$\boxed{\text{Lecture 9}}$ ③ Fürstenberg; Cass–Wildenberg ①

article in Wikip. | better version, unjustly neglected

$X \subset \mathbb{Z}$ is *periodic*: $\exists a \in \mathbb{N}$ s.t. $X + a := \{x + a \mid x \in X\} = X$, $a$ is the *period of $X$*. $\forall b \in \mathbb{N}$: $ab$ is also a period of $X$. Properties of periodic sets:

- $X \subset \mathbb{Z}$ per. $\Leftrightarrow$ $\mathbb{Z} \setminus X$ per. | • $\emptyset \neq X \subset \mathbb{Z}$ per. $\Rightarrow$ $X$ is infinite
- $a + d\mathbb{Z} = \{a + dn \mid n \in \mathbb{Z}\}$ is per., $a \in \mathbb{Z}$, $d \in \mathbb{N}$.
- $X, Y \subset \mathbb{Z}$ per. $\Rightarrow$ $X \cup Y$ also per. (exercise for you)

For the contrary, let the set $\mathbb{P}$ of primes be finite. Then

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \in \mathbb{P}} (0 + p \cdot \mathbb{Z}) \quad \text{– } -1 \text{ and } 1$$

are the only integers not divisible by any prime. But $\mathbb{Z} \setminus \{-1,1\}$ is non-periodic, by above properties, and $\bigcup_{p \in \mathbb{P}} (0 + p\mathbb{Z})$ is periodic, by above properties ↯ ☒

④ Thue, by counting | We assume that $\mathbb{P}$ is finite and show that there are not enough prime factorizations to create all natural numbers. Let $\mathbb{P} = \{2 \leq p_1 < p_2 < \dots < p_q\}$, $q \in \mathbb{N}$. (2)

Then the $\#\{(a_1, \dots, a_q) \in \mathbb{N}_0^q \mid p_1^{a_1} p_2^{a_2} \cdots p_q^{a_q} \leq x\} \leq$ (for $x > 1$)

$\leq (1 + \log_2 x)^q = o(\lfloor x \rfloor)$, where $\lfloor x \rfloor = \#\{n \in \mathbb{N} \mid n \leq x\}$ — $\lightning$ $\boxtimes$

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$\boxed{\text{5. Erdős I}}$ Let, for any real $x > 1$,

$$\pi(x) := |\mathbb{P}_x| = |\mathbb{P} \cap \mathbb{N}_x| = |\mathbb{P} \cap \{1, 2, \dots, \lfloor x \rfloor\}|$$

the prime counting function; $\mathbb{P}_x$: the primes $\leq x$. Further, let

$S_x = \{1, 4, 9, 16, \dots, \lfloor \sqrt{x} \rfloor^2\}$ $(\square \leq x)$.

Consider the map

$$\boxed{\mathbb{N}_x \ni n \mapsto (A, b^2) \in \mathcal{P}(\mathbb{P}_x) \times S_x}$$ where $A =$

$=$ the primes with odd exponent in the prime fact. of $n$; $b^2$ is given by $n = a b^2$, $a = \prod_{p \in A} p$. So $b = \sqrt{n/a} \in \mathbb{N}$. It is clear that the map is an injection, because $n = \prod_{p \in A} p \cdot b^2$.

$\rightsquigarrow x-1 < |N_x| \le |\mathcal{P}(P_x)| \cdot |S_x| = 2^{|P_x|} \cdot |S_x| \le$

$\le 2^{\pi(x)} \sqrt{x} \rightsquigarrow \pi(x) > \dfrac{\log(\sqrt{x}-1)}{\log 2} \to +\infty$ with $x \to +\infty$ ☒

## 6 P. Erdős II

Now we prove that $\boxed{\pi(x) > \frac{cx}{\log x} \ \forall x \ge 2}$ and some constant $c > 0$. Let, for $n \in \mathbb{N}$, $b_n := \binom{2n}{n} = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ (the prime factorization)

$\boxed{\text{Lemma}}$ Each $p_i^{a_i} \le 2n$. Proof: Below.

$\rightsquigarrow b_n \le (2n)^{\pi(2n)}$. On the other hand, induction shows that $\forall n$: $b_n \ge 2^n$. Thus $\pi(2n) \ge \dfrac{n(\log 2)}{\log(2n)}$, which give the bound. ☒

But we have to prove the lemma. For $n \in \mathbb{N}$ and $p$ a prime, let $v_p(n) :=$ the exponent of $p$ in the prime factorization of $n$. For example, $v_3(60) = 1$ as $60 = 2^2 3^1 5^1$. It is not hard to see that $\forall n \in \mathbb{N}$: $\mathrm{Ord}_p(n!) =$

$= \sum_{j=1}^{\infty} \lfloor \frac{n}{p^j} \rfloor$. Indeed $\operatorname{ord}_p(n!) = \sum_{k=1}^{n} \operatorname{ord}_p(k) =$ (4)

(only finitely many $\neq 0$ summands) (additivity of $\operatorname{ord}_p(\cdot)$)

$= \#\{(p^j, k) \mid j, k \in \mathbb{N}, p^j \mid k, k \leq n\} =$

grouped by $k$

$= \sum_{j \geq 1} \lfloor \frac{n}{p^j} \rfloor$. grouped by $p^j$. (this formula is due to Legendre: Adrien-Marie Legendre (1752–1833) – portrait debâcle)

So, for $n \in \mathbb{N}$, $\operatorname{ord}_p(\binom{2n}{n}) = \operatorname{ord}_p((2n)!) - 2 \operatorname{ord}_p(n!) \underset{\text{Legendre}}{=} \sum_{j \geq 1} \underbrace{\left( \lfloor \frac{2n}{p^j} \rfloor - 2 \lfloor \frac{n}{p^j} \rfloor \right)}_{\in \{0,1\}}$ because

$\forall a \in \mathbb{R}: \lfloor 2a \rfloor - 2 \lfloor a \rfloor \in \{0,1\}$.

