Matematické struktury lecture and tutorial 7 on April 3, 2017: basic topological notions and examples and Kleene's first recursion theorem

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Basic topological notions and examples

I went through the beginning of Chapter 5 "Topologie" of the lecture notes

- A. Pultr, *Matematické struktury*, 2005, 155 pp., available at `http://kam.mff.cuni.cz/~pultr/`

up to definition 3.1 of continuous maps, pp. 95–101. I left as an exercise the following (solution is available in the lecture notes):

Exercise. *Show that introducing topology via neighborhoods and via open sets is equivalent in the sense that if we define*

$$(X, \{\mathcal{U}(x) \mid x \in X\}) \rightsquigarrow (X, \tau) \rightsquigarrow (X, \{\mathcal{V}(x) \mid x \in X\})$$

then $\mathcal{V}(x) = \mathcal{U}(x)$ *for every* $x \in X$*, and if we define*

$$(X, \tau) \rightsquigarrow (X, \{\mathcal{U}(x) \mid x \in X\}) \rightsquigarrow (X, \sigma)$$

then $\sigma = \tau$.

Also, I mentioned one thing that is not in the lecture notes, characterization of set systems that are bases of topologies. Every set system on X is a subbase of a topology on X but this is not true for bases.

Exercise. *Suppose* X *is a set and* $\mathcal{B} \subseteq \exp(X)$ *is a set system on* X*. Prove that* $\mathcal{B}$ *is a base of a topology on* X *(namely one whose open sets are exactly all unions of elements of* $\mathcal{B}$*) if and only if*

1. $\bigcup \mathcal{B} = X$ *and*

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2. for every $a \in U \cap V$, $U, V \in \mathcal{B}$, there is a $W \in \mathcal{B}$ with $a \in W \subseteq U \cap V$.

In other words, we can express X and every intersection of two members of $\mathcal{B}$ as a union of elements of $\mathcal{B}$.

Thus in example 2.8 "Intervalová topologie" it is not quite true that for every linearly ordered set $(X, \leq)$ the set system $\mathcal{B} = \{(a, b) \mid a, b \in X, a < b\}$ (with $(a, b) = \{x \in X \mid a < x < b\}$) is a base of a topology on X — this holds if (and only if) X has neither minimum nor maximum. Or we do get base of a topology for every linear order $(X, \leq)$ if we add to $\mathcal{B}$ the sets $(-\infty, a)$ and $(a, +\infty)$, $a \in X$, with obvious definition, this is called the *order topology on X*.

Kleene's first recursion theorem

This is an application of Bourbaki's fixed point theorem, see pp. 41/42 of the lecture notes. I naturally mentioned *Kleene's second recursion theorem,* more precisely its special case called *Rogers' fixed point theorem.* Suppose φ_e, $e \in \mathbb{N}$, is an admissible enumeration of all partial computable (recursive) functions, i.e. an effective enumeration of all computer programs (Turing machines) that compute functions from $\mathbb{N}^k$ to $\mathbb{N}$, which may not halt on some inputs and thus the domains of definition are subsets of $\mathbb{N}^k$. Clearly, there is a tremendous redundancy in the list φ_e, $e \in \mathbb{N}$, in the sense that a particular partial computable function is computed by φ_e for many different indices e. Is it true that $\varphi_n = \varphi_{n+1}$ (as partial functions) for some $n \in \mathbb{N}$? Yes, it is and much more holds.

Theorem 1 (Rogers' fixed point theorem) *For every total (i.e. everywhere defined) computable function $f : \mathbb{N} \to \mathbb{N}$ there is an $n \in \mathbb{N}$ such that*

$$\varphi_n = \varphi_{f(n)} \text{ (as partial functions, not as programs, of course)} .$$

Exercise. *Strengthen this to: for every total computable function $f : \mathbb{N} \to \mathbb{N}$ and every $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that*

$$\varphi_n = \varphi_{f(n)} \text{ and } n > m .$$

See for example the Wikipedia entry for the proof of the theorem. For more information on Kleene's second recursion theorem see the article

- Y. N. Moschovakis, Kleene's amazing second recursion theorem. Extended abstract, 16 pp., available on the web,

or the full version Y. N. Moschovakis, Kleene's amazing second recursion theorem, *Bull. Symbolic Logic* **16** (2010), 189–239, which is not so easily available.