cdot (1 + q + \cdots)$.

Exercise 4.1.24 Every converging one, so iff $q \in (-1, 1)$.

Exercise 4.1.28 This is immediate from the divergence of the harmonic series.

Exercise 4.1.29 Iff $s > 1$, again by CCC.

Exercise 4.2.1 Let M and A be as stated and let 1 hold, so that $A \in L(M)$ by the given definition. We chose for every n an $a_n \in P(A, \frac{1}{n}) \cap M$ and get a sequence $(a_n) \subset M \setminus \{A\}$ such that $\lim a_n = A$. Hence 2 holds. For every m there is an ε such that $a_1, \ldots, a_m \notin U(A, \varepsilon)$. So we can choose from (a_n) an injective subsequence and 3 holds. Suppose that 3 holds and let $(b_n) \subset M$ be an injective sequence with $\lim b_n = A$. For given n we have $b_m \in U(A, \frac{1}{n})$ for every large m. From these for only one m it holds that $b_m = A$, hence $P(A, \frac{1}{n}) \cap M \neq \emptyset$ and 4 holds. It is clear that $4 \Rightarrow 1$.

Exercise 4.2.3 Suppose that M with $\emptyset \neq M \subset \mathbb{R}$ is finite and let $x \in \mathbb{R}$ be arbitrary. We take an ε that is smaller than the minimum distance between x and an element of $M \setminus \{x\}$ (if $M \setminus \{x\} = \emptyset$, we set $\varepsilon \equiv 1$). Then $P(x, \varepsilon) \cap M = \emptyset$ and x is not a limit point of M. Since M is bounded, neither $-\infty$ nor $+\infty$ is a limit point of M.

Exercise 4.2.4. Let $M \subset \mathbb{R}$ be infinite. If it is not bounded then $-\infty$ or $+\infty$ is in $L(M)$. Suppose that M is bounded. It is infinite and so we chose from M with the help of AC an injective sequence $(a_n) \subset M$. Using Proposition 2.3.12 we chose from (a_n) a monotone subsequence (b_n). Since (b_n) is bounded, we have the finite limit $b \equiv \lim b_n$. Since $(b_n) \subset M$ and $b_n \neq b$ for every n, by Exercise 4.2.1 we see that $b \in L(M)$.

Exercise 4.2.5 This is immediate from part 2 of Proposition 4.2.2.

Exercise 4.2.8. $L(\mathbb{N}) = \{+\infty\}$.

Exercise 4.2.12 For instance $A = 0$, $M = \{\pm\frac{1}{n} \mid n \in \mathbb{N}\}$, $X = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, $f = 0$ on $M \setminus X$ and $f = 1$ on X.

Exercise 4.2.14 We have chosen an element from any set $\{x \in P(K, 1/n) \cap M : f(x) \notin U(L, \varepsilon)\}$, $n \in \mathbb{N}$.

Exercise 4.2.15. 1. Due to the transformation ($x < 0$, hence $x = -|x|$) $\frac{x}{\sqrt{1+x^2-1}} = \frac{1}{-\sqrt{1/x^2+1-1/|x|}}$ we get for $x \to -\infty$ the limit $\frac{1}{-\sqrt{1/(+\infty)+1-0}} = -1$.

2. The transformation $\frac{1}{\sqrt{1+x}-\sqrt{x}} = \sqrt{1+x} + \sqrt{x}$ gives for $x \to +\infty$ the limit $\sqrt{1+(+\infty)} + \sqrt{+\infty} = +\infty$.

3 a 4. These limits are trivial, the first does not exist and the second equals 0.

Exercise 4.3.2 As many as real numbers, $k_c \mapsto c$ is a bijection from the set of constants to $\mathbb{R}$.

Exercise 4.3.3 We take an m such that $m \geq 2|x|$. Then for every $n \geq m$ we have that $|x^n/n!| \leq (|x|^m/m!) \cdot (1/2)^{n-m} = (2|x|)^m/m! \cdot (1/2)^n$. Then we use geometric series.

Exercise 4.3.6 1. $\exp 0 = 1$ is trivial and the rest follows from the exponential identity. 2. For $x < y$ we have that $\mathrm{e}^y - \mathrm{e}^x = \mathrm{e}^x(\mathrm{e}^{y-x} - 1) > 0$, due to the exponential identity. 3. For $x > n$ it holds that $\mathrm{e}^x > n$, so that $\lim_{x\to+\infty} \mathrm{e}^x = +\infty$. Also, $\lim_{x\to-\infty} \mathrm{e}^x = \frac{1}{\lim_{x\to+\infty} \mathrm{e}^x} = \frac{1}{+\infty} = 0$.

Exercise 4.3.8 For contradiction let $\sum_{j\geq 0} \frac{1}{j!} = \frac{n}{m}$ with $n, m \in \mathbb{N}$. Following the hint we get that $r \equiv \sum_{j>m} \frac{m!}{j!} = n \cdot (m-1)! - \cdots \in \mathbb{N}$. This is impossible because $0 < r \leq \frac{1}{m+1} \sum_{j=0}^{\infty} \frac{1}{(m+2)^j} = \frac{m+2}{(m+1)^2} < 1$.

Exercise 4.3.9 1. $\log 1 = 0$ follows from $\exp 0 = 1$. By flipping the graph over the line $y = x$ we get from the increasing function $\exp x$ the increasing function $\log x$. For $x, y > 0$ we have by the exponential identity the equality $\exp(\log x + \log y) = x \cdot y$, so that $\log x + \log y = \log(xy)$. 2. These limits are again obtained by flipping the graph of the exponential over $y = x$. 3. This follows from part 4 of the previous proposition.

Exercise 4.3.13 We begin with a^x, $a > 0$. For $x = 0$ we have that $\exp(x \log a) = \exp 0 = 1$. For $x \in \mathbb{N}$ it holds that $\exp(x \log a) = \exp(\log a + \cdots + \log a)$, with x factors $\log a$. By the exponential identity this equals $\exp(\log a) \cdot \ldots \cdot \exp(\log a) = a \cdot \ldots \cdot a$, with x factors a. For every $x \in \mathbb{N}$ it holds due to the exponential identity that $\exp((-x) \log a) = \frac{1}{\exp(x \log a)}$. Thus a^x agrees with x^m. We continue with x^b. Let $b \in \mathbb{N}$, $x > 0$. Then again $\exp(b \log x) = \exp(\log x + \cdots + \log x) = \exp(\log x) \cdot \ldots \cdot \exp(\log x) = x \cdot \ldots \cdot x$, with b factors x. Also $0^b = 0 = 0 \cdot \ldots \cdot 0$. Let $b = 0$ and $x > 0$. Then $x^b = 1$. Let $b \in \mathbb{Z}$ with $b < 0$ and $x > 0$. Then we again get by the exponential identity that $x^b = \frac{1}{x^{-b}}$. Hence x^b agrees with x^m. Finally, $0^x = 0$ for $x \in \mathbb{N}$ also agrees with x^m.

Exercise 4.3.14 By Definition 4.3.11 one has that $\mathrm{e}^x = \exp(x \log \mathrm{e})$. Since $\mathrm{e} = \exp 1$, it equals to $\exp x$.

Exercise 4.3.16 With $A \equiv 1 + x$, $B \equiv 1 + x + x^2$, $C \equiv 1 + x^3$ and $D \equiv 1 + x^2 + x^4$, for which by the hint $AD = BC \equiv E$, we should show that $(A^y + B^y)^x \cdot (C^x + D^x)^y = (A^x + B^x)^y \cdot (C^y + D^y)^x$. Equivalently, that $E^{xy}(1 + (\frac{B}{A})^y)^x(1 + (\frac{C}{D})^x)^y = E^{xy}(1 + (\frac{A}{B})^x)^y(1 + (\frac{D}{C})^y)^x$. But this holds because $\frac{B}{A} = \frac{D}{C}$ and $\frac{C}{D} = \frac{A}{B}$, and multiplication is commutative.

Exercise 4.3.17 For $A = 0$ we set $a_n \equiv \frac{1}{n^n}$ and $b_n \equiv \frac{1}{n}$. For $0 < A < 1$ we set $a_n \equiv A^n$ and $b_n \equiv \frac{1}{n}$. For $A = 1$ we set $a_n = b_n \equiv \frac{1}{n}$. For $1 < A < +\infty$ we set $a_n \equiv \frac{1}{A^n}$ and $b_n \equiv -\frac{1}{n}$. For $A = +\infty$ we set $a_n \equiv \frac{1}{n^n}$ and $b_n \equiv -\frac{1}{n}$. No, it could not, $a^b < 0$ only for $b \in \mathbb{Z} \setminus \{0\}$.

Exercise 4.3.18 Proceed as in Exercise 4.3.3.

Exercise 4.3.20 1. The runner runs one lap in time 2π and gets in the same position. 2. This is the behavior of the y-coordinate of the runner in the first quarter of the lap. 3. The track is symmetric according to the y-axis, and according to the origin $(0, 0)$. 4. The counter-clockwise rotation of S around the origin by $\frac{\pi}{2}$ is equivalent to the exchange of the coordinate axes. The second relation says that the points on S have distance 1 from the origin. 5. Search for pictures for "geometric proof of summation formulae for sinus and cosinus".

