ENUMERATIVE and extremal results for ordered hypergraphs

Martin Klazar

Institute for Theoretical Computer Science
and
Department of Applied Mathematics
Charles University, Prague, Czech Republic

Overview:

1. Containment of ordered hypergraphs and basic (sub)classes of hypergraphs:

   permutations, matchings, graphs, and partitions.

2. Examples of enumerative results for the (sub)classes and one extremal result for hypergraphs.

Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $[n] = \{1, 2, \ldots, n\}$. By a *hypergraph* we mean a finite collection

$$H = (E_i : \ i \in I)$$

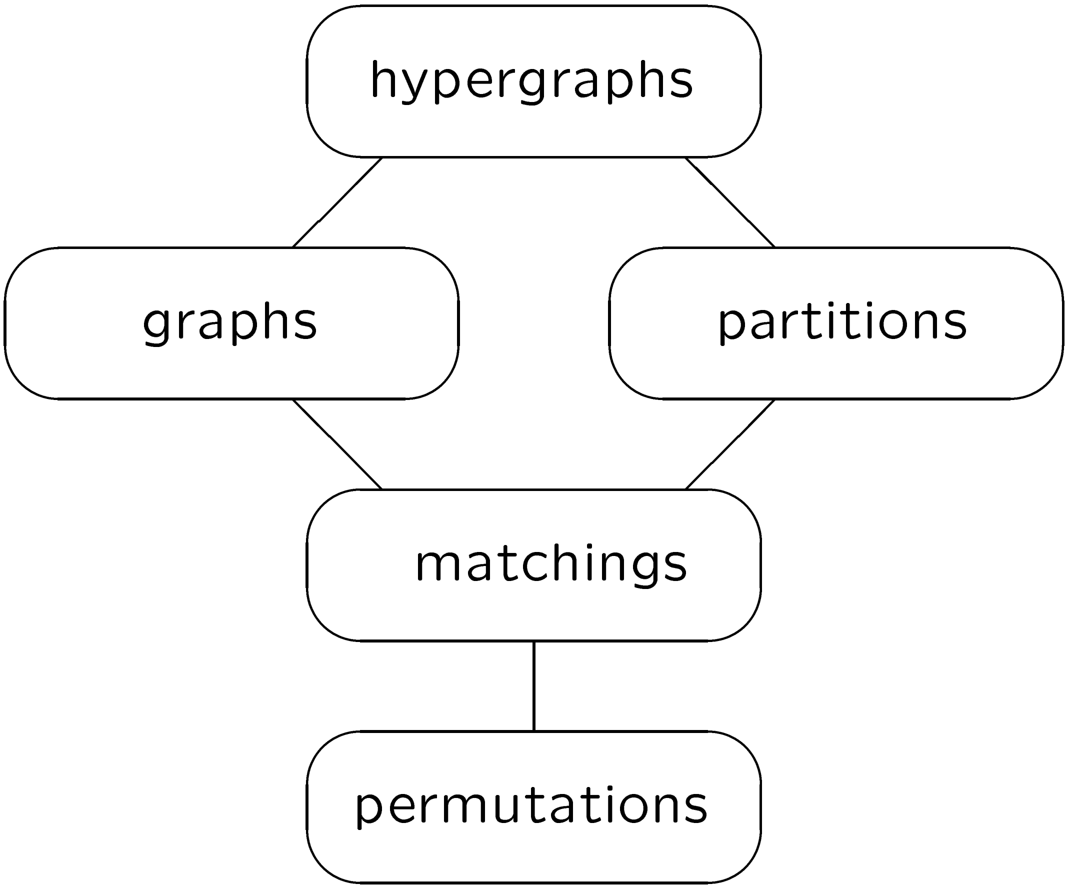
of finite nonempty *edges* $E_i \subset \mathbf{N}$. The *vertex set* is $V(H) = \bigcup_{i \in I} E_i$ (here always $\subset \mathbf{N}$).

$H = (E_i : \ i \in I)$ and $H' = (E'_i : \ i \in I')$ are *order-isomorphic* if there is an *increasing* (to $<$) bijection $f : V(H) \to V(H')$ and a bijection $g : I \to I'$ such that $f(E_i) = E'_{g(i)}$ for every $i \in I$.

$H' \prec H$, H' is *contained* in H, if one can delete edges from H and vertices from the edges so that the resulting hypergraph is order-isomorphic to H'.

Examples. $(\{1\}, \{1\}) \prec H$ iff H has two intersecting edges. $\mathrm{Cr}_2 = (\{1, 3\}, \{2, 4\}) \prec H$ iff $a, c \in E_i$ and $b, d \in E_j$ for 4 vertices $a < b < c < d$ and two edges E_i and E_j, $i \neq j$, in H.

