Matematické struktury tutorial 4 on March 13, 2017: finishing the proof of Müller's theorem and a proof of the Cantor–Bernstein theorem by a fix-point theorem

Martin Klazar*

March 15, 2017

Recall PIE (the principle of inclusion and exclusion): if $X_1,\dots,X_n\subseteq X$ are finite sets then

$$\left|X\backslash\bigcup_{i=1}^{n}X_i\right|=\sum_{I\subseteq[n]}(-1)^{|I|}\left|\bigcap_{i\in I}X_i\right|$$

where $[n]=\{1,2,\dots,n\}$ and for $I=\emptyset$ the intersection is interpreted as X.

Exercise. *Prove this identity by a double counting argument.*

We introduce this notation: if K,L are graphs and $D\subseteq E(K)$ then

$$\mathrm{inj}(K,L)=\#\{f:\ V(K)\to V(L)\mid f\text{ inj.},\forall\,e\in E(K):\ f(e)\in E(L)\}$$

and $\mathrm{inj}_D(K,L)$ is

$$\#\{f:\ V(K)\to V(L)\mid f\text{ inj.},\forall\,e\in E(K):\ f(e)\in E(L)\iff e\notin D\}$$

("inj." abbreviates "injective"). Thus $\mathrm{inj}(K,L)=\mathrm{inj}_{\emptyset}(K,L)$ and $\mathrm{inj}_D(K,L)$ counts injections from $V(K)$ to $V(L)$ that send edges in D outside $E(L)$, and the remaining edges in $E(K)$ to $E(L)$.

We denote by bar the complementary graph, so if $H=(V,F)$ then $\overline{H}=(V,\binom{V}{2}\backslash F)$. We return to our graphs G and H on the same vertex set V and with m-element edge sets $E,F\subseteq\binom{V}{2}$ (and equal decks). PIE for any fixed subset $D\subseteq E$ gives equality:

$$\mathrm{inj}_D(G,\overline{H})=\sum_{A\subseteq E\backslash D}(-1)^{|A|}\mathrm{inj}((V,D\cup A),H)\ .$$

Exercise. *What are the $X_1,\dots,X_n$ and X in this application of PIE?*

*klazar@kam.mff.cuni.cz

For integers d, $0 \le d \le m$, we set

$$\mathrm{inj}_d(G, \overline{H}) = \sum_{D \subseteq E,\, |D| = d} \mathrm{inj}_D(G, \overline{H}) \;.$$

Summing the above equalities over all $D \subseteq E$ with d elements we get equality

$$\mathrm{inj}_d(G, \overline{H}) = \sum_{B \subseteq E,\, |B| \ge d} (-1)^{|B|-d} \binom{|B|}{d} \mathrm{inj}((V, B), H) \;.$$

Here the binomial coefficient $\binom{|B|}{d}$ just counts d-element subsets D in a fixed set $B = D \cup A$. For H in place of G we have a similar equality

$$\mathrm{inj}_d(H, \overline{H}) = \sum_{B \subseteq F,\, |B| \ge d} (-1)^{|B|-d} \binom{|B|}{d} \mathrm{inj}((V, B), H) \;.$$

The lemma proved by double counting at the end of the previous lecture gives a bijection

$$\gamma :\; \{B \subseteq E \mid d \le |B| < m\} \to \{B \subseteq F \mid d \le |B| < m\}$$

with the property that always $(V, B) \cong (V, \gamma(B))$ — for any fixed $A \subseteq E$, $A \ne E$, we can pair A-copies in (V, E) with those in (V, F) because their numbers are the same. Then, trivially, always $|B| = |\gamma(B)|$ and $\mathrm{inj}((V, B), H) = \mathrm{inj}((V, \gamma(B)), H)$. Thus except for the last terms with $B = E$ and $B = F$, the summands on the right sides of both equalities coincide and differ at most by order. Subtracting the two equalities we thus get

$$\mathrm{inj}_d(G, \overline{H}) - \mathrm{inj}_d(H, \overline{H}) = (-1)^{m-d} \binom{m}{d} (\mathrm{inj}(G, H) - \mathrm{inj}(H, H)) \;.$$