Exercise 4.3.22 Relate the three power series for the exponential, cosine and sine.

Exercise 4.3.23 We know from the properties of cosine and sine what zeros they have.

Exercise 4.3.24 This follows from the properties of sine and cosine and from the fact (proven in lecture 6) that continuous functions attain all intermediate values.

Exercise 4.3.25 The proof is similar to the previous one.

Exercise 4.4.2 This follows from the commutativity, associativity and distributivity of the operations $+$ and $\cdot$ on $\mathbb{R}$ and from the fact that the operation of intersection of two sets enjoys these properties too: $M \cap N = N \cap M$, $(M \cap N) \cap P = M \cap (N \cap P)$ and $M \cap (N \cap P) = (M \cap N) \cap (M \cap P)$. In $\mathbb{R}$ the number 0, respectively 1, is neutral in addition, respectively multiplication, and always $\mathbb{R} \cap M = M$. No function $f \in \mathcal{R}$ with $M(f) \neq \mathbb{R}$ has additive or multiplicative inverse.

Exercise 4.4.4 The functions $f - g$ and $f + f_{-1} \cdot g$ have equal values and also equal definition domains: $M(f) \cap M(g) = M(f) \cap (\mathbb{R} \cap M(g))$.

Exercise 4.4.6 $|x| = (x \cdot x)^{1/2}$.

Exercise 4.4.7 It is the empty function $\emptyset$.

Exercise 4.4.8 Yes, by the previous exercise. But also, for example, $x^{1/2} + (-x - 1)^{1/2} = \emptyset$.

Exercise 4.4.9 $g = k_{-1} \cdot f$.

Exercise 4.4.11 For instance $f(x) \equiv \sqrt{\sin(\pi x)} + \sqrt{-\sin(\pi x)}$ and $g(x) \equiv \frac{1}{\sin(\pi/x)}$; here π is k_π and x is the identity id.

Exercise 4.4.12 For $a > 0$ we have the expression $a^x = \exp(x \log a)$. For $b \in \mathbb{N}$ we have the expression $x^b = x \cdot x \cdot \ldots \cdot x \,|\, [0, +\infty)$ (Proposition 4.4.10). For $b = 0$

we have the expression $x^0 = k_1 \,|\, (0, +\infty)$ (Proposition 4.4.10). For $b \in \mathbb{Z}$ with $b < 0$ and $x > 0$ we have the expression $x^b = \exp(b \log x)$. For 0^x we have the expression $0^x = k_0 \,|\, (0, +\infty)$. The expression of the functions x^m, $m \in \mathbb{Z}$, from other BEF is clear.

Exercise 4.4.16 For instance $\arcsin x$, $|x|$ or $\arcsin(\sin x)$.

Exercise 4.5.3 Suppose that $p \neq k_0$ has the canonical from $\sum_{j=0}^{n} a_j x^j$. We show by induction on $\deg p = n$ that $|Z(p)| \leq n$. For $n = 0$ it holds, then $p = k_{a_0}$ with $a_0 \neq 0$ and p has no zero. Let $n > 0$. If p has no zero, the inequality holds. Let $p(a) = 0$ for some $a \in \mathbb{R}$. Then we divide the polynomial p in its canonical form with remainder by the polynomial $x - a = \mathrm{id}_{\mathbb{R}} - k_a$ and get the expression $p = (x - a)q$, for some polynomial q with degree $n - 1$. For every $b \neq a$ with $p(b) = 0$ we have that $q(b) = 0$. Induction gives that $|Z(p)| = 1 + |Z(p) \setminus \{a\}| \leq 1 + |Z(q)| \leq 1 + (n - 1) = n$.

Exercise 4.5.4 Proposition 4.4.3 shows that it remains to prove the existence of additive inverses and the defining property of integral domains. The inverses exist due to Exercise 4.4.9 and because every polynomial has the definition domain $\mathbb{R}$. By multiplying two polynomials with canonical forms (hence nonzero) we see that the result has a canonical form and thus is nonzero.

Exercise 4.5.9 For $f_1 = f_2 = g_1 \equiv k_1$ $(= 1)$ and $g_2 \equiv \mathrm{id}$ $(= x)$ the function $\frac{f_1/f_2}{g_1/g_2}$ is $\mathrm{id} \,|\, \mathbb{R} \setminus \{0\}$ and the function $\frac{f_1 g_2}{f_2 g_1}$ is id.

Exercise 4.5.12 Reflexivity and symmetry of $\sim$ are clear. We prove transitivity. Let $r \sim s$ and $s \sim t$. We take these rational functions in canonical forms: $r = \frac{a}{b}$, $s = \frac{c}{d}$ and $t = \frac{e}{f}$. By the assumption $r = s$ on $M(r) \cap M(s) = \mathbb{R} \setminus Z(bd)$ and $s = t$ on $\mathbb{R} \setminus Z(df)$. Thus $r = t$ on $\mathbb{R} \setminus (Z(bd) \cup Z(df)) = (\mathbb{R} \setminus Z(bf)) \setminus Z(d) = (M(r) \cap M(t)) \setminus Z(d)$. By the continuity of r and t in every point $x \in M(r) \cap M(t) \cap Z(d)$ the functions r and t are equal also on $M(r) \cap M(t)$. Hence $r \sim t$.

Exercise 4.5.13 Let $r, s, r', s' \in \mathrm{RAC} \setminus \{\emptyset\}$. It is not hard to see, using continuity of rational functions, that $r \sim r'$ and $s \sim s'$ imply that also $r + s \sim r' + s'$ and $r \cdot s \sim r' \cdot s'$. Thus we can add and multiply the whole equivalence blocks. Commutativity and associativity of addition and multiplication and the distributive law in $\mathbb{R}(x)$ follow from the arithmetic in $\mathbb{R}$. For the existence of inverses we need equivalence blocks. The block $[r]_\sim = [p/q]_\sim$ has the additive inverse $[(-p)/q]_\sim$ because the sum is $[k_0/q]_\sim = [k_0]_\sim$. Similarly, $[r]_\sim = [p/q]_\sim \neq [k_0]_\sim$ (so $p \neq k_0$) has the multiplicative inverse $[q/p]_\sim$ because the product is $[pq/qp]_\sim = [k_1]_\sim$.

5 Limits of functions. Asymptotic notation

Exercise 5.1.1 Let $b \in L^-(M)$. For every $n \in \mathbb{N}$ we chose from $P^-(b, 1/n) \cap M$ a point a_n and get the required sequence (a_n). If $b \notin L^-(M)$ then for some ε we have that $P^-(b, \varepsilon) \cap M = \emptyset$ and the required sequence does not exist. For $L^+(M)$ the proof is similar.

Exercise 5.1.2 The first two implications follow from the fact that any one-sided deleted neighborhood is contained in the ordinary one. We prove the third implication. Let b be a limit point of M. Then there is a sequence $(a_n) \subset M \setminus \{b\}$ such that $\lim a_n = b$. It has a subsequence (a_{m_n}) such that for every n it holds that $a_{m_n} > b$ or for every n it holds that $a_{m_n} < b$. Thus b is a one-sided limit point of M. 4. For instance $0 \in L([0, 1))$ but $0 \notin L^-([0, 1))$.

Exercise 5.1.3 For example, it is the set $\mathbb{N} \subset \mathbb{R}$.

Exercise 5.1.4 See the solution of Exercise 4.2.4.

Exercise 5.1.6 1. Here a is a limit point of M. Thus, as we know, it is the left or right limit point of M. 2. This follows from the inclusion $P(a,\delta) \subset P^-(a,\delta') \cup P^+(a,\delta'')$ for $\delta = \min(\{\delta', \delta''\})$. 3. If $\lim_{x\to a} f(x) = A$, it suffices to take ε so small that $U(A,\varepsilon)$ and $U(K,\varepsilon)$, or $U(A,\varepsilon)$ and $U(L,\varepsilon)$, are disjoint and we get a contradiction.

Exercise 5.1.8 The proof is similar to the proof of Proposition 4.2.10.

Exercise 5.1.10 The proof is similar to the proof of Theorem 4.2.13.

Exercise 5.1.12 This follows from the equality $f[P^\pm(b,\delta)] = (f \mid I^\pm(b))[P(b,\delta)]$.

Exercise 5.2.2 Iff there is a sequence $(b_n) \subset M(f)$ such that $\lim b_n = b$, but $\lim f(b_n)$ does not exist or is not equal to $f(b)$. Or, by Exercise 5.2.10, iff there is a sequence $(b_n) \subset M(f)$ with $\lim b_n = b$ such that $\lim f(b_n) = A \neq f(b)$.