$\rightsquigarrow \underbrace{\operatorname{ord}_p(\binom{2n}{n})}_{\nu} \leq$ max $j \in \mathbb{N}$ s.t. $p^j \leq 2n$.

$\rightsquigarrow p^\nu \leq 2n$, which was to be proved (of Lemma) $\boxtimes$

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the last 4 proofs elaborate ~~the~~ the Euler identity $\boxed{\prod \frac{1}{1-p^{-1}} = \sum \frac{1}{n}}$.

## 7 Infinite distributive law

L. Euler:

$$\prod_{p\in\mathbb{P}} \frac{1}{1-p^{-1}} = \prod_{p\in\mathbb{P}} \left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right) = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty,$$

which is a contradiction, if $|\mathbb{P}|<+\infty$, for the product is finite and cannot be equal to the $\infty$. The $1^{st}$ and $3^{rd}$ equality are fine, but what about the $2^{nd}$ $=$? It is in fact an instance of the distributive law for absolutely convergent series. ~~Infinite!~~

## 8 J. Sylvester

• James J. Sylvester (1814–1897)

– finitization of 7

If $\boxed{\forall \text{ real } x>1 : \prod_{p\le x} \frac{1}{1-p^{-1}} \ge \sum_{n\le x} \frac{1}{n}}$, then

$|\mathbb{P}|=\infty$ follows since the $\sum$ goes to $+\infty$ with $x$.

**Proof.** (of $\geq$) Since $\frac{1-y^{q+1}}{1-y} = 1+y+y^2+\cdots+y^q$, We have for every real $q\in(0,1)$ and $k\in\mathbb{N}_0$ that $(1-q)^{-1} > 1+q+q^2+\dots+q^k$. We set $q:=p^{-1}$, and $k\in\mathbb{N}_0$ to be maximum one with $p^k\le x$.

$$\leadsto \prod_{p\le x}(1-p^{-1})^{-1} > \prod_{p\le x}\left(1+p^{-1}+p^{-2}+\dots+p^{-k_p}\right) =$$

$$= \underbrace{n_1^{-1}+n_2^{-1}+\dots+n_l^{-1}}_{} \ge \sum_{n\le x} n^{-1}.$$

$n_i\in\mathbb{N}$ are exactly the numbers ~~with~~ whose prime factorizations have every prime power $\le x$. $\boxtimes$

$\boxed{\text{9 L. Euler – analytical proof.}}$ Let $s\in\mathbb{R}$ with $s>1$. Then

$$\prod_{p\in\mathbb{P}}\frac{1}{1-p^{-s}} = \prod_{p\in\mathbb{P}}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\dots\right) = \sum_{n=1}^{\infty}\frac{1}{n^s} < +\infty$$

(Euler's most famous identity). If $\mathbb{P}$ were finite, then for $s\to 1^+$ the LHS goes to the finite limit $\prod_{p\in\mathbb{P}}\frac{1}{1-1/p}$, but the RHS goes to $+\infty$, ↯ $\boxtimes$

## (10) An algebraic proof by formal Dirichlet series.

We need the <u>Möbius function</u> $\mu : \mathbb{N} \to \{-1, 0, 1\}$,

$$\mu(n) = \begin{cases} 1 & \dots\ n = 1 \\ (-1)^r & \dots\ n = p_1 p_2 \cdots p_r,\ p_1 < p_2 < \dots < p_r \text{ are primes} \\ 0 & \dots \text{ else.} \end{cases}$$

$\boxed{\text{Lemma}}$ For $n \in \mathbb{N}$,

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \dots\ n = 1 \\ 0 & \dots\ n > 1. \end{cases}$$

$\boxed{\text{Proof.}}$ Exercise for you. $\boxtimes$

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In the ring of formal Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ where $a_n \in \mathbb{C}$ and $s$ is a <u>formal</u> variable, the Lemma $\iff$ $\boxed{\sum \frac{1}{n^s} \cdot \sum \frac{\mu(n)}{n^s} = 1}$. ~~Now we have in this ring also that~~ Alternatively:

$$\boxed{\underbrace{\zeta(s) = \sum_n \frac{1}{n^s}}_{\text{formal zeta-function}} \underbrace{= \prod_{p \in P} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \prod_{p \in P} \frac{1}{1 - p^{-s}}}_{\text{formal variant of Euler's id.}} = \frac{1}{\prod_{p \in P}(1 - 1/p^s)} = \frac{1}{\sum \mu(n)/n^s}}$$

If $\|P\| < \infty$ then $\sum \frac{\mu(n)}{n^s} = \sum_{i=1}^{\ell} \frac{\mu(n_i)}{n_i^s}$,

where $n_1, n_2, \dots, n_\ell$ are all $\square$-free numbers, $(j \in \mathbb{N})$ $=$

$= \sum_{n=1}^{j} \frac{a_n}{n^s}$, $a_1 = \mu(1) = 1 \neq 0$. But in

$$1 = \sum_{n=1}^{j} \frac{a_n}{n^s} \cdot \sum \frac{1}{n^s} = \sum \Big( \sum_{\substack{m \mid n \\ m \le j}} a_m \Big) \frac{1}{n^s}$$

the coefficient is $\neq 0$

for $\infty$ many $n$, e.g. for $n = j!+1,\ 2\cdot j!+1,\ 3\cdot j!+1, \dots$ when it equals $a_1 \neq 0$. We have a $\lightning$. $\boxtimes$

Thank~~s~~ you!!

[author(s) of the note

proof: R. Kleber and (later) P. Pollack.