Counting set partitions

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Overview:

1. Stirling and Bell numbers

2. Sequences, graphs, and proofs

3. Counting "partitions" with intersections

4. The generating function $B(x)$

5. Arithmetic properties of Stirling and Bell numbers

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1. Stirling and Bell numbers

$P = \{A_1, A_2, \ldots, A_k\}$ is a **partition** of a finite set X if $A_i \neq \emptyset$, $A_1 \cup A_2 \cup \ldots \cup A_k = X$, and A_i have empty pairwise intersections. We denote $V(P) = X$ and call the sets A_i **blocks** of P.

Agreement. If it is not said else, our partitions P satisfy $V(P) \subset \mathbf{N}$ where $\mathbf{N} = \{1, 2, 3, \ldots\}$. Thus we have on each $V(P)$ the usual linear ordering $<$.

Stirling number (of the second kind) $S(n, k)$ is the number of partitions P such that $V(P) = [n] = \{1, 2, \ldots, n\}$ and $|P| = k$. **Bell number** B_n is $\sum_{i=1}^{n} S(n, i)$, i.e., the number of all partitions of $[n]$ (or of any other fixed n-element set). We set $S(n, 0) = 0$ if $n > 0$ and $S(0, 0) = B_0 = 1$.

In other words, $S(n, k)$ is the number of distributions of n labelled balls in k unlabelled boxes such that each box contains at least one ball.

Example. $S(3,2) = 3$ and $B_3 = 5$:

$$1 \mid 2 \mid 3, \quad 1\,2 \mid 3, \quad 1 \mid 2\,3, \quad 1\,3 \mid 2, \quad 1\,2\,3.$$

Recurrences. Consider a generic partition of $[n]$, delete n, and see what happens with the partition. This gives recurrences

$$\begin{aligned} S(n,k) &= S(n-1,k-1) + kS(n-1,k) \\ B_n &= \sum_{i=0}^{n-1} \binom{n-1}{i} B_i \end{aligned}$$

where $B_0 = 1$.

Few values. We have, for $n = 1, 2, \ldots$, $B_n =$

$1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, \ldots$

and, for example, for $k = 1, 2, \ldots, 7$, $S(7,k) =$

$$1,\ 63,\ 301,\ 350,\ 140,\ 21,\ 1.$$

Note that $S(n,1) = S(n,n) = 1$, $S(n,2) = (2^n - 2)/2 = 2^{n-1} - 1$, and $S(n,n-1) = \binom{n}{2}$.

"Bell's triangle" (Aitken, 1933)

$$\begin{array}{ccccccccc}
 & & & & \mathbf{1} & & & & \\
 & & & 1 & & \mathbf{2} & & & \\
 & & 2 & & 3 & & \mathbf{5} & & \\
 & 5 & & 7 & & 10 & & \mathbf{15} & \\
15 & & 20 & & 27 & & 37 & & \mathbf{52}
\end{array}$$

Explicit formulae. We have

$$\begin{aligned}
S(n,k) &= \sum_{i=0}^{k-1} \frac{(-1)^i}{i!(k-i)!} \cdot (k-i)^n \\
B_n &= \frac{1}{\mathrm{e}} \sum_{i=0}^{\infty} \frac{i^n}{i!} \quad \text{(Dobiňski, 1877)}
\end{aligned}$$

(or replace ∞ by $2n$ and round the finite sum to the closest integer).

Generating functions for $S(n,k)$. We have

$$\begin{aligned}
\sum_{n\geq 0} \frac{S(n,k)x^n}{n!} &= \frac{(\mathrm{e}^x - 1)^k}{k!} \\
\sum_{n\geq 0} S(n,k)x^n &= \frac{x^k}{(1-x)(1-2x)\dots(1-kx)}
\end{aligned}$$

