THE THEORY OF THE RIEMANN ZETA-FUNCTION

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I

THE FUNCTION $\zeta(s)$ AND THE DIRICHLET SERIES RELATED TO IT

1.1. Definition of $\zeta(s)$. The Riemann zeta-function $\zeta(s)$ has its origin in the identity expressed by the two formulae

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.1.1}$$

where n runs through all integers, and

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{1.1.2}$$

where p runs through all primes. Either of these may be taken as the definition of $\zeta(s)$; s is a complex variable, $s = \sigma + it$. The Dirichlet series (1.1.1) is convergent for $\sigma > 1$, and uniformly convergent in any finite region in which $\sigma \geqslant 1+\delta$, $\delta > 0$. It therefore defines an analytic function $\zeta(s)$, regular for $\sigma > 1$.

The infinite product is also absolutely convergent for $\sigma > 1$; for so is

$$\sum_p \left|\frac{1}{p^s}\right| = \sum_p \frac{1}{p^\sigma},$$

this being merely a selection of terms from the series $\sum n^{-\sigma}$. If we expand the factor involving p in powers of p^{-s}, we obtain

$$\prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right).$$

On multiplying formally, we obtain the series (1.1.1), since each integer n can be expressed as a product of prime-powers p^m in just one way. The identity of (1.1.1) and (1.1.2) is thus an analytic equivalent of the theorem that the expression of an integer in prime factors is unique.

A rigorous proof is easily constructed by taking first a finite number of factors. Since we can multiply a finite number of absolutely convergent series, we have

$$\prod_{p \leqslant P} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) = 1 + \frac{1}{n_1^s} + \frac{1}{n_2^s} + \dots,$$

where $n_1, n_2, \dots$, are those integers none of whose prime factors exceed P.

Since all integers up to P are of this form, it follows that, if $\zeta(s)$ is defined by (1.1.1),

$$\left|\zeta(s)-\prod_{p\leqslant P}\left(1-\frac{1}{p^s}\right)^{-1}\right| = \left|\zeta(s)-1-\frac{1}{n_1^s}-\frac{1}{n_2^s}-\dots\right|$$
$$\leqslant \frac{1}{(P+1)^\sigma}+\frac{1}{(P+2)^\sigma}+\dots.$$

This tends to 0 as $P\to\infty$, if $\sigma>1$; and (1.1.2) follows.

This fundamental identity is due to Euler, and (1.1.2) is known as Euler's product. But Euler considered it for particular values of s only, and it was Riemann who first considered $\zeta(s)$ as an analytic function of a complex variable.

Since a convergent infinite product of non-zero factors is not zero, we deduce that $\zeta(s)$ *has no zeros for* $\sigma>1$. This may be proved directly as follows. We have for $\sigma>1$

$$\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\dots\left(1-\frac{1}{P^s}\right)\zeta(s) = 1+\frac{1}{m_1^s}+\frac{1}{m_2^s}+\dots,$$

where m_1, m_2,..., are the integers all of whose prime factors exceed P. Hence

$$\left|\left(1-\frac{1}{2^s}\right)\dots\left(1-\frac{1}{P^s}\right)\zeta(s)\right| \geqslant 1-\frac{1}{(P+1)^\sigma}-\frac{1}{(P+2)^\sigma}-\dots>0$$

if P is large enough. Hence $|\zeta(s)|>0$.

The importance of $\zeta(s)$ in the theory of prime numbers lies in the fact that it combines two expressions, one of which contains the primes explicitly, while the other does not. The theory of primes is largely concerned with the function $\pi(x)$, the number of primes not exceeding x. We can transform (1.1.2) into a relation between $\zeta(s)$ and $\pi(x)$; for if $\sigma>1$,

$$\log\zeta(s) = -\sum_p \log\left(1-\frac{1}{p^s}\right) = -\sum_{n=2}^{\infty}\{\pi(n)-\pi(n-1)\}\log\left(1-\frac{1}{n^s}\right)$$
$$= -\sum_{n=2}^{\infty}\pi(n)\left\{\log\left(1-\frac{1}{n^s}\right)-\log\left(1-\frac{1}{(n+1)^s}\right)\right\}$$
$$= \sum_{n=2}^{\infty}\pi(n)\int_n^{n+1}\frac{s}{x(x^s-1)}\,dx = s\int_2^\infty \frac{\pi(x)}{x(x^s-1)}\,dx. \qquad (1.1.3)$$

The rearrangement of the series is justified since $\pi(n)\leqslant n$ and

$$\log(1-n^{-s}) = O(n^{-\sigma}).$$

Again
$$\frac{1}{\zeta(s)} = \prod_p\left(1-\frac{1}{p^s}\right),$$

and on carrying out the multiplication we obtain

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \quad (\sigma>1), \qquad (1.1.4)$$

where $\mu(1)=1$, $\mu(n)=(-1)^k$ if n is the product of k different primes, and $\mu(n)=0$ if n contains any factor to a power higher than the first. The process is easily justified as in the case of $\zeta(s)$.

