$\boxed{\text{Lecture 8}}$ Another geometric argument, in this case, as it turns out, in reality not geometric.

For $n \in \mathbb{N}$, $\tau(n) := \sum_{d \in \mathbb{N}, d|n} 1 = |\{(a,b) \in \mathbb{N}^2 \mid ab = n\}|$

= the # of divisors of n. For instance, $\tau(1) = 1$, $\tau(5) = 2$, $\tau(12) = 6$ $(1, 2, 3, 4, 6, 12)$. A formula for τ: if $n = p_1^{a_1} \cdots p_k^{a_k}$ ($p_1 < \cdots < p_k$ primes, $a_i \in \mathbb{N}$) then $\tau(n) = (1+a_1)(1+a_2)\cdots(1+a_k)$ $\left(12 = 2^2 \cdot 3^1 \rightsquigarrow \tau(12) = (1+2)(1+1) = 6\right)$

Average value of τ? That is, $\frac{1}{x} \sum_{n \le x} \tau(n) = ?$, for $x \to +\infty$.

Let $\gamma := \lim_{x \to +\infty} \left(\sum_{n \le x} \frac{1}{n} - \log x \right)$ - the Euler (-Mascheroni) constant.

$\boxed{\text{Lemma}}$ For $x \to +\infty$, $\sum_{n \le x} \frac{1}{n} = \log x + \gamma + O(1/x)$, where $\gamma = 0.57722\ldots$ is a constant.

$\boxed{\text{OPEN PROBLEM}}$

$\gamma \in \mathbb{R} \setminus \mathbb{Q}$? Not known!!

Proof. (of the Lemma) For $n \in \mathbb{N}$, $\int_n^{n+1} \frac{dt}{t} = \log\left(1 + \frac{1}{n}\right) = \frac{1}{n} + z(n)$, $z(n) = -\frac{1}{2n^2} + \frac{1}{3n^3} - \dots = O(n^{-2})$

$$\rightsquigarrow \sum_{n=1}^{N} \frac{1}{n} = \sum_{n=1}^{N} \left(\int_n^{n+1} \frac{dt}{t} - z(n) \right) = \int_1^{N+1} \frac{dt}{t} -$$

$(N \in \mathbb{N})$

$$- \underbrace{\sum_{n=1}^{\infty} z(n)}_{=:\,\gamma > 0} + \underbrace{\sum_{n=N+1}^{\infty} z(n)}_{=O(1/N)} = \underbrace{\log(N+1)}_{} + \gamma + O\left(\frac{1}{N}\right) =$$

$\log\left(N\left(1+\frac{1}{N}\right)\right) = \log N + \log\left(1 + \frac{1}{N}\right)$

$$= \log N + O\left(\frac{1}{N}\right) + \gamma + O\left(\frac{1}{N}\right)$$

$$= \log N + \gamma + O\left(\frac{1}{N}\right).$$

Why? $|z(n)| \le \frac{1}{2n^2}$, hence

$$\left| \sum_{n=N+1}^{\infty} z(n) \right| \le \sum_{n=N+1}^{\infty} |z(n)| \le \frac{1}{2} \sum_{n=N+1}^{\infty} \frac{1}{n^2} < \frac{1}{2} \int_N^{+\infty} \frac{dx}{x^2} =$$

$$= \frac{1}{2} \frac{1}{N}.$$

Now we can state the asymptotic theorem on the summatory function $\sum_{n \le x} \tau(n)$.

$\boxtimes$

Theorem (Dirichlet, 1849) For $x \to +\infty$,

$$\sum_{n\le x} \tau(n) = x\log x + (2\gamma-1)x + O(x^{1/2}),$$

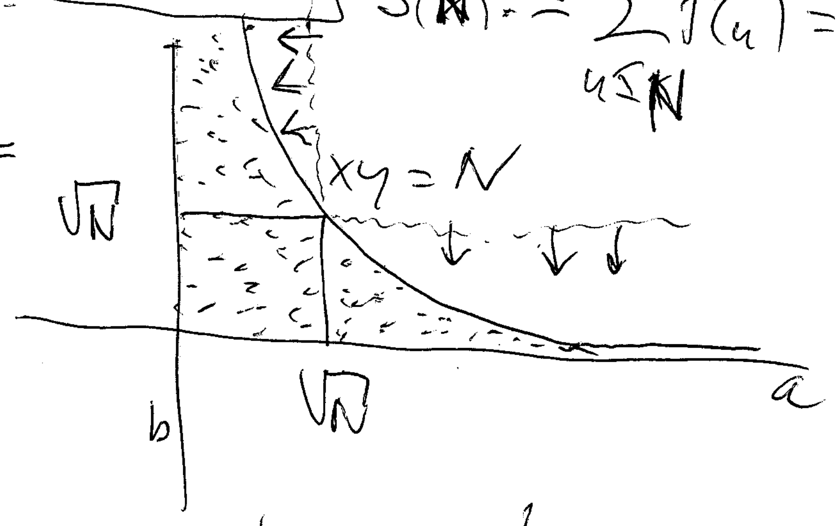
where $\gamma = 0.57722\ldots$ is the Euler constant.

