Lecture 2

Some remarks on continued fractions

$\forall \alpha \in \mathbb{R}$, $\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\vdots}}}}$ where

$a_0 = \lfloor \alpha \rfloor \in \mathbb{Z}$ but $a_n \in \mathbb{N} = \{1, 2, \dots\}$ for $n \geq 1$.

We write $\alpha = [a_0, a_1, a_2, \dots]$. We saw last time that $\sqrt{2} = [1, 2, 2, 2, \dots]$. A finite example: $-\frac{119}{27} =$

$= [-5, 1, 1, 2, 5] = -5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \frac{1}{5}}}} = [-5, 1, 1, 2, 4, 1]$

n-th convergent of α is $\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}}$

[sblížený zlomek]

Then $p_0 = a_0$, $p_1 = a_0 a_1 + 1$,

$q_0 = 1$, $q_1 = a_1$ and for $n \geq 2$,

$p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$

thm. Let $\frac{p_n}{q_n} \in \mathbb{Q}$ be the n-th convergent of $\mathbb{R} \ni \alpha = [a_0, a_1, \dots]$. Then

1) $\forall n \geq 0$: $\frac{p_{2n}}{q_{2n}} \leq \alpha \leq \frac{p_{2n+1}}{q_{2n+1}}$

2) $[a_0, a_1, \dots]$ is finite $\iff \alpha \in \mathbb{Q}$.

3) $\alpha \in \mathbb{Q} \Rightarrow \alpha =$ the last convergent

4) $\alpha \in \mathbb{R} \setminus \mathbb{Q} \Rightarrow \frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \dots < \alpha < \dots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}$

and $\lim_{n \to \infty} \frac{p_n}{q_n} = \alpha$.

5) $\forall n \in \mathbb{N}: \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}$. – Dirichlet's thm.

6) $[a_0, a_1, \dots]$ is eventually periodic or finite $\Leftrightarrow$

$\Leftrightarrow \exists a, b, c \in \mathbb{Z}: a\alpha^2 + b\alpha + c = 0$. $[\mathbb{Q}(\alpha):\mathbb{Q}] \le 2$

$\frac{1+\sqrt{5}}{2} = [1, 1, 1, 1, \dots]$, $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \dots, 1, 1, 2n, \dots] = [2, \overline{1, 2n, 1, 1}]$

for $n = 1, 2, \dots$. But the c. fraction of π,

$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, \dots]$ does not follow any known rule.

An unsolved problem Are the entries in the c. fraction of $\sqrt[3]{2}$, $\sqrt[3]{2} = [1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 14, 12, 2, 3, 2, 1, 3, 4, 1, 1, 2, 14, 3, 12, 1, 15, 3, 1, 4, 534, 1, 1, \dots]$ bounded?

there are many identities involving c. fractions. In conclusion I mention just two.

<u>Rogers-Ramanujan c. fraction</u>:

$$\cancel{q^{1/5}} \prod_{n=1}^{\infty} \frac{(1-q^{5n-1})(1-q^{5n-4})}{(1-q^{5n-2})(1-q^{5n-3})} = \cfrac{\cancel{q^{1/5}}\, 1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ddots}}}}$$

FLAJOLET

One of <u>P. Flajolet's</u> → <u>Anal. Comb.</u> c. fractions:

if B_n is the n-th Bell number (the # of partitions of $\{1,2,\dots,n\}$) then ($B_0 := 1$)

<u>[my c. fr. 2]</u>

$$\sum_{n=0}^{\infty} B_n x^n = \cfrac{1}{1-x-\cfrac{x^2}{1-2x-\cfrac{2x^2}{1-3x-\cfrac{3x^2}{\ddots}}}}$$

See the on-line available slides by <u>X. G. Viennot: Combin. aspects of c-fractions and applications</u>

I defined the <u>Farey fractions</u> F_n, $n \in \mathbb{N}$, last time and today we can prove the main theorem on them.

Theorem (Farey–Cauchy, 1816) If $\frac{a}{b} < \frac{c}{d}$ are two consecutive elements of F_n, then $\frac{c}{d} - \frac{a}{b} = \frac{1}{db} \iff bc - ad = 1$. ($\iff \frac{a}{b}$ and $\frac{c}{d}$ are as close as possible).