For contradiction we assume that $G \not\cong H$. Then $\mathrm{inj}(G, H) = 0$ (since G and H have equal numbers of edges, an isomorphism between them is the same thing as an injective homomorphism). But always $\mathrm{inj}(H, H) \ge 1$ (at least the identity on V is an injective homomorphism from H to H). Hence

$$|\mathrm{inj}_d(G, \overline{H}) - \mathrm{inj}_d(H, \overline{H})| \ge \binom{m}{d} \;.$$

Summation over $d = 0, 1, \ldots, m$ and the binomial expansion of $(1 + 1)^m$ give

$$2^m = \binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{m} \le \sum_{d=0}^{m} |\mathrm{inj}_d(G, \overline{H}) - \mathrm{inj}_d(H, \overline{H})| \le 2 \cdot n! \;.$$

The last inequality follows from the triangle inequality and the equalities

$$\sum_{d=0}^{m} \mathrm{inj}_d(G, \overline{H}) = \sum_{d=0}^{m} \mathrm{inj}_d(H, \overline{H}) = n!$$

— every permutation of V lies in exactly one set of injective mappings from V to V counted by $\mathrm{inj}_D(G, \overline{H})$, and the same for $\mathrm{inj}_D(H, \overline{H})$.

Exercise. *In which one?*

Now we get to contradiction quickly:

$$2^m \leq 2 \cdot n! < 2(n/2)^n \text{ because } n! < (n/2)^n \text{ for } n \geq 6$$

(but not for $n < 6$). Taking binary logarithm we get a contradiction, the inequality

$$m < 1 + n(\log_2 n - 1)$$

that negates the assumed lower bound $m \geq 1 + n(\log_2 n - 1)$. Thus $G \cong H$. □

For example, every graph on $n = 1024$ vertices can be reconstructed from its deck if it has at least $1+n(\log_2 n-1) = 9217$ edges (maximally it may have more than 500.000 edges). The previous theorem and proof are adapted Theorem 2.16 and its proof from the book

- P. Hell and J. Nešetřil, *Graphs and Homomorphisms*, Oxford University Press, Oxford, 2004.

What are the adaptations? Besides other things, I add the bound $n \geq 6$ on n and replace the condition $m > n(\log_2 n - 1)$ (this appears both in the book and in Müller's article), which is clearly not always negated by $m < 1+n(\log_2 n-1)$, with the correct lower bound $m \geq 1 + n(\log_2 n - 1)$.

Applications of fix-point theorems for posets

We prove the following classical result of naive set theory.

Theorem 1 (Cantor–Bernstein, 1887 and 1897) *If $f: X \to Y$ and $g: Y \to X$ are injections, then there is a bijection $h: X \to Y$ and moreover $h(x) = f(x)$ or $h(x) = g^{-1}(x)$ for every $x \in X$.*

Proof. Consider the mapping $F: \exp(X) \to \exp(X)$ (from the power set of X to itself) given by $F(M) = X \backslash g(Y \backslash f(M))$, for any $M \subseteq X$. Note that $M \subseteq N \subseteq X$ implies $F(M) \subseteq F(N)$ (complementation reverts $\subseteq$ and is applied twice). Also, the poset $(\exp(X), \subseteq)$ is a complete lattice (join is $\bigcup$ and meet is $\bigcap$). By the Tarski–Knaster theorem, F has a fix point: a set $A \subseteq X$ such that $F(A) = A$. This means that $X \backslash A = g(Y \backslash f(A))$ and

$$g^{-1}(X \backslash A) = Y \backslash f(A)\ .$$

We define $h: X \to Y$ by $h(x) = f(x)$ for $x \in A$ and by $h(x) = g^{-1}(x)$ for $x \in X \backslash A$. The displayed equality shows that h is everywhere defined on X and that it is surjective. If $x, y \in X$ are distinct then $h(x) \neq h(y)$ if $x, y \in A$ or if $x, y \notin A$, because of injectivity of f and of g^{-1}. But if $x \in A$ and $y \in X \backslash A$ then again $h(x) \neq h(y)$ by the displayed equality (and definition of h). Thus h is injective and is the desired bijection. □

In the next lecture I will show you a combinatorial proof of C.–B. theorem by directed and undirected graphs.