Permutations as hypergraphs

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Overview:

1. Containment of ordered hypergraphs

2. Why this containment?

3. Why permutations?

4. Concrete enumerative results for noncrossing and nonnesting hypergraphs

5. General enumerative results

Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $[n] = \{1, 2, \ldots, n\}$. By a *hypergraph* we mean a collection

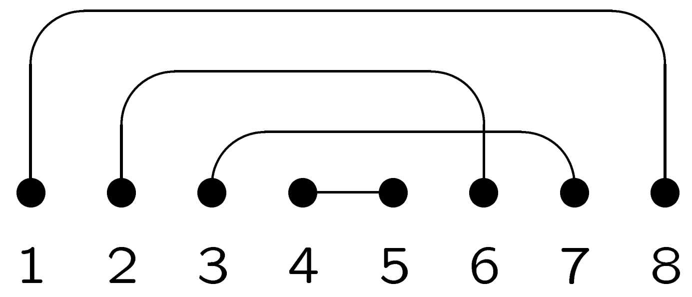
$$H = (E_i : \ i \in I), \ \emptyset \neq E_i \subset \mathbf{N};$$

the *edges* E_i and the *index set* I are finite. We denote $V(H) = \bigcup_{i \in I} E_i$.

$H' = (E'_i : \ i \in I') \prec H = (E_i : \ i \in I)$, H' is **contained in** H, if there are injections $f : V(H') \to V(H)$ and $g : I' \to I$, where f is *increasing* (with respect to $<$), such that $f(E'_i) \subset E_{g(i)}$ for every $i \in I'$.

For a permutation $\pi = a_1 a_2 \ldots a_n$ of $[n]$ we define $G(\pi) = (\{i, n + a_i\} : \ i \in [n])$.

Example. $G(4231)$ is



Why to care about the containment. It was considered before in many instances for several classes of hypergraphs.

Classes of hypergraphs. *Partitions* have pairwise disjoint edges. *Graphs* have only two-element edges. *Matchings* are 1-regular graphs (i.e., the edges are disjoint). *Permutations* are matchings M satisfying $M \not\succ (\{1,2\},\{3,4\})$ (i.e., order-isomorphic to some $G(\pi)$).

Noncrossing hypergraphs are those not containing $G(12) = (\{1,3\},\{2,4\})$. Noncrossing partitions (Kreweras, 1972; Poupard, 1972; survey by Simion, 2000) are thoroughly studied. Enumeration of noncrossing graphs goes back to Cayley and Kirkman; more recent results: Deutsch, Flajolet, Noy, Rogers, ...

Containment of permutations. Let $<^*$ be the usual containment of permutations (Tarjan, 1972; Pratt, 1972; Simion and Schmidt, 1985). $\prec$ generalizes $<^*$: $\pi <^* \sigma$ iff $G(\pi) \prec G(\sigma)$.

Modeling other containments. For two words $u = a_1a_2\ldots a_k$ and $v = b_1b_2\ldots b_l$ over the alphabet $\mathbf{N}$, one defines that u is contained in v if there is a subsequence $b_{i_1}b_{i_2}\ldots b_{i_k}$ in v such that $a_r < a_s \iff b_{i_r} < b_{i_s}$.
We can model this by $\prec$. Associate with u a hypergraph $H(u)$, $u \rightsquigarrow H(u)$, as in this example:

$$22112 \rightsquigarrow (\{0\}, \{-1, 0, 1, 2, 5\}, \{-2, 0, 3, 4\})$$

(translate $V(H(u))$ to $\mathbf{N}$). Then u is contained in v iff $H(u) \prec H(v)$.

What is so special about permutations? Well, $G(\pi)$'s are (more or less) the only forbidden (hypergraph) patterns F for which one can hope for the exponential bound

$$\#(H:\ V(H) = [n], H \text{ simple}, H \not\prec F) < c^n.$$

More precisely, these "exponential" F's are exactly the partitions of the form "$G(\pi)$ plus some singleton edges".

Theorem (Klazar, 2003/4). The exponential bound

$$\#(H:\ V(H)=[n], H \text{ simple}, H \not\succ F) < c^n$$

holds if and only if F is a partition that has no edge of size 3 or more and does not contain $(\{1,2\},\{3,4\})$.

