(L3) Conclusion of the proof of the C.-B. theorem (1)

$f: A\to B$, $g: B\to A$ injections, $A\cap B=\emptyset$, $V=A\cup B$,

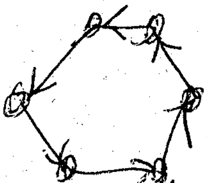
$G=(V,E)$, $E\subset V\times V$, $(a,b)\in E \iff f(a)=b$ or $g(a)=b$.

$\bar G=(V,\bar E)$, $\bar E\subset \binom{V}{2}$, $\{a,b\}\in\bar E \iff (a,b)\in E$ or $(b,a)\in E$.

The components of $\bar G$ are of three types:

(1) $v_0 \to \bullet \to \bullet\to\bullet\to\bullet \cdots$

(2a) $\cdots \to\bullet\to\bullet\to\bullet\to\bullet\to\bullet \cdots$

(2b)  and the arrows $\xrightarrow{f}$ and $\xrightarrow{g}$ always alternate.

It suffices to define h for each component. For type (1) we take every other arrow, starting from v_0:

$\bullet\xrightarrow{h}\bullet\quad \bullet\xrightarrow{h}\bullet\quad \bullet\xrightarrow{h}\bullet\ \dots$ For the types 2a and 2b we take every other edge arbitrarily:

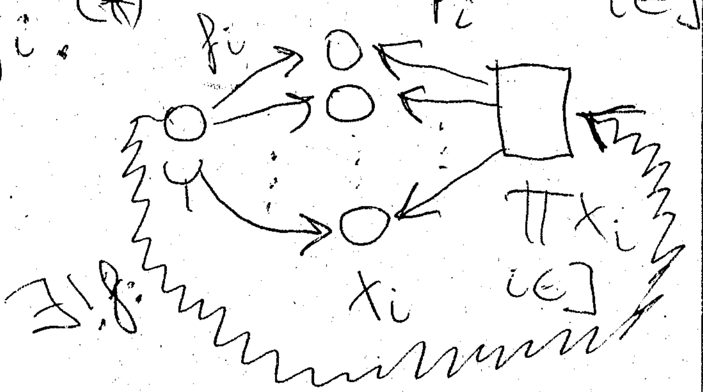
$\cdots\xrightarrow{h}\bullet\quad \bullet\xrightarrow{h}\bullet\quad \bullet\xrightarrow{h}\bullet\ \cdots;\quad \bullet\xleftarrow{h}\bullet\quad \bullet\xleftarrow{h}\bullet\quad \bullet\xrightarrow{h}\bullet$

The arrows form a perfect matching of G, and if we forget their orientation of $\bar G$: $M\subset\bar E$ and $\forall v\in V$ lies in exactly one $e\in M$ (a perf. matching M), we take the order in every $\{a,b\}\in M$ s.t. $(a,b)\in A\times B$ or $(b,a)\in A\times B$ and get the desired bijection $h: A\to B$. ☒

(3.6) More on the Cartesian product

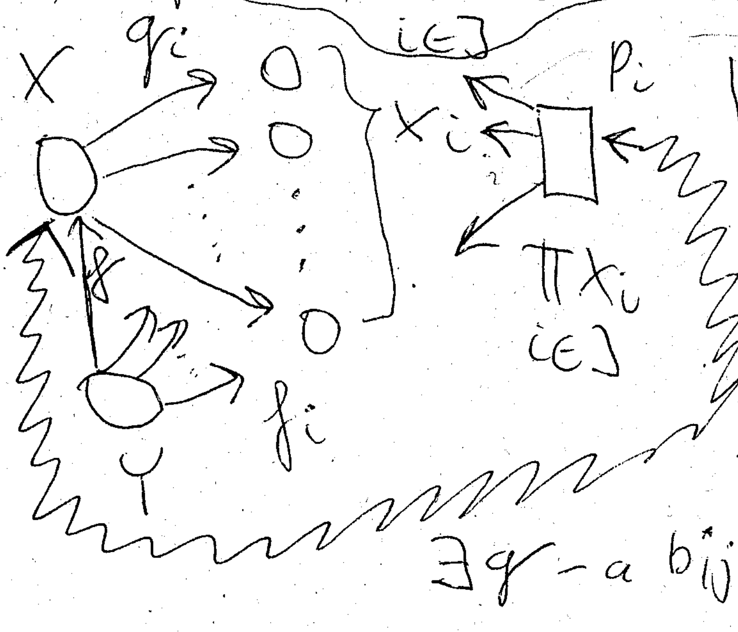
$p_j : \prod_{i\in I} X_i \to X_j$, $(x_i)_{i\in I} \mapsto x_j$, $j\in I$ is a projection (or j-th projection)

$\boxed{\text{Proposition}}$ For any system of maps $f_i : Y \to X_i$, $i\in I$, $\exists!\, f : Y \to \prod_{i\in I} X_i$ s.t. $\forall i \in I: p_i \circ f = f_i$. (*)



Proof.
If f satisfies condition (*) and $f(y) = (x_i)_{i\in I}$, then by (*) we have $x_i = f_i(y)$. And the map $y \mapsto (f_i(y))_{i\in I}$ satisfies (*) ⊠

$\boxed{\text{Proposition}}$ Let $q_i : X \to X_i$, $i \in I$, be such maps that for $\forall$ system of maps $f_i : Y \to X_i$, $i\in I$, $\exists!\, f : Y \to X$ s.t. (**) $\forall i \in I: q_i \circ f = f_i$. Then $\exists$ bijection $q : X \to \prod_{i\in I} X_i$ s.t. $p_i \circ q = q_i$.