Classes of hypergraphs. $H = (E_i : \ i \in I)$.



Graphs: $|E_i| = 2$ for every i.

Partitions: $E_i \cap E_j = \emptyset$ for every $i \neq j$.

Matchings: graphs $\cap$ partitions.

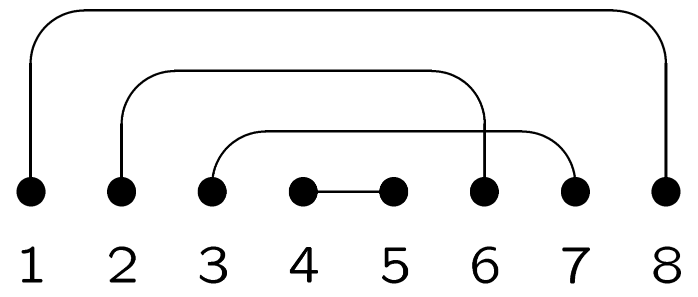
Permutations: matchings $\not\succ (\{1,2\},\{3,4\})$.

1. Enumerative results for permutations. A permutation P is any 1-regular graph that is order-isomorphic to some

$$P(\pi) = (\{i, n + a_i\} : \; i \in [n])$$

where $\pi = a_1 a_2 \dots a_n$ is a permutation of $[n]$.

Example. $P(4231)$ is



Let $S_n(\pi) = \#$ (σ: perm. of $[n]$, $P(\sigma) \not\succ P(\pi)$)

and, more generally,

$S_n(X) = \#$ (σ: perm. of $[n]$, $P(\sigma) \not\succ P(\pi)$ for every $\pi \in X$)

where X is a fixed set of permutations; w.l.o.g. X is an antichain to $\prec$ (it may be infinite).

If π is any of the 6 perm-s $123, \dots, 321$ then

$S_n(\pi) = C_n = \frac{1}{n+1}\binom{2n}{n}$ (Catalan numbers)

(Knuth, 1968; D. G. Rogers, 1978). $S_n(X)$ is determined for every subset $X \subset \{123, \dots, 321\}$ (Simion and Schmidt, 1985). If $\iota_k = 12 \dots k$ is the identical permutation, then for $n \to \infty$

$S_n(\iota_k) \sim cn^{k-k^2/2} \cdot (k-1)^{2n}$ (Regev, 1981).

$\forall k$, sequence $(S_n(\iota_k))_{n \geq 1}$ is P-recursive, i.e., satisfies a linear recurrence with polynomial coefficients (Gessel, 1990; Zeilberger, 1990). Follows also from

$$\sum_{n \geq 0} \frac{S_n(\iota_{k+1})}{n!^2} x^{2n} = \det(I_{|r-s|})_{r,s=1,\dots,k+1}$$

where $I_r(2x) = \sum_{n \geq 0} \frac{x^{2n+r}}{n!(n+r)!}$ (Gessel, 1990).

Another approach, for $k = 4$, via the OGF $\sum S_n(\iota_k) x^n$ due to Bousquet-Mélou (2003).

For X an antichain of perm-s and $n \to \infty$, let

$$c(X) = \text{lim(sup) } S_n(X)^{1/n}.$$

Does the limit always exists? For $|X| = 1$ it does (Arratia, 1999); let $c(\pi) = c(\{\pi\})$.

Examples. $c(\iota_k) = c(12\ldots k) = (k-1)^2$ (by Regev's asymptotics). $c(12453) = 9 + 4\sqrt{2} \approx 14.65685$ (Bóna, 2004).

Let

$$d_k = \min_{|\pi|=k} c(\pi) \text{ and } e_k = \max_{|\pi|=k} c(\pi)$$

$(c(\pi) = \lim S_n(\pi)^{1/n})$. Then

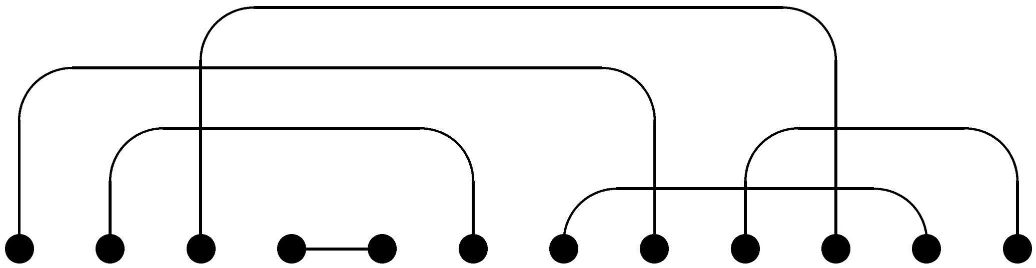
$$\liminf_{k \to \infty} \frac{d_k}{k^2} \geq e^{-3} \approx 0.04978$$

(Valtr, 2003). On the other hand,

$$e_k \leq 15^{2k^4\binom{k^2}{k}}$$

(Marcus and Tardos, 2003); thus $c(X) < +\infty$ for every $X \neq \emptyset$. Is it true that $e_k \leq (k-1)^2$? (asked by Arratia, 1999)

2. Enumerative results for matchings. Matchings M look like



$\#(M \text{ on } [2n]) = (2n-1)!! = 1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)$.