Thus the average is, for $x \to +\infty$, $\frac{1}{x}\sum_{n\le x}\tau(n) =$

$$= \log x + \underbrace{2\gamma-1}_{0.15444\ldots} + O(x^{-1/2}).$$

Proof, Geometry:

$$S(N) := \sum_{n\le N}\tau(n) = \#\{(a,b)\in\mathbb{N}^2 \mid ab\le N\} =$$



$$= 2\#\{(a,b)\in\mathbb{N}^2 \mid a\le\sqrt{N} \ \&\ ab\le N\} - \#\{(a,b)\in\mathbb{N}^2 \mid a\le\sqrt{N} \ \&\ b\le\sqrt{N}\}$$

$$= 2\sum_{a\le\sqrt{N}} \underbrace{\left\lfloor \frac{N}{a}\right\rfloor}_{\#\text{ of } b\text{'s}} - \underbrace{\lfloor\sqrt{N}\rfloor^2}_{=(\sqrt{N}-\{\sqrt{N}\})^2=\ldots = N+O(\sqrt{N})} = 2\sum_{a\le\sqrt{N}}\left(\frac{N}{a} - \underbrace{\left\{\frac{N}{a}\right\}}_{\in[0,1)}\right) -$$

$$=$$

$$= 2N \sum_{a \le \sqrt{N}} \frac{1}{a} + O(\sqrt{N}) - N + O(\sqrt{N}) =$$ 4

$$\underset{\text{Lemma}}{=} 2N \left(\log(\sqrt{N}) + \gamma + O(1/\sqrt{N}) \right) - N + O(\sqrt{N}) =$$

$$= N \log N + (2\gamma - 1) N + O(\sqrt{N}) + O(\sqrt{N}) =$$

$$= N \log N + (2\gamma - 1) N + O(\sqrt{N}).$$ ☒

[hyperbola method]

• Georgii F. Voronoj (1868 – 1908): $O(N^{1/3})$;

Voronoy
Voronoï

more precisely, $O(N^{1/3} \log N)$

Chapter 4 – Prime numbers

$p \in \mathbb{N} = \{1, 2, 3, \ldots\}$ is prime $\iff$ $p > 1$ and $\forall a, b \in \mathbb{N}: p = ab \Rightarrow \{a, b\} = \{1, p\}$.

$\mathbb{P} = \{\text{primes}\} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\}$

Theorem (Euclid, $\doteq -300$) $|\mathbb{P}| = \infty$ – there are infinitely many prime numbers.

I will show you 10 (ten) proofs of the thm.

(1) Euclid's proof. We show that for every finite set of primes A $\exists$ a prime $p \notin A$. Indeed, we may take for p any prime divisor of the number $n := \underbrace{\prod_{q \in A} q}_{:= 1 \text{ if } A = \emptyset} + 1$. Then $p \notin A$, for else we would have that $p \mid 1 = n - (n-1)$ $\lightning$ $\boxtimes$

• Euclid (fl. -300)

(2) Goldbach's proof. • Christian Goldbach (1690 – 1764)

If $1 \le n_1 < n_2 < \dots$ are integers s.t. for any $i<j$, $(n_i, n_j) = 1$, i.e. n_i and n_j are coprime numbers. We select some prime divisors $p_i \mid n_i$, $i = 1, 2, \dots$ $\rightsquigarrow$ $(p_1, p_2, \dots)$ is an ∞-te sequence of mutually different primes. $\boxtimes$ For the n_i we may for example take the numbers $n_1 = 2$ and $n_i = n_1 n_2 \cdots n_{i-1} + 1$ for $i > 1$. So

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$u_1 = 2$, $u_2 = 2+1 = 3$, $u_3 = 2\cdot 3 + 1 = 7$, $u_4 = 2\cdot 3\cdot 7 + 1 = 43$, $u_5 = 2\cdot 3\cdot 7\cdot 43 + 1 = 42\cdot 43 + 1 = 1807$, ...

$\boxtimes$

Interestingly, the numbers $F_n := 2^{2^n} + 1$, $n \in \mathbb{N}_0$, so $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 65537$, ... have the "Goldbach property" too, because

$F_0 F_1 F_2 \cdots F_n = F_{n+1} - 2 \leadsto i < j \Rightarrow (F_i, F_j) = 1.$

F_0, F_1, F_2, F_3, F_4 are all prime numbers.

Fermat's conjecture: $\forall n \in \mathbb{N}_0 : F_n \in \mathbb{P}$.

L. Euler: $F_5 = 2^{32} + 1 \notin \mathbb{P}$ because $641 \mid F_5$:

Open problem: $\forall n > 4 : F_n \notin \mathbb{P}$.

Modulo 641 we have

$2^{32} = 2^4 \cdot 2^{28} = 16 \cdot 2^{28} = (641 - 625)\, 2^{28} = (641 - 5^4)\, 2^{28} =$

$\equiv -(5 \cdot 2^7)^4 = -(5 \cdot 128)^4 = -640^4 =$

$= -(641 - 1)^4 = -1^4 + 641(\dots) \equiv -1$.

Thanks you!