The function $\mu(n)$ is known as the Möbius function. It has the property

$$\sum_{d|q}\mu(d) = 1\ (q=1), \quad 0\ (q>1), \qquad (1.1.5)$$

where $d\,|\,q$ means that d is a divisor of q. This follows from the identity

$$1 = \sum_{m=1}^{\infty}\frac{1}{m^s}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \sum_{q=1}^{\infty}\frac{1}{q^s}\sum_{d|q}\mu(d).$$

It also gives the 'Möbius inversion formula'

$$g(q) = \sum_{d|q} f(d), \qquad (1.1.6)$$

$$f(q) = \sum_{d|q}\mu\left(\frac{q}{d}\right)g(d), \qquad (1.1.7)$$

connecting two functions $f(n)$, $g(n)$ defined for integral n. If f is given and g defined by (1.1.6), the right-hand side of (1.1.7) is

$$\sum_{d|q}\mu\left(\frac{q}{d}\right)\sum_{r|d}f(r).$$

The coefficient of $f(q)$ is $\mu(1)=1$. If $r<q$, then $d=kr$, where $k\,|\,q/r$. Hence the coefficient of $f(r)$ is

$$\sum_{k|q/r}\mu\left(\frac{q}{kr}\right) = \sum_{k'|q/r}\mu(k') = 0$$

by (1.1.5). This proves (1.1.7). Conversely, if g is given, and f is defined by (1.1.7), then the right-hand side of (1.1.6) is

$$\sum_{d|q}\sum_{r|d}\mu\left(\frac{d}{r}\right)g(r),$$

and this is $g(q)$, by a similar argument. The formula may also be derived formally from the obviously equivalent relations

$$F(s)\zeta(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \qquad F(s) = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{g(n)}{n^s},$$

where

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Again, on taking logarithms and differentiating (1.1.2), we obtain, for $\sigma > 1$,

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= -\sum_p \frac{\log p}{p^s}\left(1-\frac{1}{p^s}\right)^{-1} \\ &= -\sum_p \log p \sum_{m=1}^{\infty} \frac{1}{p^{ms}} \\ &= -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}, \end{aligned} \tag{1.1.8}$$

where $\Lambda(n) = \log p$ if n is p or a power of p, and otherwise $\Lambda(n) = 0$. On integrating we obtain

$$\log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda_1(n)}{n^s} \quad (\sigma > 1), \tag{1.1.9}$$

where $\Lambda_1(n) = \Lambda(n)/\log n$, and the value of $\log \zeta(s)$ is that which tends to 0 as $\sigma \to \infty$, for any fixed t.

1.2. Various Dirichlet series connected with $\zeta(s)$. In the first place

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \quad (\sigma > 1), \tag{1.2.1}$$

where $d(n)$ denotes the number of divisors of n (including 1 and n itself). For

$$\zeta^2(s) = \sum_{\mu=1}^{\infty} \frac{1}{\mu^s} \sum_{\nu=1}^{\infty} \frac{1}{\nu^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\mu\nu=n} 1,$$

and the number of terms in the last sum is $d(n)$. And generally

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \quad (\sigma > 1), \tag{1.2.2}$$

where $k = 2, 3, 4,\ldots$, and $d_k(n)$ denotes the number of ways of expressing n as a product of k factors, expressions with the same factors in a different order being counted as different. For

$$\zeta^k(s) = \sum_{\nu_1=1}^{\infty} \frac{1}{\nu_1^s} \cdots \sum_{\nu_k=1}^{\infty} \frac{1}{\nu_k^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\nu_1\ldots\nu_k=n} 1,$$

and the last sum is $d_k(n)$.