Proof. We solve the Diophantine equation $bx - ay = 1$, $x, y \in \mathbb{Z}$?

$(a,b) = 1 \rightsquigarrow$ solution exists. We show that c, d is a sol. x, y a sol. $\Rightarrow x + ra, y + rb$ a sol., for any $r \in \mathbb{Z} \rightsquigarrow$ $\underline{\exists \text{ a solution } x_1, y_1 \in \mathbb{Z} \text{ s.t.}}$

$\underline{n - b < y_1 \leq n}$. $bx_1 - ay_1 = 1 \rightsquigarrow \frac{x_1}{y_1} = \frac{1}{by_1} + \frac{a}{b}$.

$\rightsquigarrow \frac{x_1}{y_1} \in F_n$ $\left((x_1, y_1) = 1, 0 < y_1 \leq n \rightsquigarrow 0 < x_1 < y_1\right)$.

$\rightsquigarrow \frac{x_1}{y_1} \geq \frac{c}{d}$. $\frac{x_1}{y_1} > \frac{c}{d}$? $\frac{x_1}{y_1} - \frac{c}{d} \geq \frac{1}{dy_1}$ (trivi

$\frac{c}{d} - \frac{a}{b} \geq \frac{1}{bd}$ ineq-s)

$\left(\frac{1}{by_1} =\right) \frac{x_1}{y_1} - \frac{a}{b} \geq \frac{1}{dy_1} + \frac{1}{bd} =$

$= \frac{b + y_1}{bdy_1} \rightsquigarrow d \geq b + y_1 > n$ ↯ Dirichlet's thm.? ☒

$\alpha \in \mathbb{C}$ je algebraické číslo
is an algebraic number $: \exists f \in \mathbb{Q}[x], f \neq 0,$
s.t. $f(\alpha)=0$. $\deg(\alpha)$:= minimum degree of such polynomial $f(x)$. Transcendental numbers := $\mathbb{C} \setminus \{\text{algebraic \#s}\}$. J. Liouville proved in 1844 existence of tr. numbers, as a corollary of the next thm.

$\boxed{\text{Theorem (J. Liouville, 1844)}}$

$\forall$ algebraic $\alpha \in \mathbb{R}$, $\deg(\alpha) \geq 2 =: n$ $\exists\, c = c(\alpha) > 0$ s.t.
$$\frac{p}{q} \in \mathbb{Q} \Rightarrow \left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^n}.$$

Proof. Let $P(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ be s.t. $a_n \neq 0$ and $P(\alpha)=0$. Let $I := [\alpha-1, \alpha+1]$ and $c := \min\left(1, \frac{1}{\max_{x \in I} |P'(x)|}\right)$. Two cases:

1) $\frac{p}{q} \notin I \rightsquigarrow$ trivially $\left|\alpha - \frac{p}{q}\right| \geq 1 \geq \frac{1}{q^n} \geq \frac{c}{q^n}$.

2) $\frac{p}{q} \in I \rightsquigarrow$ Lagrange: $\frac{P(\alpha) - P(p/q)}{\alpha - p/q} = P'(x_0)$ for some $x_0 \in \mathbb{R}$ between p/q and α. Since $P(\alpha)=0$,

we have $\left|\alpha - \frac{p}{q}\right| = \left|\frac{P(p/q)}{P'(t_0)}\right| \geq \frac{c}{q^n}$ because

$\frac{1}{|P'(t_0)|} \leq c$ and $|P(p/q)| = \frac{\overbrace{|a_n p^n + \dots + a_1 p q^{n-1} + a_0 q^n|}^{\neq 0,\ \geq 1}}{q^n}$

$\geq \frac{1}{q^n}$. $P(p/q) \neq 0$ because $P(p/q) = 0 \rightsquigarrow \alpha$ is a root

of $\frac{P(x)}{x - \frac{p}{q}} \in \mathbb{Q}[x]$ and so $\deg(\alpha) < n$?? $\boxtimes$

$\boxed{\text{Corollary}}$ $\alpha := \sum_{n=1}^{\infty} 10^{-n!}$ is transcendental #.

Proof. ~~mmmm~~ - exercise for you!

Thank you!