Exercise 5.2.4 The solution of the inequalities $|x-b| < \delta$, respectively $|f(x)-f(b)| < \varepsilon$, is exactly $U(b,\delta)$, respectively $U(f(b),\varepsilon)$. Non-strict inequalities are equivalent, it suffices to decrease δ or ε a little.

Exercise 5.2.6 If $b \in M \setminus L(M)$, there is a δ such that $U(b,\delta) \cap M(f) = \{b\}$. The only sequences in $M(f)$ with the limit b are then eventually constant sequences (a_n) with $a_n = b$ for $n \geq n_0$. Then $\lim f(a_n) = f(b)$ which agrees with continuity of f in such point b (trivially or by Proposition 5.2.9).

Exercise 5.2.7 If $b \in M$ is isolated, it is not limit and there is a ε such that $M \cap P(b,\varepsilon) = \emptyset$. Then $M \cap U(b,\varepsilon) = \{b\}$. If $b \in M$ is not isolated, it is limit and for every ε we have that $M \cap P(b,\varepsilon) \neq \emptyset$. Hence for every ε we have that $M \cap U(b,\varepsilon) \neq \{b\}$.

Exercise 5.2.8 This follows from the definition of isolated points.

Exercise 5.2.10 This is immediate from Heine's definition of pointwise continuity and from part 3 of Theorem 2.2.5.

Exercise 5.2.11 Let f be continuous in $b \in M(f)$ and let an ε be given. Then for some δ it holds that $f[U(b,\delta)] \subset U(f(b),\varepsilon)$. Thus $f[U^-(b,\delta)] \subset U(f(b),\varepsilon)$ and $f[U^+(b,\delta)] \subset U(f(b),\varepsilon)$ because $U^-(b,\delta)$ and $U^+(b,\delta)$ is contained in $U(b,\delta)$. So f is both left- and right-continuous in b.

Let f be both left- and right-continuous in $b \in M(f)$ and let an ε be given. Then there exist δ' and δ'' such that $f[U^-(b,\delta')] \subset U(f(b),\varepsilon)$ and $f[U^+(b,\delta')] \subset U(f(b),\varepsilon)$. We set $\delta \equiv \min(\{\delta',\delta''\})$. We get that $f[U(b,\delta)] \subset U(f(b),\varepsilon)$, because $U(b,\delta) \subset U^-(b,\delta') \cup U^+(b,\delta'')$. Hence f is continuous in b.

Exercise 5.2.13 Every number in M is positive because x is irrational. $U(x,1)$, which is an interval of length 2, contains only finitely many fractions with bounded denominators. Hence M is a finite set. But $U(x,1)$ contains at least one integer and so $M \neq \emptyset$.

Exercise 5.3.2 Let $f \in \mathcal{F}(M)$. For $b \in L^-(M)$ and f that weakly decreases on $P^-(b,\delta)$ we replace supremum with infimum. For $b \in L^+(M)$ and f that weakly increases, respectively weakly decreases, on $P^+(b,\delta)$ we take infimum, respectively supremum. For $+\infty \in L(M)$ and f that weakly decreases on $U(+\infty,\delta)$ we replace supremum with infimum. For $-\infty \in L(M)$ and f that weakly increases, respectively weakly decreases, on $U(-\infty,\delta)$ we take infimum, respectively supremum.

Exercise 5.3.4 $M(f/g) = M(f) \cap M(g) \setminus Z(g)$ and there is a δ such that $Z(g) \cap U(A, \delta) = \emptyset$.

Exercise 5.3.5 This is a particular case of part 3 of the theorem with $f = k_1$.

Exercise 5.3.8 1. The previous proof of the theorem is easily modified, for $K < L$ there is an ε and real numbers a, b such that $U(K, \varepsilon) < \{a\} < \{b\} < U(L, \varepsilon)$. 2. Again the reversal of an implication.

Exercise 5.3.10 The ordinary limits are only changed to one-sided. The proofs are basically reductions to ordinary limits by means of Proposition 5.1.13.

Exercise 5.4.3 Let a sequence $(a_n) \subset M(f(g)) \setminus \{A\}$ have $\lim a_n = A$. By Heine's definition of limits of functions (HDLF), $\lim g(a_n) = K$. Suppose that condition 1 holds. Then for the n with $g(a_n) = K$ we have that $f(g)(a_n) = f(g(a_n)) = f(K) = L$. If there are infinitely many n such that $g(a_n) \neq K$ then for the corresponding subsequence (a_{m_n}) we have by HDLF that $\lim f(g(a_{m_n})) = L$. Thus $\lim f(g)(a_n) = L$ and HDLF says that $\lim_{x \to A} f(g)(x) = L$. Suppose that condition 2 holds. Then, deleting from $(g(a_n))$ finitely many terms, we may assume that $(g(a_n)) \subset M(f) \setminus \{K\}$. By HDLF, $\lim f(g(a_n)) = L$. Again by HDLF, $\lim_{x \to A} f(g)(x) = L$. The case that none of the conditions holds is resolved via Heine's definition of limits of functions already in the original proof.

Exercise 5.4.6 We use Theorem 5.4.1 withe outer function $\frac{1}{x}$, inner function g, $A \equiv A$ and $K \equiv B$. Since $\frac{1}{x}$ is continuous, condition 1 is satisfied.

Exercise 5.5.2 Reflexivity and symmetry of $\doteq$ is easy to see. We prove the transitivity. Let $f \doteq g$ and $g \doteq h$. Then $M(f)\Delta M(g)$ and $M(g)\Delta M(h)$ are finite sets. If $x \in M(f)$ and $x \notin M(h)$ then either $x \notin M(g)$ (but $x \in M(f)$) or $x \in M(g)$ (but $x \notin M(h)$). Similarly if $x \in M(h)$ and $x \notin M(f)$. Thus the set $M(f)\Delta M(h)$ is finite because it is a subset of the union of two finite sets. In the similar vein if $x \in M(f) \cap M(h)$ and $f(x) \neq h(x)$ then either $x \notin M(g)$, or $x \in M(g)$ but $f(x) \neq g(x)$ or $g(x) \neq h(x)$. We see that $\{x \in M(f) \cap M(h) : \ f(x) \neq h(x)\}$ is finite.

Exercise 5.5.5 It is not. In [25] one has that $\mathbf{N} = \{0, 1, 2, \dots\}$. If $g(n) \equiv n$ and $f(n) \equiv 0$ then by our definition trivially $g = O(f)$ (on $\mathbf{N}$) because $M(g/f) = \emptyset$. But $g(n) = n \leq cf(n) + c = c$ does not hold for every $n \in \mathbf{N}$ for any constant c.

Exercise 5.5.6 1. Yes. 2. No (problem near 0). 3. No (problem near $\pm\infty$). 4. Yes. 5. No (problem near 0). 6. Yes.

Exercise 5.5.8 If $y = \frac{f_0}{g_0}(x) \in \frac{f_0}{g_0}[N]$ but $y \notin \frac{f}{g}[N]$ then $x \in M(f_0) \setminus M(f)$ or $x \in M(g_0) \setminus M(g)$ or ($x \in M(g_0) \cap M(g)$ but $g_0(x) \neq g(x) = 0$) or ($x \in M(g_0) \cap M(g)$ but $g_0(x) \neq g(x)$) or ($x \in M(f_0) \cap M(f)$ but $f_0(x) \neq f(x)$). The set of corresponding x is a subset of a finite union of finite sets and is therefore finite.

Exercise 5.5.11 1. Yes. 2. Yes. 3. No. 4. No. 5. Yes. 6. Yes.

Exercise 5.5.14 Let f, g, h, N and A be as stated. 1. We assume that, for a constant $c \geq 0$, for every $x \in M(f) \cap M(h) \cap N \setminus Z(h)$ it holds that $|\frac{f}{h}(x)| \leq c$, and the same holds with g in place of f. Hence it holds for every $x \in M(f) \cap M(g) \cap M(h) \cap N \setminus Z(h)$ that $|\frac{f+g}{h}(x)| \leq |\frac{f}{h}(x)| + |\frac{g}{h}(x)| \leq 2c$. 2. For f we have a bound as previously and for every $x \in M(g) \cap N$ it holds that $|g(x)| \leq c$. Hence it holds for every $x \in M(f) \cap M(g) \cap M(h) \cap N \setminus Z(h)$ that $|\frac{fg}{h}(x)| = |\frac{f}{h}(x)| \cdot |g(x)| \leq c^2$. 3. This is similar to 2. 4. We have that $\lim_{x \to A} \frac{f}{h}(x) = 0$ and $\lim_{x \to A} \frac{g}{h}(x) = 0$. Hence the limit

$\lim_{x\to A} \frac{f+g}{h}(x)$, which is defined because A is a limit point of $M((f+g)/h)$, equals $\lim_{x\to A} \frac{f}{h} + \lim_{x\to A} \frac{g}{h} = 0 + 0 = 0$. 5. Similarly, from $\lim_{x\to A} \frac{f}{h}(x) = 0$ and a bound that $|g(x)| \le c$ for every $x \in M(g) \cap P(A,\theta)$ we easily deduce that also $\lim_{x\to A} \frac{fg}{h}(x) = 0$. 6. This is similar to 5. 7. Like in part 4 we get from the assumptions on f, g, h and A that $\lim_{x\to A} \frac{f+g}{h}(x) = \lim_{x\to A} \frac{f}{h}(x) + \lim_{x\to A} \frac{g}{h}(x) = 1 + 0 = 1$. 8. We get from the assumptions on f, g, h and A that $\lim_{x\to A} \frac{fg}{h}(x) = \lim_{x\to A} \frac{f}{h}(x) \cdot \lim_{x\to A} g(x) = 1 \cdot 1 = 1$. 9. This is similar to 8.