Proof. By the prev. prop. we have a map $q : X \to \prod_{i\in I} X_i$ s.t. $q \circ p_i = q_i$. By the assumption (we take $\prod_{i\in I} X_i$ in the role of Y) $\exists$ $p : \prod_{i\in I} X_i \to X$

s.t. $\forall i \in J: q_i \circ p = p_i$. Thus: $p_i \circ (q \circ p) = p_i$ and $q_i \circ (p \circ q) = q_i$ $(\forall i \in J)$. By the unique solvability of (*) and (**) we have that $qp = id$ and $pq = id$, thus q is a bijection. $\boxtimes$

(4) Ordinals ~~numbers~~, cardinals, AC (the axiom of choice). $X \precsim Y$: $\exists$ injection from X to Y

same cardinality → $X \approx Y$: $\exists$ bijection —"—. It is trivial that $X \approx Y \Rightarrow X \precsim Y$ & $Y \precsim X$. We have proven the C.-B. theorem that $X \precsim Y$ & $Y \precsim X \Rightarrow X \approx Y$.

A relation $\leq$ on a set X is a well ordering (or X is well ordered by $\leq$) if
- $\leq$ is transitive
- $a \leq b$ & $b \leq a \Rightarrow a = b$
- $\forall a, b \in X$: $a \leq b$ or $b \leq a$
- $\forall M \subseteq X$, $M \neq \emptyset$ has the $\leq$ least element ($m \in M$ s.t. $n \in M \Rightarrow m \leq n$).

For example, the standard ordering $(\mathbb{N}, \leq)$ of the natural numbers is a well ordering - the principle of induction

Axiom of choice (AC)

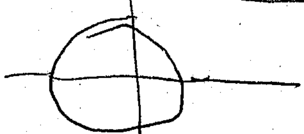
$\forall$ surjection $f: X \to Y$ $\exists g: Y \to X$: $f \circ g = id_Y$.

Problem 3.1 Prove that the next is equivalent with the AC:

$\forall$ set X s.t. $\emptyset \notin X$ $\exists f: X \to \bigcup X$: $f(A) \in A$ (for $\forall A \in X$).

$\boxed{\text{Thm. (Zermelo)}}$ AC $\Rightarrow$ every set has a well ordering

$\boxed{\text{AC} \Rightarrow \text{There exist unmeasurable sets}}$

$S = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$  the unit circle in the Euclidean plane. We would like to have a function $F: \mathcal{P}(S) \to [0,+\infty)$ s.t.:

a) $F(S) = 1$, b) $A_1, A_2, \ldots \subseteq S$ are mutually disjoint $\Rightarrow F(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} F(A_n)$, c) $\forall$ rotation r around the origin and $\forall A \subseteq S$: $F(A) = F(r(A))$.

$\boxed{\text{Theorem}}$ AC $\Rightarrow$ no such F exists.

Proof. An angle $\varphi \in [0, 2\pi)$ is rational if $\frac{\varphi}{\pi} \in \mathbb{Q}$, $r = r_\varphi$ is rotation around $\bar{O}$ by the angle φ. We define a bin. rel. $\sim$ on S by: $a \sim b \Leftrightarrow \exists$ rat angle φ s.t. $a = r_\varphi(b)$. $\boxed{\text{Problem 3.2.}}$ Prove that $\sim$ is an equivalence relation on S.

Let $P = S/\sim$ be the partition of S by $\sim$ and $X \subseteq S$ be a set such that $|X \cap A| = 1$ for $\forall A \in P$. X exists by the AC. We claim that $\{r_\varphi[X] \mid \varphi \in [0, 2\pi) \text{ is rational}\}$ is a partition of S. Indeed, if $\varphi \neq \psi \in [0, 2\pi)$ is rat. and $a \in r_\varphi[X] \cap r_\psi[X]$ then $a = r_\varphi(b) = r_\psi(c)$ for some $b, c \in X$, $b \neq c$. Then $b \sim a \sim c$ and $b \sim c$, contrary to the def. of X (the elements of X are mutually non-equivalent). Also, if $a \in S$ then $a \in A \in S/\sim$ for some $A \in P = S/\sim$. Thus $a = V_{\varphi}(b)$ where $\{b\} = X \cap A$ and φ is a rational angle ~~and~~ So $a \in V_{\varphi}[X] \subseteq \bigcup_{\substack{\varphi \in [0, 2\pi) \\ \text{is rat.}}} V_{\varphi}[X] = S$. It follows from the presumed properties of F (~~and~~ $\{\varphi \in [0, 2\pi) \mid \varphi \text{ is rational}\} = (\varphi_1, \varphi_2, \ldots)$ is countable) that

$$F(X) + F(X) + \cdots \overset{c)}{=} \sum_{n=1}^{\infty} F(V_{\varphi_n}[X]) \overset{b)}{=} F(S) \overset{a)}{=} 1$$

$$\left[\begin{array}{l} 0 + 0 + \cdots = 0 \quad (F(X) = 0) \\ c + c + \cdots = +\infty \quad (c = F(X) > 0) \end{array}\right.$$

Contradiction

$\boxtimes$