Since we have also

$$\zeta^2(s) = \prod_p \left(1-\frac{1}{p^s}\right)^{-2} = \prod_p \left(1+\frac{2}{p^s}+\frac{3}{p^{2s}}+\ldots\right), \tag{1.2.3}$$

on comparing the coefficients in (1.2.1) and (1.2.3) we verify the elementary formula

$$d(n) = (m_1+1)\ldots(m_r+1) \tag{1.2.4}$$

for the number of divisors of

$$n = p_1^{m_1} p_2^{m_2} \ldots p_r^{m_r}. \tag{1.2.5}$$

Similarly from (1.2.2)

$$d_k(n) = \frac{(k+m_1-1)!}{m_1!\,(k-1)!} \cdots \frac{(k+m_r-1)!}{m_r!\,(k-1)!}. \tag{1.2.6}$$

We next note the expansions

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} \quad (\sigma > 1), \tag{1.2.7}$$

where $\mu(n)$ is the coefficient in (1.1.4);

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\nu(n)}}{n^s} \quad (\sigma > 1), \tag{1.2.8}$$

where $\nu(n)$ is the number of different prime factors of n;

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} \quad (\sigma > 1), \tag{1.2.9}$$

and

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\{d(n)\}^2}{n^s} \quad (\sigma > 1). \tag{1.2.10}$$

To prove (1.2.7), we have

$$\frac{\zeta(s)}{\zeta(2s)} = \prod_p \frac{1-p^{-2s}}{1-p^{-s}} = \prod_p \left(1+\frac{1}{p^s}\right),$$

and this differs from the formula for $1/\zeta(s)$ only in the fact that the signs are all positive. The result is therefore clear. To prove (1.2.8), we have

$$\begin{aligned} \frac{\zeta^2(s)}{\zeta(2s)} &= \prod_p \frac{1-p^{-2s}}{(1-p^{-s})^2} = \prod_p \frac{1+p^{-s}}{1-p^{-s}} \\ &= \prod_p (1+2p^{-s}+2p^{-2s}+\ldots), \end{aligned}$$

and the result follows. To prove (1.2.9),

$$\frac{\zeta^3(s)}{\zeta(2s)} = \prod_p \frac{1-p^{-2s}}{(1-p^{-s})^3} = \prod_p \frac{1+p^{-s}}{(1-p^{-s})^2}$$
$$= \prod_p \{(1+p^{-s})(1+2p^{-s}+3p^{-2s}+\dots)\}$$
$$= \prod_p \{1+3p^{-s}+\dots+(2m+1)p^{-ms}+\dots\},$$

and the result follows, since, if n is (1.2.5),

$$d(n^2) = (2m_1+1)\dots(2m_r+1).$$

Similarly

$$\frac{\zeta^4(s)}{\zeta(2s)} = \prod_p \frac{1-p^{-2s}}{(1-p^{-s})^4} = \prod_p \frac{1+p^{-s}}{(1-p^{-s})^3}$$
$$= \prod_p (1+p^{-s})\{1+3p^{-s}+\dots+\tfrac{1}{2}(m+1)(m+2)p^{-ms}+\dots\}$$
$$= \prod_p \{1+4p^{-s}+\dots+(m+1)^2p^{-ms}+\dots\},$$

and (1.2.10) follows.

Other formulae are

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \quad (\sigma > 1), \tag{1.2.11}$$

where $\lambda(n) = (-1)^r$ if n has r prime factors, a factor of degree k being counted k times;

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} \quad (\sigma > 2), \tag{1.2.12}$$

where $\phi(n)$ is the number of numbers less than n and prime to n; and

$$\frac{1-2^{1-s}}{1-2^{-s}}\zeta(s-1) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad (\sigma > 2), \tag{1.2.13}$$

where $a(n)$ is the greatest odd divisor of n. Of these, (1.2.11) follows at once from

$$\frac{\zeta(2s)}{\zeta(s)} = \prod_p \left(\frac{1-p^{-s}}{1-p^{-2s}}\right) = \prod_p \left(\frac{1}{1+p^{-s}}\right) = \prod_p (1-p^{-s}+p^{-2s}-\dots).$$