Summing the first formula over k, we get a formula for the **EGF of Bell numbers:**

$$\sum_{n\geq 0} \frac{B_n x^n}{n!} = \mathrm{e}^{\mathrm{e}^x - 1}.$$

Extracting the coefficient of x^n in the second formula, we get

$$S(n,k) = \sum_{a_1+\cdots+a_k=n-k} 1^{a_1} 2^{a_2} \ldots k^{a_k}.$$

Summing it over k, we get

$$\sum_{n\geq 0} B_n x^n = \sum_{k\geq 0} \frac{x^k}{(1-x)(1-2x)\ldots(1-kx)}.$$

If we substitute $\frac{x}{1-x}$ for x and multiply by it, the k-th summand transmutes in the $(k+1)$-th one. Hence the **OGF of Bell numbers** $B(x) = \sum_{n\geq 0} B_n x^n$ satisfies the equation

$$B(x) = 1 + \tfrac{x}{1-x} \cdot B(x/(1-x)).$$

$B(x)$ diverges for every $x \neq 0$ and therefore this is only formal power series identity.

The asymptotics of B_n was found by Moser and Wyman in 1955: for $n \to \infty$,

$$B_n \sim \frac{\lambda^{n+1/2}}{n^{1/2} \cdot \mathrm{e}^{n+1-\lambda}}$$

where $\lambda = \lambda(n)$ is given by $\lambda \log \lambda = n$. Another formulation (Odlyzko, 1995):

$$B_n \sim \frac{\mathrm{e}^w w^{n-w}}{(\log w)^{1/2}}$$

where $w = w(n)$ is given by $w \log(w+1) = n$.

Later we will need the corollary

$$\frac{B_{n-1}}{B_n} \sim \frac{\mathrm{e} \cdot \log n}{n}.$$

An explicit asymptotic expansion is

$$\frac{\log B_n}{n} = L - L_2 - 1 + \frac{L_2 + 1}{L} + \frac{(L_2)^2}{2L^2} + O(L_2/L^2)$$

where $L = \log n$ and $L_2 = \log \log n$ (de Bruijn, 1958). Compare it with $\log n!/n = L - 1 + L/2n + c/n + o(1/n)$.

2. Sequences, graphs, and proofs

Let S be a countably infinite set of symbols, which includes $\mathbf{N}$ and symbols $a, b, c, d, \ldots$, and S^* be the set of words over S.

Representing partitions by sequences. If $u = a_1 a_2 \ldots a_n$ is in S^*, we define a partition P of $[n]$ by putting i and j in the same block of P iff $a_i = a_j$. We say that u **represents** P. Each P, $V(P) = [n]$, has the unique **canonical** representing sequence $u = a_1 a_2 \ldots a_n$: u represents P, $\{a_1, \ldots, a_n\} = [|P|]$, and the first occurrence of 1 in u precedes that of 2, which precedes that of 3, which ...

Example. $d\ b\ b\ c\ d\ d\ b\ c$ represents $P = \{\{1, 5, 6\}, \{2, 3, 7\}, \{4, 8\}\}$. The canonical representative of P is 1 2 2 3 1 1 2 3.

Other terminology: canonical sequences are also called *restricted growth functions*.

Let us prove the formula

$$\sum_{n\geq 0} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\dots(1-kx)}.$$

P. We need to count canonical $u \in [k]^*$ by length. Each such u is obtained as

$$u = 1u_1 2u_2 \dots k u_k \quad \text{where} \quad u_i \in [i]^*,$$

$i = 1, 2, \dots, k$, are selected in arbitrary way. Since the OGF of the words u_i's counted by length is $1/(1-ix) = 1 + ix + i^2x^2 + \cdots$, multiplication gives the stated formula.□

Let us prove the formula

$$S(n,k) = \sum_{i=0}^{k-1} \frac{(-1)^i}{i!(k-i)!} \cdot (k-i)^n.$$

P. If X is the set of surjections $f : [n] \to [k]$, then $|X| = k! \cdot S(n,k)$. If X_i is the set of mappings $f : [n] \to [k] \backslash \{i\}$, then $|X| = k^n - |X_1 \cup \ldots \cup X_k|$. Hence $k! \cdot S(n,k) = k^n - |X_1 \cup \ldots \cup X_k|$. Expanding the RHS by PIE, we get the stated formula.□