Exercise 5.5.16 For $k \in \mathbb{N}$ with $k \le x$ the number of pairs $(m,n) \in \mathbb{N}^2$ with $mn = k$ equals $\tau(k)$.

Exercise 5.5.17 Because $\lim_{x\to 1^+} x^{1/3} \log x = 0$.

Exercise 5.5.18 No problem in our definition of big O, only we get no upper bound on $T_{\mathrm{HH}}(1)$.

Exercise 5.5.21 This is an application of Proposition 5.5.13.

Exercise 5.5.24 Moving the denominator $\exp x - 1$ to the right, we get for $n \in \mathbb{N}_0$ that the coefficient $\sum_{k=0}^{n-1} \frac{B_k}{k!} \cdot \frac{1}{(n-k)!}$ of x^n equals 0 for $n \ne 1$, and 1 for $n = 1$. So for $n = 1$ we get that $B_0 = 1$, and for $n \ge 2$ that $B_{n-1} = -\frac{1}{n} \sum_{k=0}^{n-2} \binom{n}{k} B_k$.

The second claim follows from the identity $f(-x) = f(x)$ for the formal power series $f(x) \equiv \frac{x}{\mathrm{e}^x - 1} + \frac{x}{2}$.

6 Continuous functions

Exercise 6.1.2 Use Proposition 5.2.9, every point of the definition domain is isolated.

Exercise 6.1.3 For every $x \in \mathbb{R}$, δ and ε we have $k_a[U(x,\delta)] = \{a\} \subset U(k_a(x),\varepsilon) = U(a,\varepsilon)$.

Exercise 6.1.4 For every $a \in \mathbb{R}$ and given ε it suffices to set $\delta = \varepsilon$ because $x[U(a,\delta)] = U(a,\delta)$.

Exercise 6.1.6 Let N be dense in M & $a \in M$. Using the axiom of choice we take for every n a point b_n in $N \cap U(a,1/n)$ and get a sequence $(b_n) \subset N$ with $\lim b_n = a$. If $(b_n) \subset N$ has $\lim b_n = a \in M$ then for every δ for every large n it holds that $b_n \in U(a,\delta)$.

Exercise 6.1.7 Let $a < b$ be in $\mathbb{R}$. We take $n \in \mathbb{N}$ so large that $\frac{2\sqrt{2}}{n} < b - a$. It follows that for some $m \in \mathbb{Z}$ we have $\frac{m}{n} \in (a,b)$, and that for some $m \in \mathbb{Z} \setminus \{0\}$ we have $\frac{m\sqrt{2}}{n} \in (a,b)$. Thus every nontrivial interval contains both a fraction and an irrational number. Hence both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 6.1.8 Let $0 \le a < b \le 1$ and $n \in \mathbb{N}$ be maximum such that $\frac{1}{n} \ge \frac{a+b}{2}$. Then the nontrivial interval $(\max(\{a, \frac{1}{n+1}\}), \frac{a+b}{2})$ is contained in (a,b) and is disjoint to N.

Exercise 6.1.11 Let $b \in M$ and an ε be given. We can take a large enough k such that if any $x \in \mathbb{R}$ has the same first k digits as b then $x \in U(b,\varepsilon)$. Let $\alpha \in \mathbb{Q}$ be the finite decimal expansion formed by these first k digits of b. Then $\alpha = a_n$ for some n and certainly $X_n = \{b_n\} \ne \emptyset$. Thus $b_n \in N$ and $b_n \in U(b,\varepsilon)$.

Exercise 6.2.2 The function s is onto $\mathbb{N}$ because every natural number is a product of an odd number and a power of two. These expressions are unique: if $(2k-1)2^{l-1} = (2m-1)2^{n-1}$ then $l = n$ (else 2 would divide an odd number) hence also $k = m$ and s is injective.

Exercise 6.2.4 By Exercise 6.1.2 constants are continuous. Clearly, $k_a = k_b$ implies that $a = b$.

Exercise 6.2.5 An injection from $\mathbb{R}$ to $\mathcal{C}(M)$ is again given by constant functions. We define an injection from $\mathcal{C}(M)$ to $\mathbb{R}$ as for $M = \mathbb{R}$, we only take instead of $\mathbb{Q}$ an at most countable set $N \subset M$ dense in M, and we eventually replace the range $\mathbb{N}$ of r and the first component $\mathbb{N}$ of the definition domain of s by a nonempty initial segment of $\mathbb{N}$.

Exercise 6.3.2 Let $a < c < b$ s $a, c \in f[I]$. Then by Theorem also $c \in f[I]$. Hence $f[I]$ is an interval.

Exercise 6.3.4 We prove that for every two continuous functions $f, g\colon [0,1] \to \mathbb{R}$ satisfying $f(0) = g(1) = 0$ and $f(1) = g(0) = 1$ there is a $t \in (0,1)$ such that $f(t) = g(t)$. We set $h \equiv f - g\colon [0,1] \to \mathbb{R}$ and use the theorem on intermediate values.

Exercise 6.4.2 Their continuity is easy to show. Neither function has maximum because for every $x \in [0,1)$ and every $y \in (x,1)$ we have $f(y) > f(x)$ and $g(y) > g(x)$.

Exercise 6.4.4 Theorem 6.4.3 $\Rightarrow$ Theorem 6.4.1, as will soon be shown. Any compact set is bounded and closed and therefore has both the smallest and the largest element.

Exercise 6.4.5 1. Obvious from the definition. 2. Let $b \in \bigcup \mathcal{A}$. Then $b \in A$ for some $A \in \mathcal{A}$. Thus there is a δ such that $U(b,\delta) \subset A \subset \bigcup \mathcal{A}$. 3. Let $\mathcal{A}$ be finite and $b \in \bigcap \mathcal{A}$. Thus for every $A \in \mathcal{A}$ there exists a number δ_A such that $U(b, \delta_A) \subset A$. Then with $\delta \equiv \min(\{\delta_A : A \in \mathcal{A}\})$ we have that $U(b,\delta) \subset \bigcap \mathcal{A}$. 4 and 5 follow from 2 and 3 by the de Morgan formulas. 6. If $a \in U(b,\delta)$ then $U(a, \delta - |a-b|) \subset U(b,\delta)$.

Exercise 6.4.6 This is clear from the fact that if $b \in M$ then $U(b,\delta) \subset M$ for some δ, and therefore $P(b,\theta) \subset M$ for every $\theta \leq \delta$.

Exercise 6.4.12 C is closed because it is an intersection of closed sets. It is uncountable because it is exactly the set of those points in $[0,1]$ whose 3-adic expansions use only digits 0 and 2. C has "length" 0 because $1 - \frac{1}{3} - \frac{2}{9} - \frac{4}{27} - \frac{8}{81} - \cdots = 0$.

Exercise 6.4.14 $[a,b] \setminus P(c,\delta) = [a,b] \cap (\mathbb{R} \setminus ((c-\delta, c) \cup (c, c+\delta)))$ which is a closed and bounded set.

Exercise 6.5.1 Let $f\colon M \to \mathbb{R}$ be uniformly continuous, $c \in M$ and an ε be given. We take for this ε the δ guaranteed by uniform continuity. Then certainly $f[U(c,\delta)] \subset U(f(c), \varepsilon)$, so that f is continuous in c.

Exercise 6.5.3 If $(a_n), (b_n) \subset M$ for a compact set M, we have convergent subsequences (a_{k_n}) and (b_{m_n}). For simpler notation we denote them again by (a_n) and (b_n).

Exercise 6.5.4 Let $a_n \equiv \frac{1}{n}$ and $b_n \equiv \frac{2}{\pi(2n-1)}$. Then $\lim(a_n - a_{n+1}) = \lim(b_n - b_{n+1}) = 0$ but for every n one has that $f(a_{n+1}) - f(a_n) = 1$ and $f(b_{n+1}) - f(b_n) = 2$.

Exercise 6.5.5 For example $f \equiv 0$ on $[0, \frac{1}{\sqrt{2}}] \cap \mathbb{Q}$ and $f \equiv 1$ on $[\frac{1}{\sqrt{2}}, 1] \cap \mathbb{Q}$.

