Lecture 3 We can generalize the argument showing transcendence of 0.110001000000000000000001000... to

Corollary $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ s.t. $\forall n \in \mathbb{N}\ \exists \frac{p}{q} \in \mathbb{Q}$: $q \geq 2$ and $|\alpha - \frac{p}{q}| < \frac{1}{q^n}$. Then α is a transcendental number.

Liouville numbers

Another lower bound on approximability of real numbers by fractions is the next theorem.

Theorem (Hurwitz, 1891) a) $\forall \alpha \in \mathbb{R}\setminus\mathbb{Q}\ \exists$ infinitely many $\frac{p}{q} \in \mathbb{Q}$ s.t. $\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\,q^2}$.

b) $\forall$ real $A > \sqrt{5}$, the inequality $\left|\frac{\sqrt{5}-1}{2} - \frac{p}{q}\right| < \frac{1}{Aq^2}$ has only finitely many rat. solutions $\frac{p}{q}$.

Proof. By Farey fractions (first part), see my LNs. • Adolf Hurwitz (1859–1919)

One can prove that besides $\frac{\sqrt{5}-1}{2}$, the numbers badly approximable by b) are exactly the α

with the continued fraction $\alpha = [a_0, a_1, \dots, a_{n_0}, 1, 1, 1, 1, 1, 1, 1, \dots]$, i.e. $a_n = 1$ for $n > n_0$.

An important strengthening of Liouville's ineq. is the following Thue's ineq.

$\boxed{\text{Theorem (A. Thue, 1909)}}$ $\alpha \in \mathbb{R}$ algebraic, $\deg(\alpha) = n \geq 2$ and $\varepsilon > 0$. Then $\exists\, c = c(\alpha, \varepsilon) > 0$ s.t.

$$\frac{p}{q} \in \mathbb{Q}\ (q > 0) \Rightarrow \left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^{1+\varepsilon+n/2}}.$$

- $n = 2$ ... trivial • $n \geq 3$ ... strengthens L.'s ineq.
- Axel Thue (1863–1922)

Thue's main motivation was to deduce from his ineq. that the Diophantine equations

$x^3 - 2y^3 = 1$, or more generally

$x^3 - 2y^3 = a \in \mathbb{Z}$, or even more generally

$$\overbrace{a_n x^n + a_{n-1} x^{n-1} y + a_{n-2} x^{n-2} y^2 + \dots + a_1 x y^{n-1} + a_0 y^n}^{P(x,y)} = a,$$

where $a_i, a \in \mathbb{Z}$, $\deg P = n \geq 3$, and $P(x, 1)$ has ≥ 3 distinct roots, has only fin. many solutions $x, y \in \mathbb{Z}$?

- unfortunately, Thue's method (of solving ~~eq.~~ eq. $P(x,y)=a$) is non-effective.
- Runge's method (1889) ← effective!
  (Runge–Kutta method)
- Carl Runge (1856–1927)

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the ultimate strengthening of L.'s ineq and T.'s ineq. is the following theorem.

$\boxed{\text{Theorem (K. Roth, 1955)}}$ $\alpha\in\mathbb{R}$ algebraic, $\deg(\alpha)=n\geq 2$ and $\varepsilon>0$. Then $\exists\, c=c(\alpha,\varepsilon)>0$ s.t.

$$\frac{p}{q}\in\mathbb{Q}\ (q>0)\Rightarrow\left|\alpha-\frac{p}{q}\right|\geq\frac{c}{q^{2+\varepsilon}}.$$

(non-eff.). Proof's of Thue's ineq. and Roth's ineq. are difficult.

- Klaus Roth (1925–2015)
- Fields ~~meda~~ medal in 1958

Thue equations $P(x,y)=a$ were effectively solved in 1968 by • Alan Baker (1939-2018)

• Fields medal 1970; eff. estimates for linear forms in logarithms of algebraic numbers.

• John H. Conway... [• H. Davenport]

However, already in 1918 A. Thue could eff. solve some particular cases of "his" Dioph. equation.

the last result of the 1st chapter is the next

$\boxed{\text{Theorem (Ch. Hermite, 1873)}}$ $e = 2.71828\ldots$ is a transcendental number.