The "only if" direction is trivial. If F is not in the described form, it has two intersecting edges or an edge of size ≥ 3 or it contains $(\{1,2\},\{3,4\})$. In either case, $G(\pi) \not\succ F$ for every π and hence we have on $[n]$ at least $(n/2)!$ simple H's avoiding F.

The "if" direction. If F has the described form, then, in fact, $F \prec G(\pi)$ for some permutation π. So it suffices to prove the exponential bound only for $F = G(\pi)$. We leave the proof now and return to it in the end of our talk.

Results for noncrossing (NC) hypergraphs

The class of **NC permutations** is trivial (just the reversed identity). The number of **NC matchings** with n edges is the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$. A partition of $[n]$ is k-*sparse* if $y-x \geq k$ whenever $y > x$ are two elements of the same block. Denote $N(n,k,m)$ the number of m-sparse **NC partitions** of $[n]$ with k blocks. Then

$$\begin{aligned} N(n,k,1) &= \tfrac{1}{n-k+1}\tbinom{n}{k}\tbinom{n-1}{k-1} \\ N(n,k,2) &= \tfrac{1}{n-k+1}\tbinom{2n-2k}{n-k}\tbinom{n-1}{2n-2k} \\ N(n,k,3) &= \tfrac{1}{n-k+1}\tbinom{k-1}{2k-n-1}\tbinom{k-2}{2k-n-2} \end{aligned}$$

(Found by Kreweras, 1972; Gardy and Gouyou-Beauchamps, 1992; Klazar, 1996). For $m > 3$ probably no closed formula. Specializations give Catalan numbers: $N(n,-,1) = N(2n+1,n+1,2) = N(-,n+1,3) = \frac{1}{n+1}\binom{2n}{n}$.

Link to extremal theory (Davenport-Schinzel sequences): For fixed k, the maximum n such that $N(n,k,2) > 0$ equals $2k-1$.

An identity for NC partitions. (Simion and Ullman, 1991; Klazar, 1996). For every n, k, and $m \geq 2$, # (m-sparse NC partitions of $[n]$ with k blocks) $=$ # ($m-1$-sparse NC partitions of $[n-1]$ with $k-1$ blocks, each block has size ≤ 2).

Example. For $n = 5$, $k = 3$, and $m = 2$ we have $(\{1,5\},\{2,4\},\{3\})$ and $(\{1,3,5\},\{2\},\{4\})$ versus $(\{1,4\},\{2,3\})$ and $(\{1,2\},\{3,4\})$.

P. Simple and nice bijective proof due to W.Y.C. Chen, Deng and Du (2003) uses graphs. More complicated bijective proof using plane trees by Klazar (1998). $\square$

Enumeration of **NC graphs**: Kirkman and Cayley (1870's) and G.N. Watson (1962/3).

$a_n :=$ # (simple NC graphs on $[n]$, possibly with isolated vertices) $= 2^n S_{n-2}$

where S_n are Schröder numbers (Domb and Barrett, 1974; Flajolet and Noy, 1999).

Recall that $\sum_{n\geq 0} S_n x^n = 1 + x + 3x^2 + 11x^3 + 45x^4 + 197x^5 + \cdots = \frac{1}{4x}\Big(1 + x - \sqrt{1 - 6x + x^2}\Big)$. One can count NC graphs also by edges:

$b_n :=$ # (NC *multi*graphs with n edges and no isolated vertices) $= 2^{n-1} S_n$

(Klazar, 2000). Thus the identity $a_{n+2} = 8b_n$. Bijective proof by Podbrdský (2003) and the refinement # $(|V| = k, |E| = n$, multiple edges) $=$ # $(|V| = n+2, |E| = k-2$, isolated vertices) by Noy and Albiol (2003).

NC hypergraphs also can be counted. If $h_n :=$ # (simple NC hypergraphs on $[n]$) and $F := \sum_{n\geq 0} a_n x^n = 1 + x + 5x^2 + 109x^3 + \cdots$, then

$$A_3 F^3 + A_2 F^2 + A_1 F + A_0 = 0$$

where $A_3 = (x+1)^5$, $A_2 = -(x+1)^2(9x^2+4x+3)$, $A_1 = 23x^3 - 7x^2 + 5x + 3$, and $A_0 = 17x^2 - 1$; $h_n \approx (63.97055\ldots)^n$ (Klazar, 2000).