Exercise 6.5.7 This proof is taken from [23]. Let $M \subset \mathbb{Q}$ and f be as stated. It follows that $f[M]$ is bounded. Using completeness of $\mathbb{R}$ (with respect to at most countable real sets) we set $b' \equiv \inf(f[M])$. By the definition of infimum there is a (bounded) sequence $(b_n) \subset M$ such that $b' = \lim f(b_n)$. By Theorem 2.3.15 there exist $b \in \mathbb{R}$ and a subsequence (b_{m_n}) of (b_n) such that $\lim_{n\to\infty} b_{m_n} = b$. Hence $b \in \overline{M}$. By

Theorem 6.5.6 we have the extended value $f(b) = b'$ because $(f(b_{m_n}))$ is a subsequence of $(f(b_n))$. Since b' is a lower bound of $f[M]$, we get that $f(b) = b' \le f(a)$ for every $a \in M$.

Exercise 6.6.2 It follows from the arithmetic of continuity, from the definition of polynomials and rational functions, and from continuity of constants and identity.

Exercise 6.6.4 In the difference $a_n(x+c)^n - a_n x^n$ we use the binomial theorem, cancel $a_n x^n$ and take out c. The triangle inequality gives that $|a_n \sum_{i=1}^n \binom{n}{i} c^{i-1} x^{n-i}|$ is at most $|a_n| \sum_{i=1}^n \binom{n}{i} |c|^{i-1} |x|^{n-i}$. We replace the numbers $|c|$ and $|x|$ by $|x| + |c|$ which is not smaller. We take out its $n-1$-th power and the sum of binomial coefficients is $\le 2^n$.

Exercise 6.6.5 It suffices to show that for every $c > 0$ we have that $\lim \frac{c^n}{n!} = 0$. The sequence $(\frac{c^n}{n!})$ is nonnegative and, for large n, decreasing. Therefore it has the limit $d \ge 0$ and there is an m such that $d = \inf(\{\frac{c^n}{n!} :\ n \ge m\})$. Suppose for the contrary that $d > 0$. Then for large enough $n \ge m$ one has that $\frac{d(n+1)}{c} > \frac{c^n}{n!}$. Hence $d > \frac{c^{n+1}}{(n+1)!}$, which is a contradiction, and $d = 0$.

Exercise 6.6.10 Let $b \in M(f(g))$ and let an ε be given. There is a δ such that $f[U(g(b), \delta)] \subset U(f(g)(b), \varepsilon)$. There is a θ such that $g[U(b, \theta)] \subset U(g(b), \delta)$. Hence $f(g)[U(b, \theta)] \subset \ldots \subset U(f(g)(b), \varepsilon)$ and $f(g)$ is continuous in b according to the definition.

Exercise 6.6.12 The function f given as $f(x) = x$ on $(0, 1)$ and with the value $f(2) = 1$ is continuous and increasing but the inverse $f^{-1}\colon (0, 1] \to (0, 1) \cup \{2\}$ is not continuous in 1. The function f with the values $f(0) = 1$ and $f(n) = 1 - \frac{1}{n}$ for $n \in \mathbb{N}$ has the closed definition domain $\mathbb{N}_0 \subset \mathbb{R}$ and is injective and continuous, but the inverse $f^{-1}\colon \{1 - \frac{1}{n} \mid n \in \mathbb{N}\} \cup \{1\} \to \mathbb{N}_0$ is not continuous in 1.

Exercise 6.6.13 $\log x$ is continuous by each of parts 2–4 of the theorem, $\arccos x$ and $\arcsin x$ by each of parts 1, 2 and 4, and $\arctan x$ and $\mathrm{arccot}\, x$ by each of parts 2–4.

7 Derivatives

Exercise 7.1.2 By means of Corollary 5.4.4.

Exercise 7.1.5 It is an instance of Proposition 5.1.7.

Exercise 7.1.6 Let $f \in \mathcal{F}(M)$ and $b \in M \cap L^{\pm}(M)$. Then we have the equivalence that $f'_{\pm}(b) = L \iff (f \mid I^{\pm}(b))'(b) = L$ (equal signs). It is in fact an instance of Proposition 5.1.13.

Exercise 7.1.7 The first claim follows easily from definitions. 0 is a limit point of the interval $(0, 1)$ but it is not its two-sided limit point.

Exercise 7.1.9 It does not, the endpoints of the interval $[0, 1]$ are not its two-sided limit points.

Exercise 7.1.12 They are different, the function on the left side is defined at b, the one on the right side is not. But this is the only point where they differ.

Exercise 7.1.13 Clearly $\mathrm{sgn}'_{-}(0) = \lim_{x \to 0^-} \frac{-1-0}{x-0} = +\infty$. Similarly $\mathrm{sgn}'_{+}(0) = +\infty$. By part 2 of Exercise 7.1.5 we have that $\mathrm{sgn}'(0) = +\infty$.

Exercise 7.1.14 Clearly $(|x|)'_{-}(0) = \lim_{x \to 0^-} \frac{-x-0}{x-0} = -1$ and similarly for the right-sided derivative.

Exercise 7.1.16 For example $\operatorname{sgn} x$ has $\operatorname{sgn}'(0) = +\infty$ and $\operatorname{sgn} 0 = 0$ is not a limit point of the image $\{-1, 0, 1\}$.

Exercise 7.1.17 The function f is continuous in b.

Exercise 7.1.18 If $f \in \mathcal{R}$ and $f'_-(b) \in \mathbb{R}$ then f is left-continuous at b, and similarly for the right side. Proofs are the same as for Proposition 7.1.11.

Exercise 7.1.19 The one-sided derivative $(\sqrt{x})'_-(0)$ is not defined because 0 is not a left limit point of the definition domain $[0, +\infty)$ of $\sqrt{x}$. Other one-sided derivatives are equal to the ordinary derivatives.

Exercise 7.1.22 For every $x \neq b$ we have that $\frac{k_c(x)-k_c(b)}{x-b} = \frac{0}{x-b} = 0$. Hence $k'_c = k_0$.

Exercise 7.1.23 We have for every a that $\lim_{x\to a} \frac{x+c-(a+c)}{x-a} = 1$.

Exercise 7.1.24 This is what Proposition 7.1.11 says.

Exercise 7.1.26 This is exactly the assumption that $a_n - b_n = o(b_n)$ $(n \to \infty)$.

Exercise 7.2.2 This follows from the limit $\lim_{x\to b} \frac{f(x)-f(b)}{x-b} = f'(b)$.

Exercise 7.2.3 For $a > 0$ this equation is $y = \frac{x}{2\sqrt{a}} + \frac{\sqrt{a}}{2}$. In $a = 0$ the function $\sqrt{x}$ is not differentiable.

Exercise 7.2.4 Every pair (s, t) determines a unique non-vertical line $y = sx + t$, and these two correspondences are inverses of one another.

Exercise 7.2.6 The system $sa + t = b$ & $sa' + t = b'$, with given a, a', b and b', and unknowns s and t, has a unique solution whose s component is given by the stated formula.

Exercise 7.2.8 We assume that the line ℓ given by $y = sx + t$ is a limit tangent to G_f in $(b, f(b))$. We take any sequence $(b_n) \subset M(f) \setminus \{b\}$ such that $b_n \to b$. Let the line $\kappa(b_n, f(b_n), b, f(b))$ be given by $y = s_n x + t_n$. Then $s_n \to s$, $t_n \to t$ and always $s_n b + t_n = f(b)$. Hence $f(b) = s_n b + t_n \to sb + t$, $sb + t = f(b)$ and $(b, f(b)) \in \ell$.

Exercise 7.2.12 Let b, M and f be as stated. By Heine's definition (H) of pointwise continuity of functions and Theorem 2.2.5 we can take a sequence (a_n) that converges to b from one side and for which $\lim f(a_n) \neq f(b)$. Then it suffices to take any sequence (b_n) that converges to b from the other side and for which the limit $\lim f(b_n)$ exists.

Exercise 7.2.13 Let ℓ_n and ℓ be given by respective equations $y = s_n x + t_n$ and $y = sx + t$. By the assumption we have that $\lim s_n = s$ and $\lim t_n = t$. But $c = sb + t$ and $c_n = s_n b + t_n$, so that $\lim c_n = \lim(s_n b + t_n) = \cdots = sb + t = c$.

Exercise 7.2.14 Let d_n be the infimum of the ε such that $\frac{y_n - c}{x_n - b} \in U(A, \varepsilon)$. We may clearly assume that always $y_n \neq c$ or that $A = 0$. One then easily defines by induction an increasing sequence $(m_n) \subset \mathbb{N}$ such that always $\frac{y_n - u_{m_n}}{x_n - z_{m_n}} \in U(A, d_n + \frac{1}{n})$. We are done because $\lim d_n = 0$.

Exercise 7.2.15 We take $b \equiv 0$, $M \equiv \mathbb{R}$, $f(x) \equiv x^2 \sin(\frac{1}{x})$ for $x \neq 0$, $f(b) = f(0) \equiv 0$, $x_n \equiv \frac{2}{(4n-1)\pi}$ and $y_n \equiv \frac{2}{(4n-3)\pi}$ $(n \in \mathbb{N})$. Then $f'(0) = 0$, but the secant $\kappa(x_n, f(x_n), y_n, f(y_n))$ has a slope $\frac{y_n^2 + x_n^2}{y_n - x_n} \gg n^{-2} \cdot n^2$, so $\geq c > 0$ for every n.