Proof By D. Hilbert (1893) For the contrary, let

$a_n e^n + \cdots + a_1 e + a_0 = 0$ for some $n \in \mathbb{N}$ and $a_i \in \mathbb{Z}$ s.t. $a_0 \neq 0$ ($\Leftrightarrow a_0 a_1 \cdots a_n \neq 0$).

$$\rightsquigarrow a_n e^n \int_0^{+\infty} + a_{n-1} e^{n-1} \int_0^{+\infty} + \cdots + a_1 e \int_0^{+\infty} + a_0 \int_0^{\infty} = 0$$

where $\int_0^{+\infty} := \int_0^{+\infty} x^r \left((x-1)(x-2)\cdots(x-n)\right)^{r+1} e^{-x}\,dx$,

with $r \in \mathbb{N}$ to be determined later.

$$\rightsquigarrow \underbrace{\left(\sum_{i=0}^{n} a_i e^i \int_0^i\right)}_{P(r)} + \underbrace{\left(\sum_{i=0}^{n} a_i e^i \int_i^{+\infty}\right)}_{Q(r)\in\mathbb{Z}} = 0.$$

$\circledast \nearrow$

We shall prove that ① $\exists c>1\ \forall r\in\mathbb{N}: |P(r)|<c^r$.

② $\exists$ infinitely many $r\in\mathbb{N}$ s.t. $|Q(r)|\geq r!$

Since $\lim_{r\to\infty}\frac{c^r}{r!}=0$, ① & ② & $\circledast$ $\Rightarrow$ $\lightning$ (a contradiction)

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① is easy: $x\in[0,n] \Rightarrow |x^r((x-1)(x-2)\cdots(x-n))^{r+1}| \leq$
$\leq n^r(n^n)^{r+1}$ and $|e^{-x}|\leq 1$.

$$\rightsquigarrow \left|\int_0^i\right| = \left|\int_0^i x^r((x-1)\cdots(x-n))^{r+1}e^{-x}dx\right| \leq i\,n^r(n^n)^{r+1}$$
$$\leq (n^{n+1})^{r+1}$$

and $$|P(r)| \leq \sum_{i=0}^{n}|a_i|e^i\left|\int_0^i\right| \leq \left(\sum_{i=0}^{n}|a_i|e^i\right)(n^{n+1})^{r+1}.$$

To prove ② is more tricky. First, for any $k\in\mathbb{N}_0$ we have that (for $k>0$)

$$\int_0^{+\infty}x^k e^{-x}dx = \underbrace{\left[-x^k e^{-x}\right]_0^{+\infty}}_{=0-0=0} + k\int_0^{\infty}x^{k-1}e^{-x}dx$$

and $$\int_0^{+\infty}x^0 e^{-x}dx = \int_0^{+\infty}e^{-x}dx = \left[-e^{-x}\right]_0^{+\infty} = -e^{-\infty}-(-e^{-0}) = 1.$$

Thus $q \in \mathbb{N}_0 \Rightarrow \int_0^{+\infty} x^q e^{-x}\,dx = q!$ More generally, if $p(x) = b_n x^n + \dots + b_1 x + b_0$ then $\int_0^{+\infty} p(x) e^{-x}\,dx = \sum_{q=0}^{n} b_q\, q!$.

$\leadsto$ $p \in \mathbb{Z}[x]$, $q \in \mathbb{N}_0$ $\Rightarrow$ the integer $\int_0^{+\infty} x^q p(x) e^{-x}\,dx \equiv b_0\, q! \pmod{(q+1)!}$

By the substitution $y := x - i$ $(i = 0, 1, \dots, n)$ we get:

• Charles Hermite (1822–1901)

$$e^i \int_i^{+\infty} = \int_i^{+\infty} x^r ((x-1)(x-2)\cdots(x-n))^{r+1} e^{-(x-i)}\,dx$$

$$= \int_0^{+\infty} \underbrace{(y+i)^r ((y+i-1)(y+i-2)\cdots(y+i-n))^{r+1}} e^{-y}\,dy.$$

• David Hilbert (1862–1943)

$= (-1)^{n(r+1)} (n!)^{r+1} y^r + \bullet\, y^{r+1} + \dots$ ... for $i = 0$

$= \bullet\, y^{r+1} + \bullet\, y^{r+2} + \dots$ ... for $1 \leq i \leq n$ $(\bullet \in \mathbb{Z})$.

$$\leadsto Q(r) = \underbrace{\sum_{i=0}^{n} a_i e^i \int_i^{+\infty}}_{\in \mathbb{Z}} \equiv a_0 (-1)^{n(r+1)} (n!)^{r+1} r! \pmod{(r+1)!}.$$

Thus $r! \mid Q(r)$. If $(r+1, a_0 n!) = 1$ then $Q(r) \neq 0$ and hence $|Q(r)| \geq r!$ This proves ②.