Let us prove Dobiňski's formula

$$B_n = \frac{1}{\mathrm{e}} \sum_{i=0}^{\infty} \frac{i^n}{i!}.$$

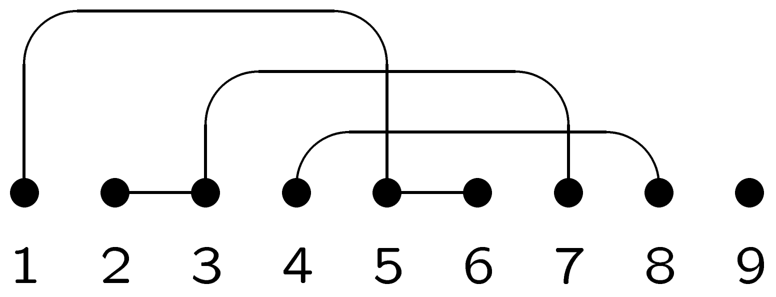
P. Denote $(x)_i = x(x-1)\dots(x-i+1)$. Then every $p \in \mathbf{Q}[x]$, $\deg(p) = n$, expands as $p(x) = \sum_{k=0}^{n} \alpha_k \cdot (x)_k$ where $\alpha_k \in \mathbf{Q}$. It follows that

$$\sum_{i=0}^{\infty} \frac{p(i)}{i!} = \mathrm{e} \cdot (\alpha_0 + \dots + \alpha_n).$$

It suffices to show that if $p(x) = x^n$, $\alpha_k = S(n,k)$. Let $r \in \mathbf{N}$. There is a 1-1 correspondence between mappings $f : [n] \to [r]$ and pairs (P, g) where P is a partition of $[n]$ and $g : [|P|] \to [r]$ is an injection: $f(x) = g(i)$ where $x \in B_i \in P$ and the blocks of P are labelled by the order of their minima. By counting, $r^n = \sum_{k=0}^{n} S(n,k) \cdot (r)_k$ for every r. Hence this identity holds even with indeterminate x in place of r and indeed $\alpha_k = S(n,k)$.$\Box$

Representing partitions by graphs. (W.Y.C. Chen, Deng and Du, 2003) Let P be a partition of $[n]$. We define a graph $G(P) = ([n], E)$ by $\{x, y\} \in E$ iff $x, y \in A \in P$ and no $z \in A$ separates x and y.

Example. If $P = \{\{1, 5, 6\}, \{2, 3, 7\}, \{4, 8\}, \{9\}\}$ then $G(P)$ is the graph



Observation. The components of $G(P)$ are monotone paths; they correspond to the blocks of P. Local characterization of such G: each vertex v has degree ≤ 2 and if deg$(v) = 2$ then v separates its two neighbors.

Sparse partitions. Let P be a partition of $[n]$ and $k \in \mathbf{N}$. We say that P is k-**sparse** if for every two numbers $x < y$ from the same block of P we have $y - x \geq k$. (So in $G(P)$ each edge jumps over at least $k-1$ vertices.)

Example. From the 15 partitions of [4] only 5 are 2-sparse, namely (represented by canonical sequences)

$$1234, \ 1232, \ 1231, \ 1213, \ \text{ and } \ 1212.$$

Theorem. (Prodinger, 1981; W. Yang, 1996) Let $n \geq k \geq 1$ and $B_{n,k}$ be the number of k-sparse partitions of $[n]$. Then $B_{n,k} = B_{n-k+1}$.

P. (Chen, Deng and Du, 2003) Let P be a k-sparse partition of $[n]$, $k \geq 2$. Transform $G = G(P)$ in the graph $G' = ([n-1], E')$ defined by $(x < y)$ $\{x, y\} \in E' \Leftrightarrow \{x, y+1\} \in E$. Then $G' = G(P')$ for some partition P' of $[n-1]$. Clearly, P is k-sparse iff P' is $k-1$-sparse and $P \mapsto P'$ is 1-1. Trivially, $B_{n,1} = B_n$. $\Box$

3. Counting "partitions" with intersections

Let $H = (E_i : i \in I)$, $\emptyset \neq E_i \subset \mathbf{N}$, be a set system with all E_i and I finite. The **weight** of H is $w(H) = \sum_{i \in I} |E_i|$. H_1 and H_2 are **order-isomorphic** if they are isomorphic via an *increasing* isomorphism (which must be unique).

How many set systems with weight n, up to order-isomorphism, are there?