Also

$$\frac{\zeta(s-1)}{\zeta(s)} = \prod_p \left(\frac{1-p^{-s}}{1-p^{1-s}}\right) = \prod_p \left\{\left(1-\frac{1}{p^s}\right)\left(1+\frac{p}{p^s}+\frac{p^2}{p^{2s}}+\dots\right)\right\}$$
$$= \prod_p \left\{1+\left(1-\frac{1}{p}\right)\left(\frac{p}{p^s}+\frac{p^2}{p^{2s}}+\dots\right)\right\},$$

and (1.2.12) follows, since, if $n = p_1^{m_1}\dots p_r^{m_r}$,

$$\phi(n) = n\left(1-\frac{1}{p_1}\right)\dots\left(1-\frac{1}{p_r}\right).$$

Finally

$$\frac{1-2^{1-s}}{1-2^{-s}}\zeta(s-1) = \frac{1-2^{1-s}}{1-2^{-s}}\prod_p \frac{1}{1-p^{1-s}}$$
$$= \frac{1}{1-2^{-s}}\,\frac{1}{1-3^{1-s}}\,\frac{1}{1-5^{1-s}}\dots$$
$$= \left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\dots\right)\left(1+\frac{3}{3^s}+\frac{3^2}{3^{2s}}+\dots\right)\dots,$$

and (1.2.13) follows.

Many of these formulae are, of course, simply particular cases of the general formula

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left\{1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\dots\right\},$$

where $f(n)$ is a multiplicative function, i.e. is such that, if $n = p_1^{m_1}p_2^{m_2}\dots$, then

$$f(n) = f(p_1^{m_1})f(p_2^{m_2})\dots.$$

Again, let $f_k(n)$ denote the number of representations of n as a product of k factors, each greater than unity when $n > 1$, the order of the factors being essential. Then clearly

$$\sum_{n=2}^{\infty} \frac{f_k(n)}{n^s} = \{\zeta(s)-1\}^k \quad (\sigma > 1). \tag{1.2.14}$$

Let $f(n)$ be the number of representations of n as a product of factors greater than unity, representations with factors in a different order being considered as distinct; and let $f(1) = 1$. Then

$$f(n) = \sum_{k=1}^{\infty} f_k(n).$$

Hence

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = 1+\sum_{k=1}^{\infty}\{\zeta(s)-1\}^k = 1+\frac{\zeta(s)-1}{1-\{\zeta(s)-1\}}$$
$$= \frac{1}{2-\zeta(s)}. \tag{1.2.15}$$

It is easily seen that $\zeta(s) = 2$ for $s = \alpha$, where α is a real number greater than 1; and $|\zeta(s)| < 2$ for $\sigma > \alpha$, so that (1.2.15) holds for $\sigma > \alpha$.

1.3. Sums involving $\sigma_a(n)$. Let $\sigma_a(n)$ denote the sum of the ath powers of the divisors of n. Then

$$\zeta(s)\zeta(s-a) = \sum_{\mu=1}^{\infty}\frac{1}{\mu^s}\sum_{\nu=1}^{\infty}\frac{\nu^a}{\nu^s} = \sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{\mu\nu=n}\nu^a,$$

i.e.

$$\zeta(s)\zeta(s-a) = \sum_{n=1}^{\infty}\frac{\sigma_a(n)}{n^s} \quad (\sigma > 1,\ \sigma > \mathbf{R}(a)+1). \tag{1.3.1}$$

Since the left-hand side is, if $a \neq 0$,

$$\prod_p\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\ldots\right)\left(1+\frac{p^a}{p^s}+\frac{p^{2a}}{p^{2s}}+\ldots\right)$$

$$= \prod_p\left(1+\frac{1+p^a}{p^s}+\frac{1+p^a+p^{2a}}{p^s}+\ldots\right) = \prod_p\left(1+\frac{1-p^{2a}}{1-p^a}\frac{1}{p^s}+\ldots\right)$$

we have

$$\sigma_a(n) = \frac{1-p_1^{(m_1+1)a}}{1-p_1^a}\cdots\frac{1-p_r^{(m_r+1)a}}{1-p_r^a}, \tag{1.3.2}$$

if n is (1.2.5), as is also obvious from elementary considerations.