Exercise 7.3.2 $(\operatorname{sgn}(x) - \sqrt{x})'(0) = \lim_{x\to 0} \frac{1-\sqrt{x}}{x} = \lim_{x\to 0^+} \frac{1}{x} = +\infty$.

Exercise 7.3.5 Use that $fg = gf$.

Exercise 7.3.6 We easily see that $f'(0) = +\infty$, $g'(0) = -\infty$, $(fg)(x) = -1$ for $x \neq 0$ and that $(fg)(0) = -\frac{1}{4}$. Hence $(fg)'_-(0) = +\infty$, $(fg)'_+(0) = -\infty$ and $(fg)'(0)$ does not exist.

Exercise 7.3.9 We change the value of f in 0 to $f(0) = \frac{1}{2}$. Then $\frac{f'(0)g(0)-f(0)g'(0)}{g(0)^2} = ((+\infty)\cdot\frac{1}{2} - \frac{1}{2}\cdot(-\infty))/\frac{1}{4} = +\infty$, but $(f/g)(x) = -1$ for $x \neq 0$ and $(f/g)(0) = 1$. So $(f/g)'(0)$ again does not exist.

Exercise 7.4.2 With g not continuous in b the formula $(f(g))'(b) = f'(g(b))\cdot g'(b)$ need not hold in the sense that the right side may be defined but the left side may be undefined. We set $f(x) \equiv x^2$, take $g(x)$ as the modified signum with the value $g(0) = \frac{1}{2}$ and $b \equiv 0$ ($M = \mathbb{R} \subset L(M) = \mathbb{R}^*$). Then $f'(g(b)) = (2x)(\frac{1}{2}) = 1$, $g'(b) = +\infty$ and the right side is $1\cdot(+\infty) = +\infty$, but $f(g)(x) = 1$ for every $x \neq 0$ and $f(g)(0) = \frac{1}{4}$, so that $(f(g))'(b)$ does not exist.

Exercise 7.4.5 Then the formula $(f^{-1})'(c) = \frac{1}{f'(f^{-1}(c))}$ may not hold, in the sense that the right side may be defined, but the left side may be undefined. For example for $f(x) \equiv x - 1$ on $(-\infty, 0)$, $f(0) \equiv 0$ and $f(x) \equiv x + 1$ on $(0, +\infty)$, which is the injectivized signum, and $c \equiv 0$, the right side equals $\frac{1}{f'(f^{-1}(c))} = \frac{1}{f'(0)} = \frac{1}{+\infty} = 0$, but the left side is undefined because $0 \notin L(M(f^{-1})) = L((-\infty,-1)\cup\{0\}\cup(1,+\infty))$.

Exercise 7.5.2 In the first inequality we have for $n = 0$ that $\frac{1}{c}(a_n(x+c)^n - a^n x^n) = 0$ and for $n \geq 1$ we get $a_n \sum_{j=0}^{n-1}(x+c)^j x^{n-1-j}$. Then n is replaced with $n+1$. In the second inequality, we use the transformation $\sum_{j=0}^{n}(x+c)^j x^{n-j} - (n+1)x^n = c\sum_{j=1}^{n}\sum_{i=1}^{j}\binom{j}{i}c^{i-1}x^{n-i}$ (by the binomial Theorem), where $n \geq 1$. Then we use the Δ-inequality and replace both $|c|$ and $|x|$ with the quantity $y = |c| + |x|$. In the third inequality, we take y^{n-1} out and the sum of binomial coefficients is $\leq 2^j$. In the fourth inequality, $\sum_{j=1}^{n} 2^j \leq 2^{n+1}$.

Exercise 7.5.4 $(\log|x|)' = \frac{1}{x}$ $(\in \mathcal{F}(\mathbb{R}\setminus\{0\}))$.

Exercise 7.5.5 1. The derivative $(a^x)' = (\exp((x\log a))' = \cdots = a^x\cdot\log a$ follows from Corollaries 7.4.3 and 7.5.3 and the derivative $(k_c\cdot \mathrm{id}_{\mathbb{R}})' = k_c$. 2. We get this derivative for $b < 1$ and $x > 0$ (because $(x^b)'(0) = +\infty$) by differentiating the function $\exp(b\log x)$ by means of Corollaries 7.4.3 and 7.5.3, the derivative of logarithm, the derivative $(k_c\cdot\mathrm{id}_{\mathbb{R}})' = k_c$ and the definition of x^b. For $b > 1$ we easily get from the definition that $(x^b)'(0) = 0$. 3. This is immediate from the definition of x^b. 4. This is immediate from the definition of 0^x. 5. We use Corollary 7.3.7, the definition of x^m and the derivative $\mathrm{id}'_{\mathbb{R}} = k_1$. 6. We use the derivative $k'_1 = k_0$.

Exercise 7.5.6 The derivative of $\tan x = \frac{\sin x}{\cos x}$ follows from the derivatives $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$, identity $\sin^2 x + \cos^2 x = k_1$ and Corollary 7.3.10. Similarly for $\cot x$.

Exercise 7.5.7 The derivative of inverse sine follows from the derivative $(\sin x)' = \cos x$, relation $\cos\mid(-\frac{\pi}{2},\frac{\pi}{2}) = \sqrt{1-(\sin\mid(-\frac{\pi}{2},\frac{\pi}{2}))^2}$ and Corollary 7.4.7. Similarly for the other three derivatives.

Exercise 7.5.8 We get $f'(x) = (2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}))\cup\{(0,0)\}$ which is a discontinuous function. The value $f'(0) = 0$ is easily calculated from the definition of derivative.

8 Applications of mean value theorems

Exercise 8.1.2 The derivative at 0 does not exist.

Exercise 8.1.5 In the proof of Rolle's theorem, f has at c global minimum or maximum, which implies that the corresponding horizontal tangent is non-cutting. Thus also in Lagrange's theorem the tangent to the graph, taken at the point of a global extreme of the auxiliary function g, is non-cutting.

Exercise 8.1.8 If $f(a) = f(b)$ and $g'(c) = \pm\infty$ then the product $z \cdot g'(c) = 0 \cdot (\pm\infty)$ and is indefinite.

Exercise 8.2.1 $F_n - F_{n-1} - F_n = 0$.

Exercise 8.2.3 Add the dummy zero coefficients $p_{k+1}(x) = 0, \dots, p_{n_0}(x) = 0$.

Exercise 8.2.4 Multiply the coefficients $p_i(x)$ by the polynomial $(x - k)(x - k - 1) \dots (x - n_0 + 1)$.

Exercise 8.2.7 First we select an injective sequence. Then we use Proposition 2.3.12.

Exercise 8.2.8 The denominator of $r(x)$ has finitely many zeros $\{z_1 < z_2 < \dots < z_l\}$ in $(k - 1, +\infty)$, and a gap (z_{i-1}, z_i), $i \in [l + 1]$, where $z_0 = k - 1$ and $z_{l+1} = +\infty$, between them contains a tail $(a_r, a_{r+1}, \dots)$ of (a_n).

Exercise 8.2.9 Suppose that (a_n) decreases. Then we can apply to $[a_{n+1}, a_n]$ and f Rolle's theorem, because f is differentiable on $[a_{n+1}, a_n]$ and $f(a_{n+1}) = f(a_n) = 0$. We get a point $b_n \in (a_{n+1}, a_n)$ with $f'(b_n) = 0$. Similarly for increasing (a_n).

Exercise 8.2.10 Clearly, $r'(x) \in \mathrm{RAC}$. Suppose that $p_j(x) \neq 0$. Then we have $(p_j(x)\log(x-j+1))' = p_j'(x)\log(x-j+1) + \frac{p_j(x)}{x-j+1}$. The latter summand is absorbed in $r'(x)$. If $\deg p_j = 0$, the former summand disappears. If $\deg p_j > 0$, then in the former summand we have $\deg p_j' = \deg p_j - 1$.

Exercise 8.2.12 We just replace the function $\log(x - j + 1)$ with $\log(x - j + 1 + c)$.

Exercise 8.2.13 Proceed as in the proof of Theorem 8.2.6.

Exercise 8.3.1 Any fraction $\alpha = \frac{a}{b}$ is a root of $x - \alpha$ and of $bx - a$. The number $\sqrt{2}$ is a root of $x^2 - 2 = 0$.

Exercise 8.3.2 We get form (i) by dividing the polynomial by the leading coefficient. We get form (ii) by multiplying it by the product of denominators of its coefficients.

Exercise 8.3.3 Exactly the integers $\mathbb{Z}$.

Exercise 8.3.4 It is, $\phi^2 - \phi + 1$.

Exercise 8.3.5 We have the (Binet) formula $F_n = \frac{1}{\sqrt{5}}\big(\phi^n - \psi^n\big)$ where ψ is the other root of $x^2 - x + 1$.