Let $\mathrm{Cr}_k := P(\iota_k)$ be the matching with k pairwise crossing edges and $\mathrm{Ne}_k := P(k, k-1, \ldots, 1)$ ($=k$ pairwise nesting edges). Classic results:

$\#(M \text{ on } [2n]:\ M \not\succ \mathrm{Cr}_2) =$
$\#(M \text{ on } [2n]:\ M \not\succ \mathrm{Ne}_2) = C_n = \frac{1}{n+1}\binom{2n}{n}$

and new ones: $\#(\ldots\ M \not\succ \mathrm{Cr}_3) = \#(\ldots\ M \not\succ \mathrm{Ne}_3) = C_n C_{n+2} - C_{n+1}^2$

(Chen, Deng and Du, 2004; independently but later by Jelínek, 2004). Enumeration of M avoiding Ne_k implicit in the work of Gouyou-Beauchamps (1989).

Let Cr(M) be the # of crossings in M (= the # of copies of Cr$_2$) and similarly for Ne(M). Then

$$\sum_{|M|=n} z^{\mathrm{Cr}(M)} = [x^n]\cfrac{1}{1-\cfrac{x}{1-\cfrac{x(1+z)}{1-\cfrac{x(1+z+z^2)}{\vdots}}}}$$

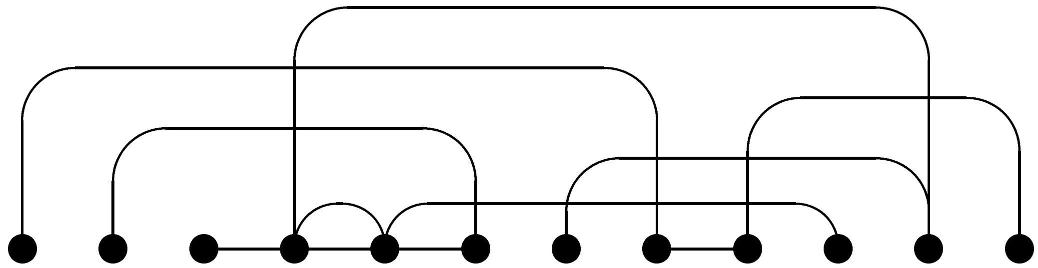
$$= \frac{1}{(1-z)^n}\sum_{k=-n}^{n}(-1)^k\binom{2n}{n+k}z^{k(k-1)/2}$$

(Touchard, 1952; Riordan, 1975).

Example. For $n = 3$ there are 15 matchings: $aabbcc$, $ababcc$, $abbacc$, $abbcac$, $\mathbf{abbcca}$, $aabccb$, $abaccb$, $abcacb$, $\mathbf{abccab}$, $abccba$, $aabcbc$, $abacbc$, $abcabc$, $abcbac$, and $\mathbf{abcbca}$. For $k = 0, 1, 2, 3$, $\#(M : \mathrm{Cr}(M) = k) = 5, 6, 3, 1$ and $\#(M : \mathrm{Ne}(M) = k) = 5, 6, \mathbf{3}, 1$.

This equidistribution of crossings and nestings holds for every n (Klazar and Noy, 2004).

3. Enumerative results for graphs. Graphs G look like



Enumeration of Cr_2-avoiding graphs (i.e., with no two crossing edges) goes back to Kirkman and Cayley (1870's) and G.N. Watson (1962).