We denote this number $b_1(n)$. If the edges E_i are not allowed to intersect, H is a partition and the answer is, of course, B_n, the Bell number. Thus $b_1(n)$ is "B_n with intersections".

Example. We have $b_1(3) = 10$: $H_1 = (1, 2, 3)$, $H_2 = (1, 1, 2)$, $H_3 = (1, 2, 2)$, $H_4 = (1, 1, 1)$, $H_5 = (12, 3)$, $H_6 = (13, 2)$, $H_7 = (23, 1)$, $H_8 = (12, 1)$, $H_9 = (12, 2)$, and $H_{10} = (123)$.

B_n versus $b_1(n)$ for $n = 2, 3, \ldots, 10$:

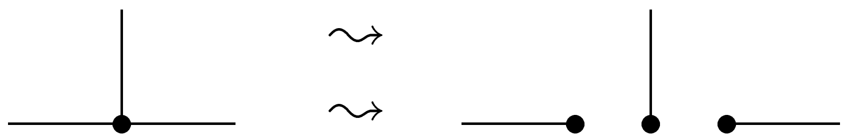
2	5	15	52	203	877	4140	21147	115975
3	10	41	192	1025	6087	39754	282241	2159916

We sketch the proof of the following result.

Theorem. (Klazar, 2004) For $n \to \infty$,

$$\begin{aligned} b_1(n) &= (1/\log 2 + o(1))^n B_n \\ &= (1.44269\ldots + o(1))^n B_n. \end{aligned}$$

The idea is to change a set system H in a partition P with the same weight. We replace each vertex v in H with an interval of $\deg(v)$ new vertices, 1-1 distributed among the edges which contained v:



Formally, distinct H's with weight n bijectively correspond to the eq. classes $W(n)/\sim$ where

- $W(n)$ is the set of pairs (Q, P) of partitions of $[n]$ such that the blocks of Q are intervals and $|A \cap B| \leq 1$ whenever $A \in Q$ and $B \in P$.

- $(Q_1, P_1) \sim (Q_2, P_2)$ iff $Q_1 = Q_2$ and there is a bijection $f : P_1 \to P_2$ such that A and $f(A)$ always intersect the same intervals of $Q_1 = Q_2$.

In particular, $b_1(n) = |W(n)/\sim|$.

Observation. For $Q = (I_1, I_2, \dots, I_r)$ the eq. class containing (Q, P) has size $\leq |I_1|! \cdot |I_2|! \cdot \ldots \cdot |I_r|!$ and $=$ holds iff H is simple (has no repeated edges).

We denote $i(n,k) = \sum_Q 1/(|I_1|! \cdot \ldots \cdot |I_r|!)$ where $Q = (I_1, \ldots, I_r)$ runs over the interval partition of $[n]$ with $|I_i| \le k$ for all i. Similarly, $i(n)$ is the same sum with no bound on $|I_i|$. The representation of H's by eq. classes implies that, for every fixed k,

$$i(n,k) \cdot B_{n,k} \le b_1(n) \le 2i(n) \cdot B_n$$

($B_{n,k}$ counts k-sparse partitions of $[n]$). The 2 comes from the fact that $i(n) \cdot B_n$ upperbounds only simple H's.

But $i(n,k) = [x^n]1/(1 - \sum_{i=1}^{k} x^i/i!)$ and $i(n) = [x^n]1/(1 - \sum_{i=1}^{\infty} x^i/i!) = [x^n]1/(2 - \mathrm{e}^x)$. So $i(n,k) \sim c_k \alpha_k^{-n}$ and $i(n) \sim c(\log 2)^{-n}$ where $\alpha_k \in (0,1)$ is the root of $1 - x - x^2/2! - \cdots - x^k/k!$. For $k \to \infty$, $\alpha_k \downarrow \log 2$.

Recall that $B_{n,k} = B_{n-k+1} \sim (\mathrm{e} \log n/n)^{k-1} B_n$. Thus $b_1(n) = (1/\log 2 + o(1))^n B_n$.