Exercise 8.3.7 Every nonzero complex polynomial with degree $d \in \mathbb{N}_0$ has at most d (complex) roots; in fact exactly d when they are counted with multiplicities. Since the set of integer polynomials is countable, the set of algebraic numbers is also countable because it is a countable union of finite sets. But the set $\mathbb{R}$ of real numbers is uncountable (Corollary 1.7.17 and Theorem 1.7.19) and therefore also the set of real transcendental numbers is uncountable, in particular nonempty.

Exercise 8.3.9 $|(\sum_{j=0}^{n} a_j x^j)'(x)|$, where $x \in [\alpha, \alpha + 1$ with $\alpha = \frac{a}{10^k}$ for $a \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, is by the triangle inequality at most $\sum_{j=0}^{n} j|a_j|(|a| + 1)^{j-1}$. This is at most

$(n+1)^2 \max(\dots)(|a|+1)^n$. We replace n by $n+1$ in order to have only positive factors in the bound, and $n-1$ by n in order to have nonnegative exponents.

Exercise 8.4.2 In the division $g(x) = \frac{f(x)}{x-\frac{p}{q}}$ the polynomial $f(x)$ loses one multiplicity of the root $\frac{p}{q}$, but $\alpha \neq \frac{p}{q}$ as it is irrational.

Exercise 8.4.3 The constant d exists because f' is continuous and I is compact.

Exercise 8.4.5 Let $\frac{a}{b}$ with $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ be any fraction. We may assume that $a \in \mathbb{N}$. Recall the school algorithm for computing the decimal expansion of $\frac{a}{b}$, for example $\frac{1}{7} = 1 : 7 = 0.142\dots$, with the residues $1, 3, 2, 6, \dots$. Once a residue repeats, the residues and the expansion start to repeat.

Exercise 8.4.6 Let $n \in \mathbb{N}$ and $c > 0$ be arbitrary. We take $m \in \mathbb{N}$ large enough so that $\frac{2}{q_m^{(m+1)/2}} < c$ and $\frac{m+1}{2} \geq n$. Then $|\lambda - \frac{z_m}{q_m}| < \frac{2}{q_m^{m+1}} = \frac{2}{q_m^{(m+1)/2}} \cdot \frac{1}{q_m^{(m+1)/2}} < \frac{c}{q_m^n}$ and Liouville's inequality is violated.

Exercise 8.4.7 Algorithm $\mathcal{L}$ determines for every input $n \in \mathbb{N}$ if $n = m!$ for some $m \in \mathbb{N}$. If it is the case, then $\mathcal{L}$ outputs the digit $\mathcal{L}(n) = 1$, and else it outputs 0. In more detail, $\mathcal{L}$ multiplies the numbers $1, 2, \dots, m$ as long as $m! \leq n$. Since $m! \geq 2^{m-1}$, $\mathcal{L}$ knows the digit $\mathcal{L}(n)$ at the latest for $m \leq \log_2(n+2) + 1 \leq \log_2(6n)$. Multiplying two numbers $\leq n$ takes time $O(\log_2(n+1)^2) = O(\log_2(6n)^2)$. Thus $\mathcal{L}$ computes the decimal digit $\mathcal{L}(n)$ ($\in \{0, 1\}$) in time $O(\log_2(6n)^3)$, which is time polynomial in the size (number of digits) of n. This (probably) cannot be achieved by the algorithm $\mathcal{A}$ in Theorem 8.3.10. In any case, the description of $\mathcal{A}$ is much more complex than that of $\mathcal{L}$.

Exercise 8.4.8 The proof is very similar to the proof of Corollary 8.4.4.

Exercise 8.5.3 We cannot. The signum function $\operatorname{sgn} x$ shows, for $b \equiv 0$, that these sets may have just one element.

Exercise 8.5.4 Let a, b and f be as stated, in particular every finite derivative $f'(c) \geq 0$ for every $c \in (a, b)$, and let $d \in (a, b)$ be such that $f'(d) = -\infty$. Then by Proposition 8.5.2 there exist points d_0 and d_1 such that $a < d_0 < d < d_1 < b$ and $f(d_0) > f(d) > f(d_1)$. But by Theorem 8.1.4 we have $\frac{f(d_1)-f(d_0)}{d_1-d_0} = f(e) \geq 0$ for some $e \in (d_0, d_1)$. Hence $f(d_0) \leq f(d_1)$, which is a contradiction.

Exercise 8.5.6 For example, $f(x) = x$ on $\mathbb{Q} \cap [0, \frac{1}{\sqrt{2}})$ and $f(x) = x - 2$ on $\mathbb{Q} \cap (\frac{1}{\sqrt{2}}, 1]$.

Exercise 8.5.8 Take, for example, $f(x) \equiv k_0(x) \,|\, [a, b]$ and for any sequence $a < a_1 < a_2 < \dots < b$ with $\lim a_n = b$ deform the flat G_f around every point $(a_n, f(a_n)) = (a_n, 0)$ in an appropriate small upward bump. Thus a function $g \in \mathcal{F}([a, b])$ arises that satisfies the assumptions of the proposition. Each bump is so low that still $g'(b) = f'(b) = 0$, but at the same time it is so steep that, say, $g'(a_n) = 1$ for every n. The bumps are also so narrow that $g'\big(\frac{a_n+a_{n+1}}{2}\big) = 0$. Then $g'(b) = 0$ but $\lim_{x \to b} g'(x)$ does not exist.

Exercise 8.5.11 We move from the interval $(b-\delta, b)$ to the interval $(b, b+\delta)$ by means of the map $x \mapsto 2b - x$. The definition domain $P(b, \delta)$ is the union $(b-\delta, b) \cup (b, b+\delta)$. We move from the interval $U(+\infty, \delta) = (\frac{1}{\delta}, +\infty)$ to the interval $(0, \delta)$ by means of the map $x \mapsto \frac{1}{x}$.

Exercise 8.5.13 Because of the definitions of $f(x)$, $g(x)$ and the derivative.

Exercise 8.6.2 The former sequence is 4-periodic: $(\overline{\sin x, \cos x, -\sin x, -\cos x})$. The n-th term of the latter sequence is $(\frac{1}{x})^{(n)} = (-1)^n n! x^{-n-1}$.

Exercise 8.6.4 Consider $f(x) \equiv x^3$ and $b \equiv 0$.

Exercise 8.6.6 The strict convexity of x^2 follows from Theorem 8.6.14 because $(x^2)'' = 2 > 0$. The claim about $|x|$ is clear from the graph which is a union of two half-lines. The strict concavity of $\log x$ follows from Theorem 8.6.14 because $(\log x)'' = -x^{-2} < 0$.

Exercise 8.6.7 This is logically clear from the definition.

Exercise 8.6.8 This is clear from the definition by applying the symmetry $(x, y) \mapsto (x, -y)$ of the plane.

Exercise 8.6.11 The argument, based on Theorem 5.3.1, is the same as in the proof of Theorem 8.6.9. Only the upper bound is now not available.

Exercise 8.6.13 It is not true because the one-sided derivative at the endpoint may be infinite, and then continuity at the point is not guaranteed. For example, the function $f \in \mathcal{F}([0, 1])$, given as $f(x) = 0$ for $x \neq 1$ and $f(1) = 1$, is convex.

Exercise 8.6.15 It follows that (c, c') lies above or on the line going through the first two points. Thus the second point (b, b') lies below or on the line going through the first point (a, a') and the third point (c, c').

Exercise 8.6.18 Indeed, the tangent at $(0, 0)$ is the x-axis.

Exercise 8.6.19 Every point of the graph is an inflection point.

Exercise 8.7.2 These functions have one-sided limits $\pm\infty$ at 0.

Exercise 8.7.4 Let $\lim_{x\to+\infty}(f(x) - sx - b) = 0$. By adding the limit $\lim_{x\to+\infty} b = b$ we get that $\lim_{x\to+\infty}(f(x) - sx) = b$. Dividing by the limit $\lim_{x\to+\infty} x = +\infty$ we get that $\lim_{x\to+\infty}(\frac{f(x)}{x} - s) = 0$, thus $\lim_{x\to+\infty} \frac{f(x)}{x} = s$.

Suppose that ($\lim_{x\to+\infty} \frac{f(x)}{x} = s$ and) $\lim_{x\to+\infty}(f(x) - sx) = b$. Subtracting from the latter limit the limit $\lim_{x\to+\infty} b = b$, we get that $\lim_{x\to+\infty}(f(x) - sx - b) = 0$.

Exercise 8.7.5 It is the axis x.

Exercise 8.7.6 $f \in \mathcal{F}(M)$ is even, resp. odd, if $M = -M$ $(= \{-x :\ x \in M\})$ and for every $x \in M$ we have $f(-x) = f(x)$, resp. $f(-x) = -f(x)$. The function f is c-periodic ($c \in \mathbb{R}$) if $M = M \pm c$ $(= \{x \pm c :\ x \in M\})$ and for every $x \in M$ we have $f(x + c) = f(x)$.