The formula†

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty}\frac{\sigma_a(n)\sigma_b(n)}{n^s} \tag{1.3.3}$$

is valid for $\sigma > \max\{1, \mathbf{R}(a)+1, \mathbf{R}(b)+1, \mathbf{R}(a+b)+1\}$. The left-hand side is equal to

$$\prod_p \frac{1-p^{-2s+a+b}}{(1-p^{-s})(1-p^{-s+a})(1-p^{-s+b})(1-p^{-s+a+b})}.$$

Putting $p^{-s} = z$, the partial-fraction formula gives

$$\frac{1-p^{a+b}z^2}{(1-z)(1-p^az)(1-p^bz)(1-p^{a+b}z)}$$

$$= \frac{1}{(1-p^a)(1-p^b)}\left\{\frac{1}{1-z}-\frac{p^a}{1-p^az}-\frac{p^b}{1-p^bz}+\frac{p^{a+b}}{1-p^{a+b}z}\right\}$$

$$= \frac{1}{(1-p^a)(1-p^b)}\sum_{m=0}^{\infty}(1-p^{(m+1)a}-p^{(m+1)b}+p^{(m+1)(a+b)})z^m$$

$$= \frac{1}{(1-p^a)(1-p^b)}\sum_{m=0}^{\infty}(1-p^{(m+1)a})(1-p^{(m+1)b})z^m.$$

† Ramanujan (2), B. M. Wilson (1).

Hence

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \prod_p\sum_{m=0}^{\infty}\frac{1-p^{(m+1)a}}{1-p^a}\frac{1-p^{(m+1)b}}{1-p^b}\frac{1}{p^{ms}},$$

and the result follows from (1.3.2). If $a = b = 0$, (1.3.3) reduces to (1.2.10).

Similar formulae involving $\sigma_a^{(q)}(n)$, the sum of the ath powers of those divisors of n which are qth powers of integers, have been given by Crum (1).

1.4. It is also easily seen that, if $f(n)$ is multiplicative, and

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$

is a product of zeta-functions such as occurs in the above formulae, and k is a given positive integer, then

$$\sum_{n=1}^{\infty}\frac{f(kn)}{n^s}$$

can also be summed. An example will illustrate this point. The function $\sigma_a(n)$ is 'multiplicative', i.e. if m is prime to n

$$\sigma_a(mn) = \sigma_a(m)\sigma_a(n).$$

Hence

$$\sum_{n=1}^{\infty}\frac{\sigma_a(n)}{n^s} = \prod_p\sum_{m=0}^{\infty}\frac{\sigma_a(p^m)}{p^{ms}},$$

and, if $k = \prod p^l$,

$$\sum_{n=1}^{\infty}\frac{\sigma_a(kn)}{n^s} = \prod_p\sum_{m=0}^{\infty}\frac{\sigma_a(p^{l+m})}{p^{ms}}.$$

Hence

$$\sum_{n=1}^{\infty}\frac{\sigma_a(kn)}{n^s} = \zeta(s)\zeta(s-a)\prod_{p|k}\left\{\sum_{m=0}^{\infty}\frac{\sigma_a(p^{l+m})}{p^{ms}}\Big/\sum_{m=0}^{\infty}\frac{\sigma_a(p^m)}{p^{ms}}\right\}.$$

Now if $a \neq 0$,

$$\sum_{m=0}^{\infty}\frac{\sigma_a(p^{l+m})}{p^{ms}} = \sum_{m=0}^{\infty}\frac{1-p^{(l+m+1)a}}{(1-p^a)p^{ms}} = \frac{1-p^{a-s}-p^{(l+1)a}+p^{(l+1)a-s}}{(1-p^a)(1-p^{-s})(1-p^{a-s})}.$$

Hence

$$\sum_{n=0}^{\infty}\frac{\sigma_a(kn)}{n^s} = \zeta(s)\zeta(s-a)\prod_{p|k}\frac{1-p^{a-s}-p^{(l+1)a}+p^{(l+1)a-s}}{1-p^a}. \tag{1.4.1}$$

Making $a \to 0$,

$$\sum_{n=0}^{\infty}\frac{d(kn)}{n^s} = \zeta^2(s)\prod_{p|k}(l+1-lp^{-s}). \tag{1.4.2}$$

1.5. Ramanujan's sums.† Let

$$c_k(n) = \sum_h e^{-2nh\pi i/k} = \sum_h \cos\frac{2nh\pi}{k}, \tag{1.5.1}$$

where h runs through all positive integers less than and prime to k. Many formulae involving these sums were proved by Ramanujan.