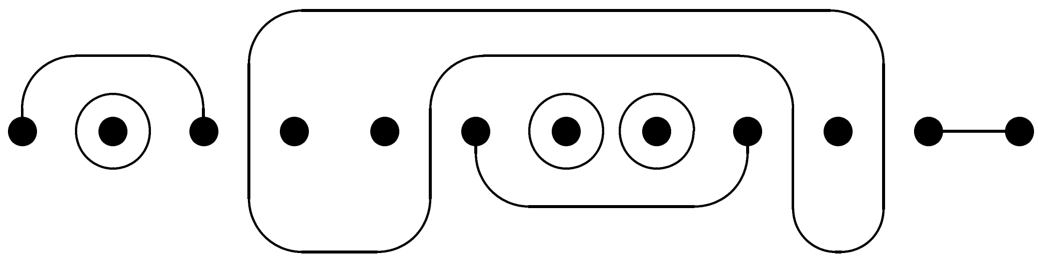
$a_n :=$ # (simple G on $[n]$, $G \not\succ \mathrm{Cr}_2$, possibly with *isolated vertices*) $= 2^n S_{n-2}$

where S_n are Schröder numbers (Domb and Barrett, 1974; Flajolet and Noy, 1999). Recall that $S_n = [x^n]\frac{1}{4x}\left(1 + x - \sqrt{1 - 6x + x^2}\right)$.

$b_n :=$ # (*multi*graphs G with n edges, $G \not\succ \mathrm{Cr}_2$, and no isolated vertex) $= 2^{n-1} S_n$

(Klazar, 2000). Thus the identity $a_{n+2} = 8b_n$.

4. Enumerative results for partitions. Partitions R look like



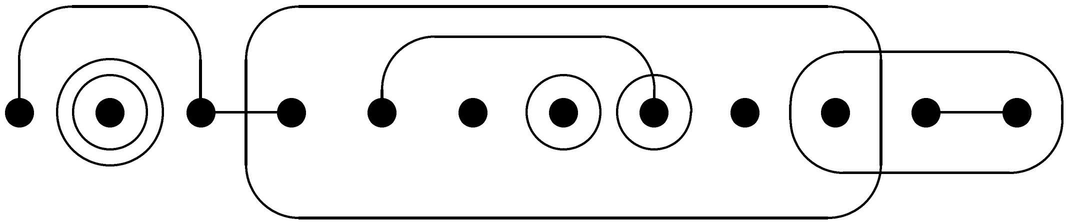
A partition R of $[n]$ is m-*sparse* if for every two elements $y > x$ of the same block we have $y - x \geq m$.

An identity for Cr$_2$**-avoiding partitions.** (Due to Simion and Ullman, 1991; Klazar, 1996):

$\forall n, k, m \geq 2$, # (m-sparse partitions $R \not\succ$ Cr$_2$ of $[n]$ with k blocks) $=$ # ($m-1$- ... $R \not\succ$ Cr$_2$ ... $[n-1]$ ... $k-1$ ..., each block has size ≤ 2).

Example. For $n = 5$, $k = 3$, and $m = 2$ we have $(\{1,5\},\{2,4\},\{3\})$ and $(\{1,3,5\},\{2\},\{4\})$ versus $(\{1,4\},\{2,3\})$ and $(\{1,2\},\{3,4\})$.

5. Enumerative and extremal results for hypergraphs. Hypergraphs H look like



Counting all hypergraphs. The # (simple H with $V(H) = [n]$) is (Comtet, 1974)

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} 2^{2^j - 1} = 1, 5, 109, 32297,$$

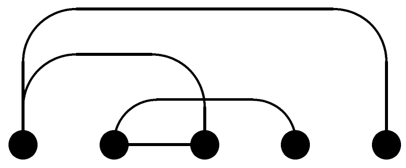
$2147321017, 9223372023970362989, \ldots$

Counting Cr_2**-avoiding hypergraphs.** If $h_n =$ # (simple H with $V(H) = [n]$ and $H \not\succ \mathrm{Cr}_2$) and $F = \sum_{n \geq 0} h_n x^n = 1 + x + 5x^2 + 109x^3 + \cdots$, then

$$A_3 F^3 + A_2 F^2 + A_1 F + A_0 = 0$$

where $A_3 = (x+1)^5$, $A_2 = -(x+1)^2(9x^2+4x+3)$, $A_1 = 23x^3 - 7x^2 + 5x + 3$, and $A_0 = 17x^2 - 1$; $h_n \approx (63.97055\ldots)^n$ (Klazar, 2000).

Let G_F be



that is, $G_F = (\{1,3\},\{1,5\},\{2,3\},\{2,4\})$. If $g(n)$ is the maximum # of edges in a simple graph G with $V(G) = [n]$ and $G \not\succ G_F$, then

$$n \cdot \log n \ll g(n) \ll n \cdot \log n$$

(Füredi, 1990).

If $h(n)$ is the maximum # of edges in a simple hypergraph H with $V(G) = [n]$ and $H \not\succ G_F$, then

$$n \cdot \log n \ll h(n) \ll n \cdot \log^2 n \cdot \log\log^3 n$$

(Klazar, 2004). The same upper bound holds for $\sum_{E \in H} |E|$ in place of $\sum_{E \in H} 1$. Is $h(n) = \Theta(n \cdot \log n)$?