Exercise 8.7.7 **0.** $r(x) \notin$ EF. **1.** $M(r) = \mathbb{R}$. **2.** $r(x)$ is an even and 1-periodic function; the only periods are integers. **3.** From Proposition 5.2.12 we know that $r(x)$ is continuous exactly at irrational numbers. We show that $r'(\alpha)$ does not exist for any $\alpha \in \mathbb{R}$. For rational $\alpha = \frac{m}{n}$ in lowest terms, $r(\alpha) = \frac{1}{n}$ and $r(x) = 0$ for x arbitrarily close to α both from the left and the right. This gives differential ratios at α going to $+\infty$ on the left, and to $-\infty$ on the right, of α. If α is irrational then these zero values of $r(x)$ give diff. ratios at α equal to 0 and arbitrarily close to α. But by the theorem of Dirichlet there exist infinitely many different fractions $\frac{m}{n}$ in lowest terms such that $|\alpha - \frac{m}{n}| < \frac{1}{n^2}$. These fractions give at α diff. ratios in absolute value $> \frac{1/n}{1/n^2} = n \to +\infty$. **4.** It is not hard to see that $\lim_{x\to\alpha} r(x) = 0$ for every $\alpha \in \mathbb{R}$.

5. The function $r(x)$ intersects the axis x exactly in the points $(\alpha, 0)$ for irrational α, and the axis y in the point $(0, 1)$. **6.**

Exercise 8.7.8 0. $f(x) \in \mathrm{SEF}$. **1.** $M(f(x)) = \cdots = M(\log x) = (0, +\infty)$ because $f(x) \in \mathrm{EF}$. **2.** This function is not of a special form. **3.** Now $f(x) \in \mathcal{C}$ and $D(f) = M(f)$ because $f(x) \in \mathrm{SEF}$. The derivative is $f'(x) = (1 + \log x)f(x) = (1 + \log x)x^x$. **4.** $\lim_{x\to 0} f(x) = \cdots = 1$ because $\lim_{x\to 0} x \log x = 0$. Clearly, $\lim_{x\to+\infty} f(x) = +\infty$. **5.** Here G_f is disjoint to both coordinate axes. **6.** Since $f(x) \in \mathrm{SEF}$, we have $D(f) = M(f)$ and there is nothing to compute. **7.** We equate the f' found in part 3 to 0 and get that $f' < 0$ on $(0, \frac{1}{\mathrm{e}})$ and $f' > 0$ on $(\frac{1}{\mathrm{e}}, +\infty)$. Thus the maximal sets of monotonicity of f are the intervals $(0, \frac{1}{\mathrm{e}}]$ and $[\frac{1}{\mathrm{e}}, +\infty)$. On the former f decreases and on the latter it increases. At $x = \frac{1}{\mathrm{e}}$ f has a strict global minimum with the value $1/\mathrm{e}^{1/\mathrm{e}}$. It follows that this is the only extreme of f. In particular, there is no local maximum: if $x \in (0, \frac{1}{\mathrm{e}}]$ then $f(y) > f(x)$ for every $y \in (0, x)$, and if $x \in [\frac{1}{\mathrm{e}}, +\infty)$ then $f(y) > f(x)$ for every $y \in (x, +\infty)$. **8.** We have $f''(x) = (\frac{1}{x} + (1 + \log x)^2)x^x$. Since $f'' > 0$ on $(0, +\infty)$, f is convex on its definition domain. **9.** It follows (from the previous part) that f has no inflection. **10.** It is clear that f has no vertical asymptotes. Since $\lim_{x\to+\infty} \frac{f(x)}{x} = +\infty$, by Exercise 8.7.4 f does not have asymptote at $+\infty$. **11.** `https://www.desmos.com/calculator`.

9 Taylor expansions. Primitives

Bibliography

[1] 0.999 . . . , Wikipedia article, `https://en.wikipedia.org/wiki/0.999...`

[2] T. M. Apostol, *Mathematical Analysis. Second Edition*, Addison-Wesley, Reading, MA 1974

[3] A. Blass and Yu. Gurevich, Why sets?. In: *Pillars of computer science*, 179–198. *Lecture Notes in Comput. Sci.*, 4800 Springer, Berlin, 2008

[4] M. Borinsky, Generating asymptotics for factorially divergent sequences, *Electron. J. Combin.* **25** (2018), Paper No. 4.1, 32 pp.

[5] N. Bourbaki, *Fonctions d'une variable réelle. Théorie élémentaire*, Hermann & C[ie], Paris 1958

[6] N. Burbaki, *Funkcii dejstvitel'nogo peremennogo. Elementarnaya teoriya*, Nauka, Moskva 1965 (translated from [5] by E. I. Stechkina)

[7] F. Cajori, *A History of Mathematical Notations*, Dover, New York 1993 (first published in 1928–1929)

[8] R. Černý a M. Pokorný, *Základy matematické analýzy pro studenty fyziky* 1, MatfyzPress, Praha 2020

[9] I. Černý and M. Rokyta, *Differential and Integral Calculus of One Real Variable*, Karolinum - Nakladatelství Univerzity Karlovy, Praha 1998

[10] J. H. Conway and R. G. Guy, *The Book of Numbers*, Springer-Verlag, New York 1996

[11] E. T. Copson, *Asymptotic Expansions*, Cambridge University Press, Cambridge, UK 1967

[12] K. Devlin, *The Joy of Sets. Fundamentals of Contemporary Set Theory.* Second Edition, Springer-Verlag, New York 1993

[13] `https://www.dictionary.com/browse/elementary--my-dear-watson`

[14] A. Erdélyi, *Asymptotic Expansions*, Dover Publications, Inc., U.S.A. 1956

[15] L. Fehér, G. Kós and A. Tóth, *Mathematical Analysis — Problems and Exercises II*, Eötvös Loránd University, Faculty of Science 2014–19, 211 pp.

[16] Field with one element, Wikipedia article, `https://en.wikipedia.org/wiki/Field_with_one_element`

[17] Gottfried Wilhelm Leibniz, Wikipedia article, `https://en.wikipedia.org/wiki/Gottfried_Wilhelm_Leibniz`

[18] R. Gray, Georg Cantor and transcendental numbers, *Amer. Math. Monthly*, **101** (1994), 819–832

[19] E. Hairer and G. Wanner, *Analysis by Its History*, Springer, New York 2008

[20] G. H. Hardy, *A Course of Pure Mathematics*, Cambridge: At the University Press, Cambridge 1908

[21] J. Jost, *Postmodern Analysis*, Springer, Berlin 2005

[22] M. Klazar, Non-holonomicity of the sequence $\log 1, \log 2, \log 3, \dots$, arXiv:math/0502141v1, 2005, 3 pp.

[23] M. Klazar, Real analysis without uncountable sets, arXiv:2301.08142, ...

[24] M. Klazar, Matematická analýza 1, ...

[25] J. Krajíček, *Proof Complexity*, Cambridge University Press, Cambridge 2019

[26] I. Kriz and A. Pultr, *Introduction to Mathematical Analysis*, Springer Basel, Basel 2013

[27] T. Monteil and K. Nurligareev, Asymptotic probability for connectedness, arXiv:2401.00818v1, 2024, 24 pp.

[28] D. Normann and S. Sanders, On the uncountability of $\mathbb{R}$, *J. Symb. Logic* **87** (2022), 1474–1521

[29] OEIS Foundation Inc. (2022), The On-line Encyclopedia of Integer Sequences, `https://oeis.org`

[30] S. Ponnusamy, *Foundations of Mathematical Analysis*, Birkhäuser, Springer, New York 2012

[31] Principle of locality, Wikipedia article, `https://en.wikipedia.org/wiki/Principle_of_locality`

[32] P. Pudlák, *Logical Foundations of Mathematics and Computational Complexity. A Gentle Introduction*, Springer, Cham 2013

[33] Ch. Ch. Pugh, *Real Mathematical Analysis*, Springer, New York 2002

[34] Reverse mathematics, Wikipedia article, `https://en.wikipedia.org/wiki/Reverse_mathematics`

[35] W. Rudin, *Principles of Mathematical Analysis. Third Edition*, McGraw-Hill, New York 1976

[36] Stanford Encyclopedia of Philosophy, Propositions, `https://plato.stanford.edu/entries/propositions/`

[37] R. P. Stanley, *Enumerative Combinatorics.* Volume II, Cambridge University Press, Cambridge, UK 1999

[38] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, *Acta Arith.* **27** (1975), 199–245

[39] T. Tao, *Analysis I*, Hindustan Book Agency, New Delhi 2006

[40] T. Tao, Szemerédi's proof of Szemerédi's theorem, *Acta Mathematica Hungarica* **161** (2020), 443–487

[41] L. Wittgenstein, *Tractatus Logico-Philosophicus. Logisch-philosophische Abhandlung*, Kegan Paul, London 1922

[42] V. A. Zorich, *Mathematical Analysis I*, Springer, Berlin 2004

Index

Page number in *italic* refers to the definition of a notion or to a theorem with proof.