We shall first prove that

$$c_k(n) = \sum_{d|k, d|n} \mu\left(\frac{k}{d}\right) d. \tag{1.5.2}$$

The sum
$$\eta_k(n) = \sum_{m=0}^{k-1} e^{-2nm\pi i/k}$$
is equal to k if $k|n$ and 0 otherwise. Denoting by (r, d) the highest common factor of r and d, so that $(r, d) = 1$ means that r is prime to d,

$$\sum_{d|k} c_d(n) = \sum_{d|k} \sum_{(r,d)=1, r<d} e^{-2nr\pi i/d} = \eta_k(n).$$

Hence by the inversion formula of Möbius (1.1.7)

$$c_k(n) = \sum_{d|k} \mu\left(\frac{k}{d}\right) \eta_d(n),$$

and (1.5.2) follows. In particular

$$c_k(1) = \mu(k). \tag{1.5.3}$$

The result can also be written

$$c_k(n) = \sum_{dr=k, d|n} \mu(r) d.$$

Hence
$$\frac{c_k(n)}{k^s} = \sum_{dr=k, d|n} \frac{\mu(r)}{r^s} d^{1-s}.$$

Summing with respect to k, we remove the restriction on r, which now assumes all positive integral values. Hence‡

$$\sum_{k=1}^{\infty} \frac{c_k(n)}{k^s} = \sum_{r, d|n} \frac{\mu(r)}{r^s} d^{1-s} = \frac{\sigma_{1-s}(n)}{\zeta(s)}, \tag{1.5.4}$$

the series being absolutely convergent for $\sigma > 1$ since $|c_k(n)| \leqslant \sigma_1(n)$, by (1.5.2).

We have also

$$\sum_{n=1}^{\infty} \frac{c_k(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|k\, d|n} \mu\left(\frac{k}{d}\right) d$$
$$= \sum_{d|k} \mu\left(\frac{k}{d}\right) d \sum_{m=1}^{\infty} \frac{1}{(md)^s} = \zeta(s) \sum_{d|k} \mu\left(\frac{k}{d}\right) d^{1-s}. \tag{1.5.5}$$

† Ramanujan (3), Hardy (5).
‡ Two more proofs are given by Hardy, *Ramanujan*, 137–41.

We can also sum series of the form†

$$\sum_{n=1}^{\infty} \frac{c_k(n) f(n)}{n^s},$$

where $f(n)$ is a multiplicative function. For example,

$$\sum_{n=1}^{\infty} \frac{c_k(n) d(n)}{n^s} = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \sum_{\delta|k, \delta|n} \delta\mu\left(\frac{k}{\delta}\right)$$
$$= \sum_{\delta|k} \delta\mu\left(\frac{k}{\delta}\right) \sum_{m=1}^{\infty} \frac{d(m\delta)}{(m\delta)^s}$$
$$= \zeta^2(s) \sum_{\delta|k} \delta^{1-s} \mu\left(\frac{k}{\delta}\right) \prod_{p|\delta} (l+1-lp^{-s})$$

if $\delta = \prod p^l$. If $k = \prod p^\lambda$ the sum is

$$k^{1-s} \prod_{p|k} (\lambda+1-\lambda p^{-s}) - \sum_{p|k} \left(\frac{k}{p}\right)^{1-s} \{\lambda-(\lambda-1)p^{-s}\} \prod_{p'|k, p' \neq p} (\lambda+1-\lambda p'^{-s}) +$$
$$+ \sum_{pp'|k} \left(\frac{k}{pp'}\right)^{1-s} \{\lambda-(\lambda-1)p^{-s}\}\{\lambda-(\lambda-1)p'^{-s}\} \prod_{p''|k, p'' \neq p, p'} (\lambda+1-\lambda p''^{-s}) - \dots$$
$$= k^{1-s} \prod_{p|k} \left\{(\lambda+1-\lambda p^{-s}) - \frac{1}{p^{1-s}}\{\lambda-(\lambda-1)p^{-s}\}\right\}$$
$$= k^{1-s} \prod_{p|k} \left\{1 - \frac{1}{p} + \lambda\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{1-s}}\right)\right\}.$$

Hence

$$\sum_{n=1}^{\infty} \frac{c_k(n) d(n)}{n^s} = \zeta^2(s) k^{1-s} \prod_{p|k} \left\{1-\frac{1}{p}+\lambda\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{1-s}}\right)\right\}. \tag{1.5.6}$$

We can also sum
$$\sum_{n=1}^{\infty} \frac{c_k(qn) f(n)}{n^s}.$$

For example, in the simplest case $f(n) = 1$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\delta|k, \delta|qn} \delta\mu\left(\frac{k}{\delta}\right).$$

For given δ, n runs through those multiples of δ/q which are integers. If δ/q in its lowest terms is δ_1/q_1, these are the numbers $\delta_1, 2\delta_1, \dots$. Hence the sum is

$$\sum_{\delta|k} \delta\mu\left(\frac{k}{\delta}\right) \sum_{r=1}^{\infty} \frac{1}{(r\delta_1)^s} = \zeta(s) \sum_{\delta|k} \delta\mu\left(\frac{k}{\delta}\right) \delta_1^{-s}.$$

† Crum (1).