Results for nonnesting (NN) hypergraphs

Nonnesting hypergraphs are those not containing $G(21) = (\{1,4\},\{2,3\})$.

The class of **NN permutations** is trivial (just the identity). The number of **NN matchings** with n edges is again the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$. In fact, we can prove (Klazar and Noy, 2004) that

\# (matchings with n edges and k crossings) =
\# (matchings with n edges and k nestings).

Example. For $n = 3$ there are 15 matchings: $aabbcc$, $ababcc$, $abbacc$, $abbcac$, $\mathbf{abbcca}$, $aabccb$, $abaccb$, $abcacb$, $\mathbf{abccab}$, $abccba$, $aabcbc$, $abacbc$, $abcabc$, $abcbac$, and $\mathbf{abcbca}$. Their distribution for $k = 0, 1, 2, 3$ crossings (or nestings) is 5, 6, 3, 1. The 3 matchings with 2 nestings are marked by the bold type.

One can count **NN graphs**, by vertices and by edges. The results come out exactly the same as for NC graphs! (Klazar, 2000).

Denote $M(n,m)$ the number of m-sparse **NN partitions** of $[n]$. Then

$$\begin{aligned}\sum_{n\geq 0} M(n,1)x^n &= \tfrac{2-7x+3x^2+x\sqrt{1-2x-3x^2}}{2-8x+6x^2-2x^3}\\ \sum_{n\geq 0} M(n,2)x^n &= 1+\tfrac{x}{2}\Big(1+\sqrt{\tfrac{1+x}{1-3x}}\Big)\end{aligned}$$

(Klazar, 1996). In particular, not the same as for NC partitions! The sequence

$$(M(n,2))_{n\geq 2} = 1,2,5,13,35,96,267,\ldots$$

(A005773) counts also (i) directed animals with $n-1$ points and (ii) words over $\{-1,0,1\}$ of length $n-2$, which have each initial sum non-negative (Gouyou-Beauchamps and Viennot, 1988). A simple bijection between (i) and (ii) was given by L.W. Shapiro (1999).

NN hypergraphs? Hard to count, would be interesting to get any results.

More general results. We say that a (typically infinite) set X of partitions is **closed** if $Q \prec P \in X$ always implies $Q \in X$. For $n \in \mathbf{N}$ denote $X(n) = \#(P \in X, V(P) = [n])$.

Example. The set X of all partitions with $\leq k$ blocks is closed and

$$\sum_{n \geq 0} X(n)x^n = \sum_{i=0}^{k} \frac{x^i}{(1-x)(1-2x)\ldots(1-ix)}.$$

Theorem (Klazar, 2000). Let X be any closed set of partitions with $\max_{P \in X} |P| \leq k$. Then

$$\sum_{n \geq 0} X(n)x^n = \frac{p(x)}{(1-x)^{r_1}(1-2x)^{r_2}\ldots(1-kx)^{r_k}}$$

for some integers $r_i \geq 0$ and an integral polynomial $p(x)$.

Let us return to the proof of the bound

$$\#(H:\ V(H) = [n], H \text{ simple}, H \not\succ F) < c^n$$

when $F = G(\pi)$. We sketch the argument.

Step 1. Marcus and G. Tardos (2003) proved that for any fixed π, the simple graphs G on $[n]$ avoiding π (i.e., $G \not\succ G(\pi)$) have only $O(n)$ edges.

Step 2. This extends to hypergraphs: For any fixed π, the simple hypergraphs H on $[n]$ avoiding π have only $O(n)$ edges and even have weight $(= \sum_{i \in I} |E_i|)$ only $O(n)$. This follows by adapting the original argument of Marcus and Tardos or as an application of a theorem (which derives extremal bounds for hypergraphs from those for graphs) due to Klazar (*Eur. J. Combin.*, 2004).

Step 3. By induction, the $O(n)$ bound on weights of simple H's satisfying $V(H) = [n]$ and $H \not\succ G(\pi)$ implies that there is only exponentially many of them. This inductive argument and the implication (which was then conditional) are described in Klazar (*Electr. J. Combin.*, 2000). $\square$