4. The generating function $B(x)$

A power series $F(x) \in \mathbb{C}[[x]]$ is **differentially algebraic** (DA) if

$$p(x, F, F', F'', \ldots, F^{(r)}) = 0$$

for a nonzero polynomial $p(x, y_0, y_1, \ldots, y_r)$. If $p = p_0(x)y_0 + p_1(x)y_1 + \cdots + p_r(x)y_r$, we call F **D-finite**. If p does not use $y_1, \ldots, y_r$ (i.e., F satisfies a polynomial equation), we say that F is **algebraic**. Finally, if p does not use $y_1, \ldots, y_r$ and is linear in y_0, F is **rational**. It is a hierarchy: DA $\supset$ D-finite $\supset$ algebraic $\supset$ rational.

Examples. $\sum S(n,k)x^n = \frac{x^k}{(1-x)\ldots(1-kx)}$ is rational. $\sum S(n,k)x^n/n! = \frac{(\mathrm{e}^x-1)^k}{k!}$ is D-finite (but not algebraic). $B_e(x) = \sum B_n x^n/n! = \mathrm{e}^{\mathrm{e}^x-1}$ is not D-finite but is DA:

$$B_e''B_e - (B_e')^2 - B_e'B_e = 0.$$

What about $B(x) = \sum B_n x^n$? The Hadamard product with $\mathrm{e}^x = \sum x^n/n!$ shows that it is not D-finite (because $B_e(x)$ is not D-finite).

Theorem (Klazar, 2003). The OGF of Bell numbers $B(x) = \sum_{n\geq 0} B_n x^n$ is not differentially algebraic.

P. (sketch) Take a minimum ADE satisfied by $B(x)$. Use identity $B(x) = 1 + \frac{x}{1-x} \cdot B(x/(1-x))$ to obtain other minimum ADE for $B(x)$. Eliminate between them and get a contradiction. $\square$

We define $C(x)$ by $B(x) = \frac{1}{1-C(x)}$. $B(x) = 1 + \frac{x}{1-x} \cdot B(x/(1-x))$ yields

$$C(x) = \frac{x}{1-(1-x)\cdot C(x/(1-x))}.$$

Iterating this we get a continued fraction expansion for $C(x)$ and thus also for $B(x)$:

$$B(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x-x^2}{1-x-\cfrac{x-2x^2}{1-2x-\cfrac{x-3x^2}{\vdots}}}}}.$$

More compact continued fraction for $B(x)$ was found by Flajolet in 1980:

$$B(x) = \cfrac{1}{1 - x - \cfrac{x^2}{1 - 2x - \cfrac{2x^2}{1 - 3x - \cfrac{3x^2}{\vdots}}}}.$$

One more nice formula for B_n:

$$\det \begin{pmatrix} B_0 & B_1 & \dots & B_n \\ B_1 & B_2 & \dots & B_{n+1} \\ \vdots & \vdots & \dots & \vdots \\ B_n & B_{n+1} & \dots & B_{2n} \end{pmatrix} = 1! \cdot 2! \cdot \ldots \cdot n!$$

(Aigner, 1999).

Another one:

$$\frac{1}{n!}\sum_{k=0}^{n} k! \ \sim \ 1 + \frac{1}{n} + \frac{1}{n^2} + \frac{2}{n^3} + \frac{5}{n^4} + \frac{15}{n^5} + \frac{52}{n^6} + \frac{203}{n^7} + \cdots$$

(Comtet, 1970)

5. Arithmetic properties of Stirling and Bell numbers

Diophantine results for Stirling numbers (Klazar and Luca, 2003). $S(n, k_1), \ldots, S(n, k_m)$ behave like distinct indeterminates:

- $p, q \in \mathbf{Q}[y_1, \ldots, y_m]$, q does not divide p, and $k_1 > k_2 > \ldots > k_m > c$. Then
$$\frac{p(S(n,k_1),\ldots,S(n,k_m))}{q(S(n,k_1),\ldots,S(n,k_m))} \in \mathbf{Z}$$
for only finitely many $n \in \mathbf{N}$.

- $p \in \mathbf{Q}[y_1, \ldots, y_m]$ is not a d-th power, $d > 1$, and $k_1 > k_2 > \ldots > k_m > c$. Then
$$p(S(n, k_1), \ldots, S(n, k_m)) = x^d$$
has only finitely many solutions $(n, x) \in \mathbf{N}^2$.