Since $\delta_1 = \delta/(q,\delta)$, the result is

$$\sum_{n=1}^{\infty} \frac{c_k(qn)}{n^s} = \zeta(s) \sum_{\delta|k} \delta^{1-s}\mu\left(\frac{k}{\delta}\right)(q,\delta)^s. \tag{1.5.7}$$

1.6. There is another class of identities involving infinite series of zeta-functions. The simplest of these is†

$$\sum_p \frac{1}{p^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns). \tag{1.6.1}$$

We have
$$\log \zeta(s) = \sum_m \sum_p \frac{1}{mp^{ms}} = \sum_{m=1}^{\infty} \frac{P(ms)}{m},$$

where $P(s) = \sum p^{-s}$. Hence

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{m=1}^{\infty} \frac{P(mns)}{m} = \sum_{r=1}^{\infty} \frac{P(rs)}{r} \sum_{n|r} \mu(n),$$

and the result follows from (1.1.5).

A closely related formula is

$$\sum_{n=1}^{\infty} \frac{\nu(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns), \tag{1.6.2}$$

where $\nu(n)$ is defined under (1.2.8). This follows at once from (1.6.1) and the identity

$$\sum_{n=1}^{\infty} \frac{\nu(n)}{n^s} = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_p \frac{1}{p^s}.$$

Denoting by $b(n)$ the number of divisors of n which are primes or powers of primes, another identity of the same class is

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log \zeta(ns), \tag{1.6.3}$$

where $\phi(n)$ is defined under (1.2.12). For the left-hand side is equal to

$$\sum_{m=1}^{\infty} \frac{1}{m^s} \sum_p \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots\right),$$

and the series on the right is

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \sum_{m=1}^{\infty} \sum_p \frac{1}{mp^{mns}} = \sum_p \sum_\nu \frac{1}{\nu p^{\nu s}} \sum_{n|\nu} \phi(n).$$

Since
$$\sum_{n|\nu} \phi(n) = \nu,$$
the result follows.

† See Landau and Walfisz (1), Estermann (1), (2).

II

THE ANALYTIC CHARACTER OF $\zeta(s)$, AND THE FUNCTIONAL EQUATION

2.1. Analytic continuation and the functional equation, first method. Each of the formulae of Chapter I is proved on the supposition that the series or product concerned is absolutely convergent. In each case this restricts the region where the formula is proved to be valid to a half-plane. For $\zeta(s)$ itself, and in all the fundamental formulae of § 1.1, this is the half-plane $\sigma > 1$.

We have next to inquire whether the analytic function $\zeta(s)$ can be continued beyond this region. The result is

THEOREM 2.1. *The function $\zeta(s)$ is regular for all values of s except $s = 1$, where there is a simple pole with residue 1. It satisfies the functional equation*

$$\zeta(s) = 2^s \pi^{s-1} \sin \tfrac{1}{2}s\pi\, \Gamma(1-s)\zeta(1-s). \tag{2.1.1}$$

This can be proved in a considerable variety of different ways, some of which will be given in later sections. We shall first give a proof depending on the following summation formula.

Let $\phi(x)$ be any function with a continuous derivative in the interval $[a, b]$. Then, if $[x]$ denotes the greatest integer not exceeding x,

$$\sum_{a<n\leqslant b} \phi(n) = \int_a^b \phi(x)\, dx + \int_a^b (x-[x]-\tfrac{1}{2})\phi'(x)\, dx + (a-[a]-\tfrac{1}{2})\phi(a) - (b-[b]-\tfrac{1}{2})\phi(b). \tag{2.1.2}$$

Since the formula is plainly additive with respect to the interval $(a, b]$ it suffices to suppose that $n \leqslant a < b \leqslant n+1$. One then has

$$\int_a^b (x-n-\tfrac{1}{2})\phi'(x)\, dx = (b-n-\tfrac{1}{2})\phi(b) - (a-n-\tfrac{1}{2})\phi(a) - \int_a^b \phi(x)dx,$$