Proofs use the power sum representation of $S(n, k)$ and build on the results of Corvaja and Zannier (1998), which are based on the subspace theorem that generalizes Roth theorem.

Observation. For any $m \in \mathbf{N}$, $(B_n \bmod m)_{n\geq 0}$ is eventually periodic.

P. Recall that

$$\sum_{n\geq 0} B_n x^n = \sum_{k\geq 0} \frac{x^k}{(1-x)(1-2x)\dots(1-kx)}.$$

Reducing the RHS mod m and denoting $r = (1-x)(1-2x)\dots(1-mx)$, we get

$$\begin{aligned}\sum_{n\geq 0} B_n x^n &\equiv \frac{1}{1-x/r}\sum_{k=0}^{m-1} \frac{x^k}{(1-x)\dots(1-kx)} \\ &= \frac{p(x)}{1-a_1x-a_2x^2-\cdots-a_dx^d}\end{aligned}$$

with $a_i \in \mathbf{Z}$ and $d \leq m$. So

$$B_n \equiv a_1B_{n-1} + a_2B_{n-2} + \cdots + a_dB_{n-d}$$

whenever $n \geq \max(d, \deg(p)+1)$. It follows that $(B_n)_{n\geq 0}$ is eventually periodic mod m. $\square$

Let $\mathcal{M}$ be a class of finite relational structures on positive integers and

$$a(\mathcal{M}, n) = \#(M \in \mathcal{M}, V(M) = [n]).$$

Specker–Blatter theorem (1984) says: If $\mathcal{M}$ uses only unary and/or binary relations and is definable in the *monadic second order logic*, then $(a(\mathcal{M}, n) \bmod m)_{n \geq 0}$ is eventually periodic for every $m \in \mathbf{N}$.

Examples. This applies to Bell numbers ($= \#$ of equivalences on $[n]$), numbers of triangle-free graphs on $[n]$, numbers of connected such graphs, and many other structures.

Recently, Makowsky and Fischer obtained further interesting related results. For example, the Specker-Blatter theorem ceases to hold for quaternary relations (Fischer, 2003).

But more can be said on Bell numbers: $B_{n+p} \equiv B_{n+1} + B_n \bmod p$ (Touchard, 1933) for every n and prime p, which implies that $(B_n)_{n \geq 0}$ is purely periodic mod p with a period T_p dividing $\frac{p^p - 1}{p - 1}$.

Still open conjecture: $\forall p: \ T_p = \frac{p^p - 1}{p - 1}$.

Lunnon, Pleasant and Stephens, 1979:

$$T_p \geq \binom{2p-1}{p-1}.$$

Voloch, 2003: $T_p \geq 2^{2.54p}$.

Shparlinski, 1987:

$$\forall p: \ B_n \equiv 0 \bmod p \text{ for some } n \leq \binom{2p-2}{p-1} + 1$$

and therefore $\omega(B_1 B_2 \ldots B_n) \gg \log n / \log\log n$ and, for infinitely many n, $P(B_n) > \frac{1}{2}\log n$.

Open problem: $P(B_n) \to \infty$ as $n \to \infty$.

Shparlinski, 1991:

$$\forall a, p: \ B_n \equiv a \bmod p \text{ for some } n \leq \binom{2p-1}{p-1}.$$

6. Patterns in partitions

P is a **noncrossing** (NC) partition: there are no four numbers $a < b < c < d$ in $V(P)$ so that $a, c \in A \in P$ and $b, d \in B \in P$, $A \neq B$.

Partition containment. P, Q are partitions. P **contains** Q, $P \succ Q$, if there is an increasing injection $f : V(Q) \to V(P)$ such that x, y are in the same block of $Q \iff f(x), f(y)$ are in the same block of P. So P is a NC partition iff $P \not\succ \{\{1, 3\}, \{2, 4\}\}$.

In the sequential representation: $P = a_1 a_2 \dots a_k$ contains $Q = b_1 b_2 \dots b_l$ iff P has a subsequence representing Q.

Example. $P = a\ b\ c\ c\ b\ b\ a\ a\ d\ d\ a$ is a NC partition because it has no subsequence of the type 1 2 1 2.

See Simion (*Discrete Math.* 217 (2000) 367–409) for a survey on NC partitions.

Theorem. (Simion and Ullman, 1991; Klazar, 1996). # (m-sparse NC partitions of $[n]$ with k blocks) = # ($m-1$-sparse NC partitions of $[n-1]$ with $k-1$ blocks, each block has size ≤ 2). $(n, k, m \geq 2)$

P. (Chen, Deng and Du, 2003) Use the representing graph $G(P) = ([n], E)$ and the operation $\{x, y\} \rightsquigarrow \{x, y-1\}$ where $\{x, y\} \in E, x < y$. The associated mapping $P \mapsto P'$ is a bijection between both classes of NC partitions. $\square$

A few formulas for NC partitions. Denote $N(n, k, m)$ the number of m-sparse NC partitions of $[n]$ with k blocks. Then

$$\begin{aligned} N(n,k,1) &= \tfrac{1}{n-k+1}\tbinom{n}{k}\tbinom{n-1}{k-1} \\ N(n,k,2) &= \tfrac{1}{n-k+1}\tbinom{2n-2k}{n-k}\tbinom{n-1}{2n-2k} \\ N(n,k,3) &= \tfrac{1}{n-k+1}\tbinom{k-1}{2k-n-1}\tbinom{k-2}{2k-n-2} \end{aligned}$$

$N(n, -, 1) = N(2n+1, n+1, 2) = N(-, n+1, 3) = \frac{1}{n+1}\binom{2n}{n}$. (Kreweras, 1972; Gardy and Gouyou-Beauchamps, 1992; Klazar, 1996). No closed formula known for $m > 3$.

Partitions P satisfying $P \not\succ \{\{1,4\},\{2,3\}\}$ may be called **nonnesting** (NN); as a sequence, P has no subsequence of the type 1221.

Example. There are 13 2-sparse NN partitions of $[n]$ with 3 blocks: 12123, 121232, 1212323, 1213, 12132, 121323, 123, 1231, 12312, 123123, 12313, 1232, and 12323.

A few formulas for NN partitions. Denote $M(n,m)$ the number of m-sparse NN partitions of $[n]$. Then

$$\begin{aligned}\sum_{n\geq 0} M(n,1)x^n &= \tfrac{2-7x+3x^2+x\sqrt{1-2x-3x^2}}{2-8x+6x^2-2x^3}\\ \sum_{n\geq 0} M(n,2)x^n &= 1+\tfrac{x}{2}\Big(1+\sqrt{\tfrac{1+x}{1-3x}}\Big)\end{aligned}$$

(Klazar, 1996). The sequence

$$(M(n,2))_{n\geq 2} = 1,2,5,13,35,96,267,\ldots$$

(A005773) counts also (i) directed animals and (ii) words over $\{-1,0,1\}$ of fixed length, which have each initial sum nonnegative (Gouyou-Beauchamps and Viennot, 1988).

More general results. We say that a (typically infinite) set X of partitions is **closed** if $Q \prec P \in X$ always implies $Q \in X$. For $n \in \mathbf{N}$ denote $X(n) = \#(P \in X, V(P) = [n])$.

Theorem (Klazar, 2000). Let X be a closed set of partitions with $\max_{P \in X} |P| \le k$. Then

$$\sum_{n \ge 0} X(n)x^n = \frac{p(x)}{(1-x)^{r_1}(1-2x)^{r_2}\ldots(1-kx)^{r_k}}$$

for some integers $r_i \ge 0$ and an integral polynomial $p(x)$.

Theorem (Klazar, 2003/4). X is a closed set of partitions. Then either (i) $X(n) < c^n$ for a constant $c > 1$ or (ii) $X(n) \ge (n/2)!$.

P. For a permutation $\pi = a_1 a_2 \ldots a_n$, let $P_\pi = \{\{i, n + a_i\} : \ i \in [n]\}$. $\exists$ an antichain Y (to $\prec$) such that $X = \{P : \ P \not\succ Q \ \forall Q \in Y\}$. If a $Q \in Y$ is of the form "singleton blocks + P_π", then (i) holds (by the result of Marcus and Tardos on the SW conjecture). If no $Q \in Y$ is of this form, (ii) holds trivially (each P_π is in X). $\square$